Exercice 2 :

$X_t = B_t \Rightarrow dX_t = dB_t$

$u(x) = x^3$ $\qquad \frac{\partial u}{\partial t} = 0, \ \frac{\partial u}{\partial x} = 3x^2, \ \frac{\partial^2 u}{\partial x^2} = 6x$

$Y_t = u(B_t) = B_t^3$

$$dY_t = \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial x} dB_t + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} \underbrace{(dB_t)^2}_{dt}$$

$$= 0 + 3B_t^2 dB_t + 3B_t dt$$

$$\Rightarrow d(B_t^3) = 3B_t^2 dB_t + 3B_t dt$$

$$\Rightarrow \frac{1}{3} B_t^3 = \int_0^t B_s^2 dB_s + \int_0^t B_s ds$$

$$\Rightarrow \boxed{\int_0^t B_s^2 dB_s = \frac{1}{3} B_t^3 - \int_0^t B_s ds}$$

Exercice 3 : B_t un brownien p-dimensionel. $R(t) = \|B_t\| = \sqrt{\sum_{i=1}^{p} (B_t^{(i)})^2}$

1) $R_t^2 = \|B_t\|^2 = \sum_{i=1}^{p} (B_t^{(i)})^2$

on pose $u(x_1, \dots, x_p) = \sum_{i=1}^{p} x_i^2$ et $\mathcal{C}^2(\mathbb{R}^p)$.

• $\frac{\partial u}{\partial t} = 0$ • $\forall i = 1..p \ \frac{\partial u}{\partial x_i} = 2x_i$ • $\forall i,j = 1..p$ si $\begin{cases} i \neq j & \frac{\partial^2 u}{\partial x_i \partial x_j} = 0 \\ i = j & \frac{\partial^2 u}{\partial x_i^2} = 0 \end{cases}$

on applique la formule d'Itô

$$d(R_t^2) = \sum_{i=1}^{p} \frac{\partial u}{\partial x_i}(B_t^{(1)}, \dots B_t^{(p)}) dB_t^{(i)} + \frac{1}{2} \sum_{i,j} \frac{\partial^2 u}{\partial x_i \partial x_j}(B_t^{(1)}, \dots, B_t^{(p)}) dB_t^{(i)} dB_t^{(j)}$$

$$= 2 \sum_{i=1}^{p} B_t^{(i)} dB_t^{(i)} + \sum_{i=1}^{p} \underbrace{dB_t^{(i)} dB_t^{(i)}}_{dt}$$

$$\Rightarrow d(R_t^2) = 2 \sum_{i=1}^{p} B_i^{(i)} dB_t^{(i)} + p\,dt$$

$$\Rightarrow \boxed{R_t^2 = R_0^2 + 2 \sum_{i=1}^{p} \int_0^t B_s^{(i)} dB_s^{(i)} + pt}$$

$$\mathbb{P}(R_t < \alpha) = \frac{1}{\sqrt{2\pi t}} \int_{B^p(0,\alpha)} e^{-\frac{x_1^2 + \cdots + x_p^2}{2t}} \, dx_1 \cdots dx_p$$

On fait le changement de coordonnées suivant :

$$(x_1, \dots, x_p) \longleftrightarrow (r, \phi_1, \dots, \phi_p)$$

$r = \|x\|$

$x_1 = r \cos \phi_1$

$x_2 = r \sin \phi_1 \cos \phi_2$

$x_p = r \sin \phi_1 \cdots \sin \phi_{p-2} \sin \phi_{p-1}$

$$\begin{pmatrix} \cos\phi_1 & -r\sin\phi_1 & \cdots & & \\ \sin\phi_1\cos\phi_2 & r\cos\phi_1\cos\phi_2 & -r\sin\phi_1\sin\phi_2 & & \\ & & & & \\ & & & & \end{pmatrix}$$

$$= \frac{1}{\sqrt{2\pi t}} \int_0^\alpha \int_{\phi_1=0}^{\pi} \cdots \int_{\phi_{p-2}=0}^{\pi} \int_{\phi_{p-1}=0}^{2\pi} e^{-\frac{r^2}{2t}} \sin^{p-2}(\phi_1) \cdots \sin(\phi_{p-2}) \, r^{p-1} \, dr\, d\phi_1 \cdots d\phi_{p-1}$$

$$\boxed{\mathbb{P}(R_t < \alpha) \propto \int_0^\alpha e^{-\frac{r^2}{2t}} r^{p-1} \, dr}$$

$$\boxed{V_{S_{p-1}} = p \, \frac{\pi^{\frac{p}{2}}}{\Gamma\left(\frac{p}{2} + 1\right)}}$$