TD 6 : Intégrale Stochastique - Formule d'Itô

Exercice 1 :

1) $\mathbb{E}[B_t | \mathcal{F}_s] = \mathbb{E}[(B_t - B_s) | \mathcal{F}_s] + \mathbb{E}[B_s | \mathcal{F}_s]$

$= \underbrace{\mathbb{E}[B_t - B_s]}_{\text{indep de } \mathcal{F}_s} + \underbrace{B_s}_{\mathcal{F}_s\text{- mesurable}}$

$= B_s \Rightarrow$ martingale.

2) $\mathbb{E}[B_t^2 - t | \mathcal{F}_s] = \mathbb{E}[(B_t - B_s + B_s)^2 | \mathcal{F}_s] - t$.

$= \mathbb{E}[(B_t - B_s)^2 | \mathcal{F}_s] + \mathbb{E}[B_s^2 | \mathcal{F}_s] + 2\mathbb{E}[\overset{\text{indep}}{B_s (B_t - B_s)}] - t$

$= (t-s) + B_s^2 + 0 - t$

$= B_s^2 - s \Rightarrow$ martingale

3) $\mathbb{E}[e^{\theta B_t - \theta^2/2\, t} | \mathcal{F}_s] = e^{-\theta^2/2\, t}\, \mathbb{E}[e^{\theta(B_t - B_s)} e^{\theta B_s} | \mathcal{F}_s]$

$= e^{-\theta^2/2\, t} e^{\theta B_s} \mathbb{E}[e^{\theta(B_t - B_s)}] = e^{\theta B_s} e^{-\theta^2/2\, t} e^{+\theta^2/2\, (t-s)}$

$= e^{\theta B_s - \theta^2/2\, s} \Rightarrow$ martingale.

4) $X_t = (W_t^1 \dots W_t^d)$

$F(X_t) = F(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial F}{\partial x_i}(X_s)\, dW_s^i + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(X_s)\, dW_s^i\, dW_s^j$

$= \underbrace{F(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial F}{\partial x_i}(X_s)\, dW_s^i}_{\text{martingale}} + \underbrace{\frac{1}{2}\int_0^t \Delta F(X_s)\, ds}_{=0}$

martingale.

5) $X_t = (t, B_t)$.

(a) $dX_t^1 = dt$. , $dX_t^2 = dB_t$ donc ils s'écrivent bien sous la forme $dX_t^i = F^i\, dt + G^i\, dW_t$ avec des fonction $F^i \in \mathbb{L}^1([0,T])$ et $G^i \in \mathbb{L}^2([0,T])$.

Donc la formule d'Itô s'applique.

(b) Formule d'Itô →

$$F(t,B_t) = F(0,0) + \int_0^t \frac{\partial F}{\partial t}\, ds + \int_0^t \frac{\partial F}{\partial x}\, dB_s + \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial t^2}\, dt\, dt + \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}\, ds$$

$$+ \int_0^t \frac{\partial F}{\partial t \partial x}\, ds\, dB_s$$

$$= F(0,0) + \int_0^t \left(\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\right) ds + \int_0^t \frac{\partial F}{\partial x}\, dB_s$$

(c) si $\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} = 0$, alors $f(t, X_t) = F(0,0) + \int_0^t \frac{\partial F}{\partial x}\, dB_s$ martingale.

(d) Par la propriété de martingale,

$$\mathbb{E}[F(t,X_t)] = \mathbb{E}[\mathbb{E}[F(t,X_t) \mid \mathcal{F}_0]] = \mathbb{E}[F(0,0)] = F(0,0).$$

(e) 1^e exemple : $F(t,x) = x$ $\quad \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} = 0$.

2^e — : $F(t,x) = x^2 - t$, $\quad \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} = -1 + \frac{1}{2} \times 2 = 0$.

3^e — : $F(t,x) = e^{\theta x - \theta^2/2\, t}$

$\frac{\partial F}{\partial t} = -\frac{\theta^2}{2} F \qquad \frac{\partial F}{\partial x} = \theta F$, $\frac{\partial^2 F}{\partial x^2} = \theta^2 F \Rightarrow$ ok.

(f) $$\frac{\partial h_n}{\partial t} = \frac{n}{2} t^{n/2 - 1} H_n\left(\frac{x}{\sqrt{t}}\right) - \frac{t^{n/2}}{2} H_n'\left(\frac{x}{\sqrt{t}}\right)\frac{x}{\sqrt{t}^3}$$

$$\frac{\partial^2 h_n}{\partial x^2} = t^{n/2 - 1} H_n''\left(\frac{x}{\sqrt{t}}\right) = t^{n/2 - 1}\left(\frac{x}{\sqrt{t}} H_n'\left(\frac{x}{\sqrt{t}}\right) - n H_n\left(\frac{x}{\sqrt{t}}\right)\right).$$

$$\Rightarrow \frac{\partial h_n}{\partial t} + \frac{1}{2}\frac{\partial^2 h}{\partial x^2} = 0$$