1. Invariant sets
   - Limit cycles
   - Stable/Unstable manifolds
   - Center manifold
   - Center manifold
   - Center manifold

2. Classification of hyperbolic sets

3. Bifurcation theory
   - Definitions
   - Normal form theory

4. Codimension 1 bifurcations of equilibria of continuous systems
\[ \dot{x} = F(x), \; F \text{ smooth} \]

**Lemma (Flow box)**

Assume \( F(x_0) \neq 0 \), then there is a local diffeomorphism \( h \) such that \( h_* F \) is constant.
Suppose there is limit cycle $\Gamma$ with $x_0 \in \Gamma$ and period $T$. Consider an hypersurface $\Sigma$ transverse to $\Gamma$ en $x_0$: $\langle F(x_0), x - x_0 \rangle = 0$ It is called a cross-section.

**Definition**

The Poincare map $\Pi_{x_0}^{\Sigma}$ is the application

$$\Pi_{x_0}^{\Sigma} : \begin{cases} \Sigma \cap U & \rightarrow \Sigma \\ x & \rightarrow \phi^{T+\epsilon(x)}(x) \end{cases}$$
Poincare Map

Properties

Proposition
The stability of $\Gamma$ is equivalent to the stability of $x_0$ for $\Pi_{x_0}^\Sigma$.

Theorem
The Poincare map $\Pi_{x_0}^\Sigma$ is a local diffeomorphism on a neighborhood of $x_0$ and in a basis with first vector collinear to $F(x_0)$, we have

$$d\phi^T(x_0) = \begin{bmatrix} 1 & * \\ 0 & d\Pi_{x_0}(x_0) \end{bmatrix}$$

Corollary
The eigenvalues (multipliers) of $\Pi_{x_0}^\Sigma$ are independent of $x_0$ and $\Sigma$.

$\Rightarrow$ Use the fact $d\phi^T(x_0) \sim d\phi^T(y_0)$.
Stability of limit cycles

Proposition
The stability of $\Gamma$ is equivalent to the stability of $x_0$ for $\prod_{x_0}^{\Sigma}$.

It is not trivial!
Hyperbolic equilibria

Continuous DS

\[ \dot{x} = F(x), \text{ } F \text{ smooth with } F(x_0) = 0 \]

Let \( n_-, n_0, n_+ \) be the number of eigenvalues of \( dF(x_0) \) with negative, null, positive real part counted with multiplicity.

**definitions**

An equilibrium is:

- **hyperbolic** if \( n_0 = 0 \)
- a **hyperbolic saddle** if \( n_- n_+ \neq 0 \)

Since a generic matrix has no eigenvalues on the imaginary axis, hyperbolicity is a typical property and an equilibrium in a generic system (i.e., one not satisfying certain special conditions) is hyperbolic.
Invariant sets: Local stable manifold

Continuous case

**definitions**

Consider an equilibrium $x_0$, its stable and unstable sets are defined by

$$W^s(x_0) = \{x : \lim_{t \to \infty} \phi^t x = x_0\}$$

$$W^u(x_0) = \{x : \lim_{t \to -\infty} \phi^t x = x_0\}$$

**Theorem**

Let $x_0 \in \mathbb{R}^n$ be a **hyperbolic** equilibrium (i.e., $n_0 = 0$, $n_- + n_+ = n$). Then the intersections of $W^s(x_0)$ and $W^u(x_0)$ with a sufficiently small neighborhood of $x_0$ contain smooth **submanifolds** $W^s_{loc}(x_0)$ and $W^u_{loc}(x_0)$ of dimension $n_-$ and $n_+$, respectively.

Moreover, $W^s_{loc}(x_0)$ (resp. $W^u_{loc}(x_0)$) is tangent to $T^s$ (resp. $T^u$) where $T^s$ (resp. $T^u$) is the **generalized eigenspace** corresponding to the eigenvalues of $dF(x_0)$ of negative (resp. positive) real part.
Invaraint sets: Local stable manifold
Continuous case

Sketch of proof

- $\phi^1(T^u)$ is a manifold of dimension $n_+$ tangent to $T^u$ at $x_0$
- Restrict attention to a sufficiently small neighborhood of the equilibrium where the linear part is “dominant” and repeat the procedure.
- the iterations converge to a smooth invariant submanifold defined in this neighborhood of $x_0$ and tangent to $T^u$ at $x_0$.
- for $W^s$, apply $\phi^{-1}$ to $T^s$
Local stable manifold in the continuous case
From Kuznetsov

**Figure**: a) Saddle b) Saddle-Foci
Invariant sets: Local stable manifold

Discrete case

\[ x \rightarrow F(x), \quad F \text{ and } F^{-1} \text{ smooth with } F(x_0) = 0, \text{ diffeomorphism} \]

Let \( n_-, n_0, n_+ \) be the number of eigenvalues of \( dF(x_0) \) with modulus \( < 1, = 1, > 1 \) counted with multiplicity.

**definitions**

An equilibrium is:

- *hyperbolic* if \( n_0 = 0 \)
- a *hyperbolic saddle* if \( n_- n_+ \neq 0 \)
Local stable manifold in the discrete case

definitions
Consider an equilibrium $x_0$, its stable (unstable) set is defined by

$$W^s(x_0) = \{x : \lim_{k \to \infty} F^k(x) = x_0\}$$

$$W^u(x_0) = \{x : \lim_{k \to -\infty} F^k(x) = x_0\}$$

Theorem
Let $x_0 \in \mathbb{R}^n$ be a hyperbolic equilibrium ($n_0 = 0, n_- + n_+ = n$). Then the intersections of $W^s(x_0)$ and $W^u(x_0)$ with a sufficiently small neighborhood of $x_0$ contain smooth submanifolds $W^s_{loc}(x_0)$ and $W^u_{loc}(x_0)$ of dimension $n_-$ and $n_+$, respectively. Moreover, $W^s_{loc}(x_0)$ (resp. $W^u_{loc}(x_0)$) is tangent to $T^s$ (resp. $T^u$) where $T^s$ (resp. $T^u$) is the generalized eigenspace corresponding to the eigenvalues $\lambda$ of $dF(x_0)$ such that $|\lambda| < 1$ ($|\lambda| > 1$).

Proof analogous to continuous case if one substitutes $\phi^1$ by $F$. 
Local stable manifold in the discrete case
From Kuznetsov

Example with positive/negative multiplier

\[ \mu_2, \mu_1 \]

\[ W_u^1, W_s^2, W_u^2 \]

\[ W_u^1, W_s^2, W_u^2 \]

\[ W_u^1, W_s^2, W_u^2 \]

\[ W_u^1, W_s^2, W_u^2 \]

Moreover, one transversal intersection, if it occurs, implies an infinite number of such intersections. Indeed, let \( x_0 \) be a point of the intersection. By definition, it belongs to both invariant manifolds. Therefore, the orbit starting at this point converges to the saddle point \( x_0 \) under repeated iteration of either \( f \) or \( f^{-1} \):

\[ f^k(x_0) \rightarrow x_0 \] as \( k \rightarrow \pm \infty \).

Each point of this orbit is a point of intersection of \( W_s(x_0) \) and \( W_u(x_0) \). This infinite number of intersections forces the manifolds to "oscillate" in a complex manner near \( x_0 \), as sketched in Figure 2.10(b). The resulting "web" is called the Poincaré homoclinic structure. The orbit starting at \( x_0 \) is said to be homoclinic to \( x_0 \). It is the presence of the homoclinic structure that can make the intersection of \( W_{s,u}(x_0) \) with any neighborhood of the saddle \( x_0 \) highly nontrivial.

The dynamical consequences of the existence of the homoclinic structure are also dramatic: It results in the appearance of an infinite number of periodic points with arbitrary high periods near the homoclinic orbit. This follows from the presence of Smale horseshoes (see Chapter 1). Figure 2.11 illustrates how the horseshoes are formed. Take a (curvilinear) rectangle \( S \) near the stable manifold \( W_s(x_0) \) and consider its iterations \( f^k S \). If the homoclinic structure is present, for a sufficiently high number of iterations \( N \), \( f^N S \) will look like the folded and expanded band \( Q \) shown in the figure.
Hyperbolic limit cycles
Saddle cycles in 3d systems: (a) positive multipliers and (b) negative multipliers. From Kuznetsov.

Invariant manifolds:

\[ W^s(L_0) = \{ x : \lim_{t \to \infty} \phi^t(x) \in L_0 \} \]
\[ W^u(L_0) = \{ x : \lim_{t \to -\infty} \phi^t(x) \in L_0 \} \]

⇒ We have \( W^{s,u}(x_0, \Pi) = \Sigma \cap W^{s,u}(L_0) \)
Center manifold
Continuous case

\[ \dot{x} = Lx + R(x; \mu), \quad L \in \mathcal{L}(\mathbb{R}^n), \quad R \in C^k(\mathcal{V}_x \times \mathcal{V}_\mu, \mathbb{R}^m) \quad (1) \]

\[ R(0; 0) = 0, \quad dR(0; 0) = 0 \]

**Theorem 1/2**

Write \( \mathbb{R}^n = T^c \oplus T^h \) where \( T^h = T^s \oplus T^u \). Then, there is a neighborhood \( O = O_x \times O_\mu \) of \((0, 0)\) in \( \mathbb{R}^n \times \mathbb{R}^m \), a mapping \( \Psi \in C^q(T^c \times \mathbb{R}^m; T^h) \) with

\[ \Psi(0; 0) = 0, \quad d\Psi(0; 0) = 0 \]

and a manifold \( M(\mu) = \{u_c + \Psi(u_c, \mu), u_c \in T^c\} \) for \( \mu \in \mathcal{V}_\mu \) such that:

1. \( M(\mu) \) is **locally invariant**, i.e., \( x(0) \in M(\mu) \cap O_x \) and \( x(t) \in O_x \) for all \( t \in [0, T] \) implies \( x(t) \in M(\mu) \) for all \( t \in [0, T] \).
Center manifold
Continuous case

**Theorem 2/2**

2 $\mathcal{M}(\mu)$ contains the set of **bounded solutions** of (1) staying in $O_x$ for all $t \in \mathbb{R}$, i.e. if $x$ is a solution of (1) satisfying for all $t \in \mathbb{R}$, $x(t) \in O_x$, then $x(0) \in \mathcal{M}(\mu)$.

3 (Parabolic case) if $n_+ = 0$, then $\mathcal{M}(\mu)$ is **locally attracting**, i.e. if $x$ is a solution of (1) with $x(0) \in O_x$ and $x(t) \in O_x$ for all $t > 0$, then there exists $v(0) \in \mathcal{M}(\mu) \cap O_c$ and $\tilde{\gamma} > 0$ such that

$$x(t) = v(t) + O(e^{-\tilde{\gamma}t}) \text{ as } t \rightarrow \infty$$

where $v$ is a solution of (1) with initial condition $v(0)$. 
5.1 Center manifold theorems

1. The second statement of the theorem means that orbits staying near the equilibrium for $t \geq 0$ or $t \leq 0$ tend to $W^c$ in the corresponding time direction. If we know a priori that all orbits starting in $U$ remain in this region forever (a necessary condition for this is $n_+ = 0$), then the theorem implies that these orbits approach $W^c(0)$ as $t \to +\infty$. In this case the manifold is "attracting."

2. $W^c$ need not be unique. The system

\[
\begin{align*}
\dot{x} &= x^2, \\
\dot{y} &= -y,
\end{align*}
\]

has an equilibrium $(x, y) = (0, 0)$ with $\lambda_1 = 0$, $\lambda_2 = -1$ (a fold case). It possesses a family of one-dimensional center manifolds:

$W^c(0) = \{(x, y): y = \psi_\beta(x)\}, \quad \beta > 0 \text{ or } \beta < 0$. 

FIGURE 5.2. One-dimensional center manifold at the fold bifurcation.

FIGURE 5.3. Two-dimensional center manifold at the Hopf bifurcation.
Center manifold
Additional properties

1. The Center manifold is not unique
2. If $x_c(0) \in \mathcal{M}(\mu)$, then
   \[ \dot{x}_c = L_c x_c + P_c R(x_c + \Psi(x_c, \mu), \mu) \equiv f(x_c, \mu) \]
   where $P_c$ is the projector on $T^c$.
3. The local coordinates function satisfies
   \[ d\Psi(x_c, \mu) \cdot f(x_c) = P^h L \cdot \Psi(x_c, \mu) + P^h R(x_c + \Psi(x_c, \mu)) \]
4. There are extensions for non-autonomous systems, with symmetries...
5. Extensions to Banach spaces possible in some cases
\[ \dot{x} \to Lx + R(x, \alpha), \quad L \in \mathcal{L}(\mathbb{R}^n), \quad R \in C^k(\mathcal{V}_x \times \mathcal{V}_\alpha, \mathbb{R}^n) \]

\[ R(0, 0) = 0, \quad d_x R(0, 0) = 0 \]

Write \( \mathbb{R}^n = T^c \oplus T^h \) where \( T^h = T^s \oplus T^u \) is the hyperbolic part. Then we have the same conclusions as for the continuous case.
1 Invariant sets
   - Limit cycles
   - Stable/Unstable manifolds
   - Center manifold
   - Center manifold
   - Center manifold

2 Classification of hyperbolic sets

3 Bifurcation theory
   - Definitions
   - Normal form theory

4 Codimension 1 bifurcations of equilibria of continuous systems
A dynamical system \( \{ T, \mathbb{R}^n, \phi^t \} \) is called **locally topologically equivalent** near an equilibrium \( x_0 \) to a dynamical system \( \{ T, \mathbb{R}^n, \psi^t \} \) near an equilibrium \( y_0 \) if there exists a homeomorphism \( h : \mathbb{R}^n \rightarrow \mathbb{R}^n \) that is

1. defined in a small neighborhood \( U \subset \mathbb{R}^n \) of \( x_0 \);
2. satisfies \( y_0 = h(x_0) \);
3. maps orbits of the first system in \( U \) onto orbits of the second system in \( V = F(U) \subset \mathbb{R}^n \), preserving the direction of time.

In the discrete case \( x \rightarrow F(x) \) is equivalent to \( y \rightarrow G(y) \) if \( F = h^{-1} \circ G \circ h \) i.e they are **conjugate**.
Example:

Vector fields: \[ \begin{cases} \dot{x}_1 = -x_1 \\ \dot{x}_2 = -x_2 \end{cases}, \lambda = -1 \quad \text{and} \quad \begin{cases} \dot{x}_1 = -x_1 - x_2 \\ \dot{x}_2 = -x_2 + x_1 \end{cases}, \lambda = -1 \pm i \]

There are not smoothly equivalent!
Consider:

\[ \dot{x} = F(x), \ x \in \mathbb{R}^n, \ F \text{ smooth} \]

**Theorem**

The phase portraits of the DS near two hyperbolic equilibria, \( x_0 \) and \( y_0 \), are **locally topologically equivalent** if and only if these equilibria have the same number \( n_- \) and \( n_+ \) of eigenvalues with \( \Re \lambda < 0 \) and with \( \Re \lambda > 0 \), respectively.
Topological equivalence of hyperbolic points
Continuous case

Sketch of the proof:

1. near a hyperbolic equilibrium the system is locally topologically equivalent to its linearization: $\dot{\xi} = dF(x_0)\xi$ (Grobman-Hartman Theorem).

2. We apply it near $x_0$ and $y_0$

3. We prove the topological equivalence of two linear systems having the same numbers of eigenvalues with $\Re \lambda < 0$ and $\Re \lambda > 0$ and no eigenvalues on the imaginary axis.
Exercise:

<table>
<thead>
<tr>
<th>$(n_+, n_-)$</th>
<th>Eigenvalues</th>
<th>Phase portrait</th>
<th>Stability</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 2)</td>
<td></td>
<td>node</td>
<td>stable</td>
</tr>
<tr>
<td></td>
<td></td>
<td>focus</td>
<td></td>
</tr>
<tr>
<td>(1, 1)</td>
<td></td>
<td>saddle</td>
<td>unstable</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2, 0)</td>
<td></td>
<td>node</td>
<td>unstable</td>
</tr>
<tr>
<td></td>
<td></td>
<td>focus</td>
<td></td>
</tr>
</tbody>
</table>
Topological equivalence of hyperbolic points
Discrete case

Consider:

\[ x \rightarrow F(x), \ x \in \mathbb{R}^n, \ F, F^{-1} \text{ smooth} \quad (1) \]

**Theorem**

The phase portraits of system (1) near two hyperbolic equilibria, \( x_0 \) and \( y_0 \), are **locally topologically equivalent** if and only if these equilibria have the same number \( n_- \) and \( n_+ \) of multipliers with \( |\lambda| < 1 \) and with \( |\lambda| > 1 \), respectively, and the signs of the products of all the multipliers with \( |\lambda| < 1 \) and with \( |\lambda| > 1 \) are the same for both fixed points.

Analogous to the continuous case but one needs to be careful about preserving the direction of time.

Recall the conjugacy property...
Invariant sets: Local stable manifold

Discrete case

Example

2.2 Classification of equilibria and fixed points 51

otherwise. Two topologically equivalent maps must have the same orientation properties. The products in Theorem 2.4 are exactly the determinants of the Jacobian matrices of the map (2.12) restricted to its stable and unstable local invariant manifolds. It should be clear that one needs only account for real multipliers to compute these signs, since the product of a complex-conjugate pair of multipliers is always positive.

Let us consider two examples of fixed points.

Example 2.4 (Stable fixed points in $\mathbb{R}^1$)

Suppose $x_0 = 0$ is a fixed point of a one-dimensional discrete-time system ($n = 1$). Let $n - = 1$, meaning that the unique multiplier $\mu$ satisfies $|\mu| < 1$. In this case, according to Theorem 2.3, all orbits starting in some neighborhood of $x_0$ converge to $x_0$. Depending on the sign of the multiplier, we have the two possibilities presented in Figure 2.6. If $0 < \mu < 1$, the iterations converge to $x_0$ monotonously (Figure 2.6(a)). If $-1 < \mu < 0$, the convergence is non-monotonous and the phase point "jumps" around $x_0$ while converging to $x_0$ (Figure 2.6(b)). In the first case the map preserves orientation in $\mathbb{R}^1$ while reversing it in the second. It should be clear that "jumping" orbits cannot be transformed into monotonous ones by a continuous map. Figure 2.7 presents orbits near the two types of fixed points using staircase diagrams.

Example 2.5 (Saddle fixed points in $\mathbb{R}^2$)

Suppose $x_0 = 0$ is a fixed point of a two-dimensional discrete-time system (now $n = 2$). Assume that $n - = n + = 1$, so that there is one (real) multiplier $\mu_1$ outside the unit circle ($|\mu_1| > 1$) and one (real) multiplier $\mu_2$ inside the unit circle ($|\mu_2| < 1$). In our case, there are two invariant manifolds passing through the fixed point, namely the one-dimensional manifold $W_s(x_0)$ formed by

From Kuznetsov
Local stable manifold in the discrete case
Staircase diagrams for stable fixed points

Example

\[ \tilde{x} = f(x) \]
\[ \tilde{x} = x \]
\[ \tilde{x} = -x \]

From Kuznetsov
Topological equivalence and center manifold
Continuous case

Write the DS in an eigenbasis

\[
(1) \begin{cases}
\dot{x}_c = A_c x_c + f(x_c, x_h) \\
\dot{x}_h = A_h x_h + g(x_c, x_h)
\end{cases}
\]

**proposition**

The system (1) is locally topologically equivalent to

\[
\begin{cases}
\dot{x}_c = A_c x_c + f(x_c, x_c + \Psi(x_c)) \\
\dot{x}_h = A_h x_h
\end{cases}
\]

- the equations discouple

Note that \( \dot{x}_h = A_h x_h \) is equivalent to

\[
\begin{cases}
\dot{y}_- = -y_- \in \mathbb{R}^{n_-} \\
\dot{y}_+ = y_+ \in \mathbb{R}^{n_+}
\end{cases}
\]
Topological equivalence and center manifold
Discrete case

Write the DS in an eigenbasis

\[
\begin{align*}
    x_c & \rightarrow A_c x_c + f(x_c, x_h) \\
    x_h & \rightarrow A_h x_h + g(x_c, x_h)
\end{align*}
\]

proposition

The system (1) is locally topologically equivalent to

\[
\begin{align*}
    x_c & \rightarrow A_c x_c + f(x_c, x_c + \Psi(x_c)) \\
    x_h & \rightarrow A_h x_h
\end{align*}
\]

- the equations discouple
Outline

1. Invariant sets
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Topological equivalence of DSs

Classification of possible phase diagrams of generic systems, at least, locally. Consider two DS:

\[ \dot{x} = F(x, \alpha), \quad \dot{y} = G(y, \beta), \quad x, y \in \mathbb{R}^n, \quad \alpha, \beta \in \mathbb{R}^m \]

**definition**

The DS are called **locally topologically equivalent near the origin** if there is a map \((x, \alpha) \rightarrow (h_\alpha(x), p(\alpha))\) defined in a small neighborhood of \((x, \alpha) = (0, 0) \in \mathbb{R}^n \times \mathbb{R}^m\) such that

- \(p\) is a homeomorphism defined in a small neighborhood of \(\alpha = 0, p(0) = 0\)
- \(h : \mathbb{R}^n \rightarrow \mathbb{R}^n\) is a parameter-dependent homeomorphism defined in a small neighborhood \(U_\alpha\) of \(x = 0, h_0(0) = 0\), mapping orbits in \(U_\alpha\) of the first system at parameter values \(\alpha\) onto orbits in \(h_\alpha(U_\alpha)\) of the second system and preserving the direction of time.
Equivalence of DSs

We consider parameter dependent dynamical systems

\[ x \rightarrow F(x, \alpha) \quad \dot{x} = F(x, \alpha) \quad x \in \mathbb{R}^n, \ \alpha \in \mathbb{R}^m \]

definition

The appearance of a topologically nonequivalent phase portrait under variation of parameters is called a **bifurcation**.

definition

A **bifurcation diagram** of the dynamical system is a stratification of its parameter space induced by the topological equivalence, together with representative phase portraits for each stratum.

definition

The **codimension** is the number of independent conditions determining the bifurcation.
Example 1: the Pitchfork bifurcation
Local bifurcation, from Kuznetsov

Can be detected if we fix any small neighborhood of the equilibrium.

\[ \dot{x} = x(\alpha - x^2) \]

Two strataums \( \{\alpha \leq 0\} \) and \( \{\alpha > 0\} \), codim 1
Example 2: the Andronov-Hopf bifurcation
Local bifurcation, from Kuznetsov

Can be detected if we fix any small neighborhood of the equilibrium.

\[
\begin{align*}
\dot{x}_1 &= \alpha x_1 - x_2 - x_1(x_1^2 + x_2^2) \\
\dot{x}_2 &= x_1 + \alpha x_2 - x_2(x_1^2 + x_2^2)
\end{align*}
\]

Two strataums \(\{\alpha \leq 0\}\) and \(\{\alpha > 0\}\), codim 1
Example 3: the Heteroclinic bifurcation
Global bifurcation, from Kuznetsov

To detect this bifurcation we must fix a region covering both saddle.

\[
\begin{cases}
\dot{x}_1 = 1 - x_1^2 - \alpha x_1 x_2 \\
\dot{x}_2 = x_1 x_2 + \alpha (1 - x_1^2)
\end{cases}
\]

The system has two saddle equilibria \(x(1) = (-1, 0), \ x(2) = (1, 0)\).

Three strata: \(\{\alpha < 0\}, \ \{\alpha = 0\} \text{ and } \{\alpha > 0\}, \ \text{codim} \ 1\)
Normal form theory 1/2

Each bifurcation needs to be studied?

The idea is to find a polynomial CHV which *improves* locally a nonlinear system, in order to analyze its dynamics more easily.

\[ \dot{x} = Lx + R(x; \alpha), \quad L \in \mathcal{L}(\mathbb{R}^n), \quad R \in C^k(V_x \times V_\alpha, \mathbb{R}^m) \quad (1) \]

\[ R(0; 0) = 0, \quad dR(0; 0) = 0 \]

**Theorem 1/2**

\( \forall p \in [2, k], \) there are neighborhoods \( V_1 \) and \( V_2 \) of 0 in \( \mathbb{R}^n \) and \( \mathbb{R}^m \), respectively, such that for any \( \alpha \in V_2 \), there is a polynomial \( \Phi_\alpha : \mathbb{R}^n \to \mathbb{R}^n \) of degree \( p \) with the following properties:

- The coefficients of the monomials of degree \( q \) in \( \Phi_\alpha \) are functions of \( \alpha \) of class \( C^{k-q} \), and

\[ \Phi_0(0) = 0, \quad d\Phi_0(0) = 0 \]
Normal form theory 2/2

Each bifurcation needs to be studied?

Theorem 2/2

- For any $x \in \mathcal{V}_1$, the polynomial $CHV \ x = y + \Phi_\alpha(y)$ transforms (1) into the normal form

$$\dot{y} = Ly + N_\alpha(y) + \rho(y, \alpha)$$

- The coefficients of the monomials of degree $q$ in $N_\alpha$ are functions of $\alpha$ of class $C^{k-q}$, and

$$N_0(0) = 0, \ d_x N_0(0) = 0$$

- the equality $N_\alpha(e^{tL^*}y) = e^{tL^*}N_\alpha(y)$ holds for all $(t, y) \in \mathbb{R} \times \mathbb{R}^n$ and $\alpha \in \mathcal{V}_2$

- the maps $\rho$ belongs to $C^k(\mathcal{V}_1 \times \mathcal{V}_2, \mathbb{R}^n)$ and

$$\forall \alpha \in \mathcal{V}_2, \ \rho(y; \alpha) = o(|y|^p)$$
An example
Prelude to the Hopf bifurcation

Consider the case \( L = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}, \ \omega > 0. \)

- In the basis \((\zeta, \bar{\zeta}), \zeta = (1, -i)\): \( L = \begin{bmatrix} i\omega & 0 \\ 0 & -i\omega \end{bmatrix} \)
- Write \( x = y + \Phi_\alpha(y) \), the change of variable with \( y = A\zeta + \bar{A}\bar{\zeta} \)

**Lemma**

\[ N_\alpha(A\zeta + \bar{A}\bar{\zeta}) = AQ_\alpha(|A|^2)\zeta + \bar{AQ}_\alpha(|A|^2)\bar{\zeta}. \]

Proof: use \( N_\alpha(e^{tL^*}y) = e^{tL^*}N_\alpha(y) \). Write

\[ N_\alpha(A\zeta + \bar{A}\bar{\zeta}) = P_\alpha(A, \bar{A})\zeta + \bar{P}_\alpha(A, \bar{A})\bar{\zeta} \]

and note that \( e^{tL^*} = \text{diag} \left( e^{-i\omega t}, e^{i\omega t} \right) \) which gives

\[ P_\alpha \left( e^{-i\omega t}A, e^{i\omega t}\bar{A} \right) = e^{-i\omega t}P_\alpha(A, \bar{A}). \]

Looking for the monomials satisfying this condition allows to conclude.
Consider a continuous dynamical system with a local bifurcation:
Consider a continuous dynamical system with a local bifurcation:

- there a center manifold \( x = x_c + \Psi(x_c; \mu) \). Compute \( \Psi \) with a Taylor expansion.
Consider a continuous dynamical system with a local bifurcation:

- there a center manifold \( x = x_c + \Psi(x_c; \mu) \). Compute \( \Psi \) with a Taylor expansion.
- project the dynamics on the center manifold

\[
\dot{x}_c = Lx_c + P^c R(x_c + \Psi(x_c; \mu); \mu)
\]
Consider a continuous dynamical system with a local bifurcation:

- there a center manifold \( x = x_c + \Psi(x_c; \mu) \). Compute \( \Psi \) with a Taylor expansion.
- project the dynamics on the center manifold

\[
\dot{x}_c = Lx_c + P^c R(x_c + \Psi(x_c; \mu); \mu)
\]

- simplify the dynamics with a normal form which needs to be computed with the CHV \( x_c = v_0 + \Phi(v_0; \mu) \)
1 Invariant sets
   • Limit cycles
   • Stable/Unstable manifolds
   • Center manifold
   • Center manifold
   • Center manifold

2 Classification of hyperbolic sets

3 Bifurcation theory
   • Definitions
   • Normal form theory

4 Codimension 1 bifurcations of equilibria of continuous systems
Consider a DS

\[ \dot{x} = F(x, \alpha), \ x \in \mathbb{R}^n \]
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\[ \dot{x} = F(x, \alpha), \ x \in \mathbb{R}^n \]

- Consider an hyperbolic equilibrium \( x_0 \) for \( \alpha = \alpha_0 \). Remains hyperbolic for slight variations of \( \alpha \).
Bifurcation conditions

Consider a DS

\[ \dot{x} = F(x, \alpha), \ x \in \mathbb{R}^n \]

Consider an hyperbolic equilibrium \( x_0 \) for \( \alpha = \alpha_0 \). Remains hyperbolic for slight variations of \( \alpha \)

two ways of breaking hyperbolicity:

1. Either a simple real eigenvalue approaches zero and we have \( \lambda = 0 \), **fold** bifurcation.
2. or a pair of simple complex eigenvalues reaches the imaginary axis and we have \( \lambda_{1,2} = \pm i\omega_0, \omega_0 > 0 \), **Andronov-Hopf** bifurcation.
The Fold bifurcation
Also called Saddle-Node of Turning-point bifurcation

Consider a DS projected on the center manifold:

\[ \dot{x} = F(x, \alpha) = Lx + R(x, \alpha), \quad x \in \mathbb{R}^1 \]

with \( L = dF(x_0, \alpha_0) \). We have \( L = L^* = 0 \) (Fold) and \( e^{tL^*} = \text{Id} \): no constraint on the normal form \( N_\alpha(y) \), but we can still simplify it a bit.

- Write a Taylor expansion

\[ F(x, \alpha) = F_0(\alpha) + F_1(\alpha)x + F_2(\alpha)x^2 + O(x^3) \]

with \( F_0(0) = 0 \) and \( F_1(0) = dF(0, 0) = 0 \).

- Change variables and parameters

- Some nondegeneracy and traversality conditions will appear
Use $\xi = x + \delta(\alpha)$, $\delta$ smooth to get

$$\dot{\xi} = \cdots + (F_1(\alpha) - F_2(\alpha)\delta + O(\delta^2)) \cdot \xi + \cdots + O(\xi^3)$$

If $F_2(0) = \frac{1}{2}d^2F(0,0) \neq 0$, the Implicit Function Theorem applied to $A(\alpha, \delta) = F_1(\alpha) - F_2(\alpha)\delta + \delta^2\psi(\alpha, \delta)$, $\psi$ smooth, gives a smooth function $\delta$ such that $\delta(\alpha) = \frac{F'_1(0)}{2F_2(0)} \alpha + O(\alpha^2)$, hence:

$$\dot{\xi} = (F'_0(0)\alpha + O(\alpha^2)) + (F_2(0) + O(\alpha)) \cdot \xi^2 + O(\xi^3)$$
Define $\mu = \mu(\alpha) = F'_0(0)\alpha + \alpha^2 \phi(\alpha)$. We have $\mu(0) = 0$, $\mu'(0) = F'_0(0) = \partial_{\alpha} F(0,0)$. Hence, if $\partial_{\alpha} F(0,0) \neq 0$, the Implicit Function Theorem gives a function $\alpha(\mu)$ with $\alpha(0) = 0$ such that

$$\dot{\xi} = \mu + a(\mu)\xi^2 + O(\xi^3)$$

with $a(0) = F_2(0)$.

Finally, use $\eta = |a(\mu)|\xi$ and $\beta = |a(\mu)|\mu$ around $\mu = 0$ to find

$$\dot{\eta} = \mu + s\eta^2 + O(\eta^3), s = \text{sign}(a(0))$$
The Fold bifurcation
Truncated system, $s = 1$

$y = f(x, \alpha)$

$\alpha < 0$

$\alpha = 0$

$\alpha > 0$

Figure: From Kuznetsov
The Fold bifurcation
Topological equivalence

Lemma

The system \( \dot{y} = \alpha + y^2 + O(y^3) \) is locally topologically equivalent to the system \( \dot{x} = \alpha + x^2 \).

Proof: use IFT to show the persistence of local equilibria \( y_i(\alpha) \). Write \( x_i(\alpha) = \pm \sqrt{-\alpha} \). Use the homeomorphism:

\[
y = h(x, \alpha) = \begin{cases} 
  x & \text{for } \alpha > 0 \\
  a(\alpha) + b(\alpha)x & \text{otherwise}
\end{cases}
\]

such that \( y_i(\alpha) = h(x_i(\alpha), \alpha) \).

Theorem

If \( \dot{x} = F(x, \alpha), \ x \in \mathbb{R}^1 \) is such that \( F(0, 0) = dF(0, 0) = 0 \). Assume

1. \( d^2F(0, 0) \neq 0 \)
2. \( \partial_\alpha F(0, 0) \neq 0 \)

then the DS is topo. equivalent to \( \dot{x} = \beta + sx^2 \), \( s = \text{sign} \left( d^2F(0, 0) \right) \).
The Hopf bifurcation
Also called Andronov-Hopf bifurcation

Consider a DS projected on the center manifold

\[ \dot{x} = F(x, \alpha) = Lx + R(x, \alpha), \quad x \in \mathbb{R}^2 \]

Assume \( L \equiv dF(x_0, \alpha_0) \) has two eigenvalues \( \pm i\omega \) and \( x_0 = 0, \alpha_0 = 0 \):

\[ L\zeta = i\omega \zeta, \quad L\bar{\zeta} = -i\omega \bar{\zeta} \]

Use the CHV \( x = z\zeta + \bar{z}\bar{\zeta} + \Phi(z, \bar{z}, \alpha) \) (see previous slide)

\[ \dot{z} = i\omega z + zQ(|z|^2, \alpha) + \rho(z, \bar{z}, \alpha) \]

From \( Q(|z|^2, \alpha) = a(\alpha) + b(\alpha)|z|^2, \quad a(\alpha), b(\alpha) \in \mathbb{C} \) with \( a(0) = 0 \) we get:

\[ \dot{z} = (a(\alpha) + i\omega)z + b(\alpha)|z|^2z + O((\mu + |z|^2)^2) \]
The Hopf bifurcation
Also called Andronov-Hopf bifurcation

Lemma
Assume $\Re a'(0), \Re b(0) \neq 0$, then

$$
\dot{z} = (a(\alpha) + i\omega) z + b(\alpha)|z^2|z + O((\mu + |z|^2)^2)
$$

is equivalent to

$$
\dot{z} = (\mu + i) z + s|z^2|z + O((\mu + |z|^2)^2), \ s = \pm 1.
$$

Lemma
The system

$$
\dot{z} = (\mu + i) z + s|z^2|z + O((\mu + |z|^2)^2)
$$

is locally topologically equivalent to

$$
\dot{z} = (\mu + i) z + s|z^2|z.
$$
The Hopf bifurcation
Also called Andronov-Hopf bifurcation

**Theorem**

If \( \dot{x} = F(x, \alpha) \), \( x \in \mathbb{R}^2 \) has for all sufficiently small \( \alpha \) the equilibrium \( x = 0 \) with eigenvalues

\[
\lambda_{1,2}(\alpha) = \mu(\alpha) \pm i\omega_0(\alpha)
\]

where \( \mu(0) = 0, \omega_0(0) = \omega > 0 \). Assume that

1. \( \mu'(0) \neq 0 \)
2. \( \Re b \neq 0 \) (Lyapunov coefficient)

then the DS is topo. equivalent to

\[
\dot{z} = (\mu + i)z + s|z^2|z
\]

Note: the second condition involves a lengthy expression that is not provided here.