Design and study of a risk management criterion for an unstable anaerobic wastewater treatment process

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Abstract

In this paper we consider an unstable biological process used for wastewater treatment. This anaerobic digestion ecosystem can have two locally stable steady states and one unstable steady state. We first study the model and characterise the attraction basin associated to the normal operating mode. In a second step we estimate the size of this attraction basin by using a simplified criterion that turns out to be a good approximation. Finally we apply the approach on a real anaerobic digestion plant, and we show that the proposed criterion allows to rapidly detect the conditions of a destabilisation.

Key words: Haldane model, Anaerobic digestion, Nonlinear systems diagnosis

1 Introduction and motivation

Control of biological systems is a very delicate problem since one has to deal with highly nonlinear systems described by poor quality models. In some cases this control issue can be really crucial when perturbations can move the system from one steady-state to another. This is especially the case for the anaerobic digestion process: a more and more popular bioprocess (Angelidaki *et al.*, 2003) that treats wastewater and at the same time produces energy through methane (CH_4). This process can also produce hydrogen (H_2) under specific conditions. This complex ecosystem involves more than 140 bacterial species

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(Delbès *et al.*, 2001) that progressively degrade the organic matter into carbon dioxide (CO_2) and methane (CH_4). However this process is known to be very delicate to manage since it is unstable (Fripiat *et al.*, 1984): an accumulation of intermediate compounds can lead to the acidification of the digester.

To solve this problem, many authors have proposed controllers (Perrier and Dochain, 1993; Steyer *et al.*, 1999; Mailleret *et al.*, 2004) that were able to ensure the local or even the global stability of the system using the dilution rate of the bioreactor as input.

These control laws are however difficult to apply in practice due to the lack of available sensors on this type of process. Moreover, by essence they act on the influent flow rate, and they may therefore not accept all the incoming wastewater. It means that this type of controllers implies a storage of the wastewater to be treated. In practice storage tanks are very small and this solution is therefore difficult to setup on a long term basis. As a consequence, the controllers are often disconnected at the industrial scale and the plant manager manually operates the process trying both to avoid process destabilisation and wastewater storage.

The approach that we propose has the objective to provide the operator with a risk index associated to his management strategy. The idea is therefore to determine from the global analysis of the nonlinear system whether the process has been triggered to a dangerous working mode. This risk index can also be used in parallel to a controller that only guarantees local convergence.

The paper is organized as follows: in the second section a dynamical model of an anaerobic digestion process is recalled. The third part puts the emphasis on the analysis of the model dynamics. A simple criterion to assess the stability of the process is set in the fourth section, and finally this criterion is applied to a real process to determine its destabilisation risk.

2 Model presentation

There exists numerous dynamical models for anaerobic digestion, from the basic ones considering only one biomass (Andrews, 1968) to detailed models including several bacterial populations and several substrates. Among complex models the IWA Anaerobic Digestion Model 1 (Batstone *et al.*, 2002) has imposed itself as a useful tool to simulate a digestion plant with more insight into the process dynamics. However its excessive complexity makes any advanced mathematical analysis of the model critical.

We thus consider a simplified macroscopic model of the anaerobic process

based on 2 main reactions (Bernard *et al.*, 2001), where the organic substrate (S_1) is degraded into volatile fatty acids (VFA, denoted S_2) by acidogenic bacteria (X_1), and then the VFA are degraded into methane CH_4 and CO_2 by methanogenic bacteria (X_2):

- Acidogenesis:
$$k_1 S_1 \overset{\mu_1(S_1)X_1}{\longrightarrow} X_1 + k_2 S_2$$
- Methanogenesis:
$$k_3 S_2 \overset{\mu_2(S_2)X_2}{\longrightarrow} X_2 + k_4 CH_4$$

Where $\mu_1(S_1)$ and $\mu_2(S_2)$ represent the bacterial growth rates associated to these 2 bioreactions.

The mass balance model in the CSTR (Continuous Stirred Tank Reactor) can then straightforwardly be derived (Bastin and Dochain, 1990):

$$\dot{X}_1 = \mu_1(S_1)X_1 - \alpha D X_1 \tag{1}$$
$$\dot{S}_1 = -k_1\mu_1(S_1)X_1 + D(S_{1in} - S_1) \tag{2}$$
$$\dot{X}_2 = \mu_2(S_2)X_2 - \alpha D X_2 \tag{3}$$
$$\dot{S}_2 = -k_3\mu_2(S_2)X_2 + k_2\mu_1(S_1)X_1 + D(S_{2in} - S_2) \tag{4}$$

D is the dilution rate, S_{1in} and S_{2in} are respectively the concentrations of influent organic substrate and of influent VFA. The k_is are pseudo-stoichiometric coefficients associated to the bioreactions. Parameter $\alpha \in (0\ 1]$ represents the fraction of the biomass which is not retained in the digester. We denote by $\xi = (X_1,\ S_1,\ X_2,\ S_2)^T$ the state vector.

In the sequel, we will consider the rather generic mappings μ_1 and μ_2, satisfying the following properties:

Hypothesis 1 μ_1 **is an increasing function of** S_1**, with** $\mu_1(0) = 0$**.**

Hypothesis 2 μ_2 **is a function of** S_2 **which increases until a concentration** S_2^M **and then decreases, with** $\mu_2(S_2^M) = \mu_M$ **and** $\mu_2(0) = 0$**.**

For the numerical application in the real example, we will consider the following kinetics that verify Hypotheses 1 and 2:

$$\mu_1(S_1) = \bar{\mu}_1 \frac{S_1}{S_1 + K_{S1}} \quad (Monod)$$
$$\mu_2(S_2) = \bar{\mu}_2 \frac{S_2}{S_2 + K_{S2} + \frac{{S_2}^2}{K_{I2}}} \quad (Haldane) \tag{5}$$

In the mathematical analysis of this system, assumption is made that the environment of the bacteria remains constant and we will thus assume that D, S_{1in} and S_{2in} are positive constants. All the initial conditions are assumed to be positive.

3 Model analysis

3.1 Analysis of the acidogenic dynamics

The subsystem (1,2) is close to a classical model with a Monod kinetics but slightly modified by the term α. This makes the study of this system less straightforward than for Monod model (with $\alpha = 1$), for which the global stability has been demonstrated (see e.g. Smith and Waltman (1995)). However its behaviour remains simple as stated in the following Property:

Property 1 *System (1,2) with positive initial conditions admits a single globally stable equilibrium. If* $\alpha D < \mu_1(S_{1in})$ *this equilibrium is in the strictly positive orthant.*

Proof: a detailed proof is given in Appendix B.

It follows that the useful working point (X_1^*, S_1^*) of system (1,2) is globally asymptotically stable. As a consequence we have the following property:

Property 2 *After a transient time T, system (1,2) satisfies the inequality* $k_1\,\mu_1(S_1)X_1 \leq DS_{1in}$.

Proof: First let us note that at steady-state $k_1\mu_1(S_1^*)X_1^* = D(S_{1in} - S_1) < DS_{1in}$. So the inequality of Property 2 holds for the equilibrium.

We have shown with Property 1 that the normal working point $\xi_1^* = (X_1^*, S_1^*)$ was globally asymptotically stable. Therefore after a transient period all the trajectories will reach any neighbourhood of the steady state ξ_1^* and especially by continuity a neighbourhood where the inequality holds.

Remark: in practice this condition is often already met at initial time and in all the cases the transient time T is small.

3.2 Analysis of the methanogenic dynamics

In anaerobic digestion the accumulation of VFA might result from the imbalance between acidogenesis and methanogenesis leading thus to acidification. Since the methanogenesis is slower and can be inhibited it turns out to be the limiting step.

We now consider the methanogenic system given by equations (3) and (4) after a period greater than T (*cf. Property 2*). The total concentration of VFA available for the second step of the process is $S_{2in} + \frac{k_2}{D}\mu_1(S_1)X_1 \leq S_{2in} + \frac{k_2}{k_1}S_{1in} = \tilde{S}_{2in}$.

In order to study the methanogenesis as a stand-alone process we consider $\tilde{S}_{2in}$ as a worst-case upper bound of the total concentration of VFA in the reactor.

Thus the methanogenic system is reduced to a one-stage process independent of the acidogenic phase:

$$\begin{cases} \dot{X}_2 = \mu_2(S_2)X_2 - \alpha D X_2 \\ \dot{S}_2 = D(\tilde{S}_{2in} - S_2) - k_3\mu_2(S_2)X_2 \end{cases} \tag{6}$$

This system is close to a generic Haldane model but, as for the acidogenic subsystem, it is modified by the term α.

Property 3 *System (6) with initial conditions in $\Omega = \mathbb{R}_+^* \times \mathbb{R}_+$ admits a globally exponentially stable equilibrium in the interior domain for $\alpha D < \mu_2\left(\tilde{S}_{2in}\right)$. If $\mu_2\left(\tilde{S}_{2in}\right) < \alpha D < \mu_M$ it becomes locally exponentially stable (l.e.s) and the acidification equilibrium is also l.e.s. For $\alpha D > \mu_M$ the acidification equilibrium becomes globally exponentially stable (g.e.s.) (see Table 1 for more details)*

Proof: We first study the bounds of the variables X_2 and S_2 in the same way as for the acidogenic phase, considering the quantity $Z_2 = S_2 + k_3X_2$. It follows that $S_2 \leq \max\left(S_{20}, \tilde{S}_{2in}\right)$ and $X_2 \leq \frac{\max\left(Z_{20}, \frac{\tilde{S}_{2in}}{\alpha}\right)}{k_3}$.

The trivial steady state $\xi_2^\dagger$ corresponding to the bioreactor acidification is given by $\left(X_2^\dagger,\ S_2^\dagger\right) = \left(0,\ \tilde{S}_{2in}\right)$.

Now we are going to explore the other steady states. They are solutions of the

following system:

$$\begin{cases} \mu_2(S_2^\star) = \alpha D \\ X_2^\star = \frac{1}{\alpha k_3}(\tilde{S}_{2in} - S_2^\star) \end{cases} \tag{7}$$

Note that the steady states must verify $S_2^\star \leq \tilde{S}_{2in}$ to have $0 \leq X_2^\star$.

First remark that, if $\tilde{S}_{2in} \leq S_2^M$ then μ_2 is an increasing function on the admissible domain $\left[0, \tilde{S}_{2in}\right]$. As a consequence the study of system (6) is identical to the study of equations (1,2). We will then focus on the other case where $\tilde{S}_{2in} > S_2^M$.

As illustrated on Fig. 1, five cases are possible, depending on the parameters values.

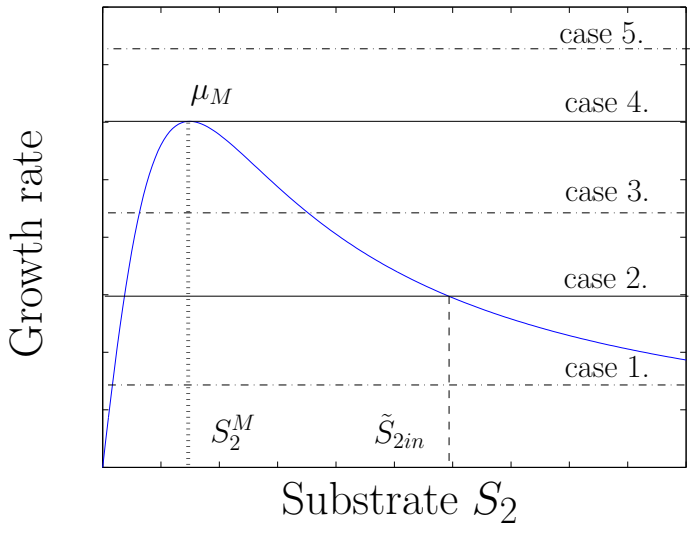

Fig. 1. Possible solutions for $\mu_2(S) = \alpha D$

cases 1. and 2. $\alpha D \in \left(0, \mu_2(\tilde{S}_{2in})\right]$: then the equation $\mu_2(S_2) = \alpha D$ has a single solution for $S_2 \in \left[0, \tilde{S}_{2in}\right)$:

$$(X_2^\star, S_2^\star) = \left(\frac{\tilde{S}_{2in} - \mu_2^{-1}(\alpha D)}{\alpha k_3}, \mu_2^{-1}(\alpha D)\right)$$

case 3. $\alpha D \in \left(\mu_2(\tilde{S}_{2in}), \mu_M\right)$: here the equation $\mu_2(S_2) = \alpha D$ has two solutions for $S_2 \in \left[0, \tilde{S}_{2in}\right)$. Let us denote $S_2^{1\star}$ and $S_2^{2\star}$ such that $\mu_2(S_2^{1\star}) = \mu_2(S_2^{2\star}) =$

αD:

$$0 < S_2^{1\star} < S_2^M < S_2^{2\star} < \tilde{S}_{2in}$$

Application to the specific example: considering μ_2 as expressed in (5) we can compute analytically these steady states:

$$S_2^{i\star} = \frac{K_{I2}}{2}\left(\frac{\bar{\mu}_2}{\alpha D} - 1\right) + (-1)^i \sqrt{\left(\frac{K_{I2}}{2}\left(\frac{\bar{\mu}_2}{\alpha D} - 1\right)\right)^2 - K_{I2}K_{S2}}, \tag{8}$$

with $i = 1$ for the useful working point $\left(X_2^{1\star}, S_2^{1\star}\right)$ and $i = 2$ for the unstable equilibrium $\left(X_2^{2\star}, S_2^{2\star}\right)$ (see Table 1 for the justification of this nomeclature).

then the two possible equilibria are:

$$\begin{cases} S_2^{1\star} < S_2^M \\ X_2^{1\star} = \frac{1}{\alpha k_3}(\tilde{S}_{2in} - S_2^{1\star}) \end{cases} \text{ and } \begin{cases} S_2^{2\star} > S_2^M \\ X_2^{2\star} = \frac{1}{\alpha k_3}(\tilde{S}_{2in} - S_2^{2\star}) \end{cases}$$

case 4. $\alpha D = \mu_M$: there is a unique solution to equation $\mu_2(S_2) = \alpha D$:

$$(X_2^\star, S_2^\star) = \left(\frac{\tilde{S}_{2in} - S_2^M}{\alpha k_3}, S_2^M\right)$$

case 5. $\alpha D > \mu_M$: here there is no solution to the equation $\mu_2(S_2) = \alpha D$. In this case there is no other equilibrium than the acidification point.

3.3 Study of equilibria stability

We compute the Jacobian matrix of system (6) at any point on the non-negative orthant:

$$\mathcal{J}(X_2, S_2) = \begin{pmatrix} \mu_2(S_2) - \alpha D & X_2 \frac{\partial \mu_2}{\partial S_2}(S_2) \\ -k_3\mu_2(S_2) & -D - k_3 X_2 \frac{\partial \mu_2}{\partial S_2}(S_2) \end{pmatrix} \tag{9}$$

The stability of system (6) is now easy to assess by computing the trace and the determinant of matrix (9) for all the considered cases:

- For the interior steady states $(X_2^{i\star}, S_2^{i\star})$:

$$\begin{aligned}\text{trace}(\mathcal{J}) &= -D - k_3 X_2^{i\star} \frac{\partial \mu_2}{\partial S_2}(S_2^{i\star}) \\ \det(\mathcal{J}) &= k_3 \alpha D X_2^{i\star} \frac{\partial \mu_2}{\partial S_2}(S_2^{i\star})\end{aligned}$$

- For the acidification steady state $(X_2^{\dagger}, S_2^{\dagger})$:

$$\begin{aligned}\text{trace}(\mathcal{J}) &= \mu_2(\tilde{S}_{2in}) - (1+\alpha)D \\ \det(\mathcal{J}) &= -D(\mu_2(\tilde{S}_{2in}) - \alpha D)\end{aligned}$$

It straightforwardly leads to the classification proposed in Table 1 [1] .

Table 1
Possible equilibria together with operating modes when $\tilde{S}_{2in} > S_2^M$

Case #	Conditions	**int.**	**acidi.**
1)	$\alpha D < \mu_2(\tilde{S}_{2in})$	g.e.s.	un.
2)	$\alpha D = \mu_2(\tilde{S}_{2in})$	l.e.s.	un†.
3)	$\alpha D \in (\mu_2(\tilde{S}_{2in}), \mu_M)$	$S_2^{1\star}$ l.e.s. $S_2^{2\star}$ un.	l.e.s.
4)	$\alpha D = \mu_M$	un†.	l.e.s.
5)	$\alpha D > \mu_M$	/	g.e.s.

Remark: the 2 cases denoted by 'un†.' corresponding to non hyperbolic equilibria are:

- Case 2: $\left(0, \tilde{S}_{2in}\right)$ for $\alpha D = \mu_2(\tilde{S}_{2in})$. Let us remark that the region $\{S_2 \leq \tilde{S}_{2in},\ X_2 \geq 0\}$ is positively invariant (any trajectory initialized in this domain stays in this domain). Moreover X_2 is increasing in the sub-domain $\{X_2 > 0,\ S_2^{1\star} \leq S_2 \leq \tilde{S}_{2in}\}$. The only way to reach the acidification $X_2^{\dagger} = 0$ from the region $\{S_2 \leq \tilde{S}_{2in}\}$ is thus to start with a zero initial condition. This proves that $\left(0, \tilde{S}_{2in}\right)$ is unstable.
- Case 4: $(X_2^{\star}, S_2^{\star})$ for $\alpha D = \mu_M$. It is clear that in this case $\dot{X}_2 \leq 0$, and therefore the point is unstable (there is however a region above $X_2 = X_2^{\star}$ converging toward this steady-state).

[1] l.e.s.: locally exp. stable, g.e.s.: globally exp. stable, un.: unstable, int.: interior, acidi.: acidification.

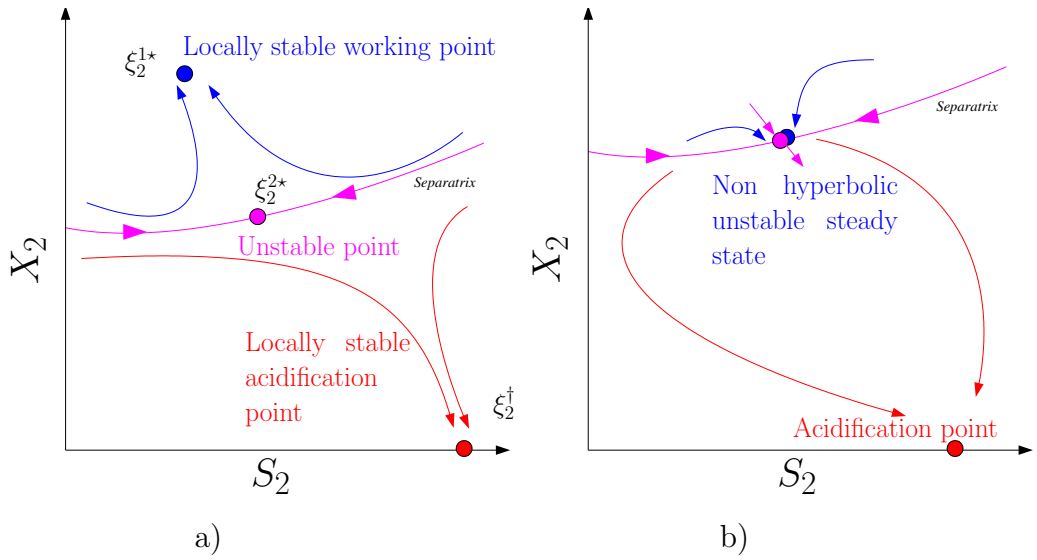

Fig. 2. Possible orbits in the phase plan: a) case 3, b) case 4.

3.4 Concluding remarks on stability

This study highlighted a special case of interest, for $\tilde{S}_{2in} > S_2^M$ and $\alpha D \in \left(\mu_2(\tilde{S}_{2in}), \mu_M\right)$. Here there are 2 steady states in the interior domain, one of which together with the acidification are stable. In this case, illustrated on Fig. 2a), the asymptotic state of the system is *a priori* not predictable, and depends on the initial conditions.

The next section will consist in characterising the size of the domain of initial conditions leading to the interior steady state $\xi_2^{1\star}$.

4 Attraction basin of the normal operating mode and stability criteria

In this section we still focus on the methanogenic step to establish a stability criterion, but we assume here specific forms for $\mu_1(S_1)$ and $\mu_2(S_2)$ given by equations (5).

4.1 Definition of the attraction basin and of the stability criterion

We have shown in the previous section that the state vector ξ_2 remains bounded. We thus consider the acceptable domain for (X_2, S_2) as follows:

$$\mathcal{K} = \left(0, \frac{\tilde{S}_{2in}}{\alpha k_3}\right] \times \left[0, \tilde{S}_{2in}\right] \tag{10}$$

Definition 1 *For $\xi_2^{1\star} = (X_2^{1\star}, S_2^{1\star})$, the interior steady state of system (6), we define its* ***basin of attraction*** Λ *as the set of initial conditions in $\mathcal{K}$ converging asymptotically towards it.*

$$\Lambda\left(D, \tilde{S}_{2in}\right) = \left\{\xi_{20} \in \mathcal{K} \mid \lim_{t \to +\infty} \xi_2(\xi_{20}, t) = \xi_2^{1\star}\right\},$$

The main idea of this paper is to characterise the stability of the system by the area of the attraction basin Λ. The process stability can then be assessed by the relative surface of Λ in $\mathcal{K}$ (11).

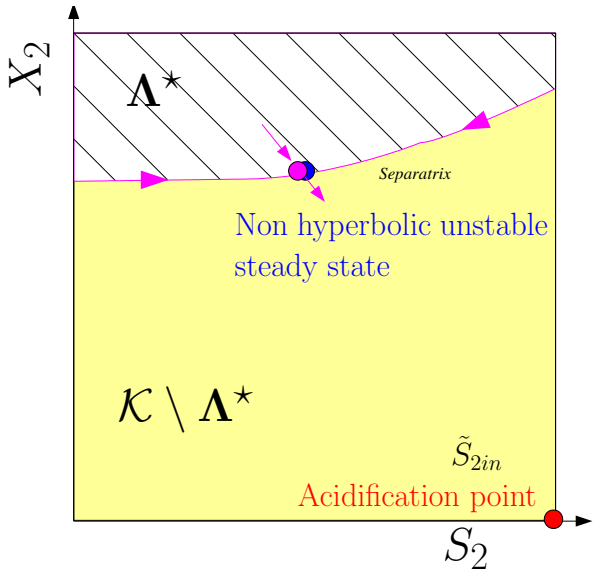

Fig. 3. Maximal set of initial conditions considered within the set $\mathcal{K}$ (case 4).

However, from the previous study (see Table 1) it is worth noting that there still exists a non empty attraction basin $\Lambda^\star = \Lambda\left(\frac{\mu_M}{\alpha}, \tilde{S}_{2in}\right)$ associated to case 4 ($\alpha D = \mu_M$) where the interior equilibrium is unstable (see Fig. 2 b)). We have seen that although this steady state is unstable any trajectory initiated in the region Λ^* above it (above the separatrix) will converge towards it. However it

is obvious that after any perturbation it will eventually converge towards $\xi_2^\dagger$. The region Λ^* was then excluded from the considered domain.

Definition 2 *We define the* ***Index of Stability*** *($\mathcal{I}_\mathcal{S}$) as the relative area of the attraction basin on the domain $\mathcal{K} \setminus \Lambda^\star$ (see Figure 3):*

$$\mathcal{I}_\mathcal{S}\left(D, \tilde{S}_{2in}\right) = \frac{\mathcal{S}\left(\Lambda\left(D, \tilde{S}_{2in}\right) \setminus \Lambda^\star\right)}{\mathcal{S}\left(\mathcal{K} \setminus \Lambda^\star\right)} \tag{11}$$

Where application $\mathcal{S}$ is the area of the considered domain.

4.2 Numerical computation of the stability index

The separatrix can be computed numerically by integrating System (6) in inverse time along the stable direction of the saddle point $(X_2^{2\star}, S_2^{2\star})$ starting very close to it. The computation of the attraction basin area follows straightforwardly.

However the numerical computation of $\mathcal{I}_\mathcal{S}$ does not provide any analytical expression of the stability index that would base a management strategy. In the following section we seek a simpler criterion related to $\mathcal{I}_\mathcal{S}$.

4.3 Overloading tolerance of the process: a simple criterion

If the dilution rate is increased from zero, the interior equilibrium will remain g.e.s. until $D = \frac{\mu_2\left(\tilde{S}_{2in}\right)}{\alpha}$ (*case 1,2*). Then the second (unstable) steady state appears in the interior domain together with a separatrix associated to the attraction basin $\Lambda\left(D, \tilde{S}_{2in}\right)$ that does no longer occupy all the domain (*case 3*). The size of $\Lambda\left(D, \tilde{S}_{2in}\right)$ will then decrease and finally vanish for $D \geq \frac{\mu_M}{\alpha}$ (*case 4*). It is worth noting that the distance between the 2 interior steady states follows a rather comparable scheme: it will decrease from a maximum distance when $D = \frac{\mu_2\left(\tilde{S}_{2in}\right)}{\alpha}$ to zero for $D = \frac{\mu_M}{\alpha}$ (see Fig. 4 a) and b)). For $D > \frac{\mu_M}{\alpha}$ the acidification steady state is the only possible equilibrium and the distance between the 2 interior steady states is no longer defined. From this consideration we define the notion of **Overloading Tolerance (OT)**:

Definition 3 *We define for $\alpha D \in \left[\mu_2(\tilde{S}_{2in}), \mu_M\right]$ the* ***Overloading Tolerance (OT)****,* ***M*** *which is simply the distance between the 2 interior steady states (see Fig. 4 a)):*

$$M\left(D\right) = \|\xi_2^{2\star} - \xi_2^{1\star}\| \tag{12}$$

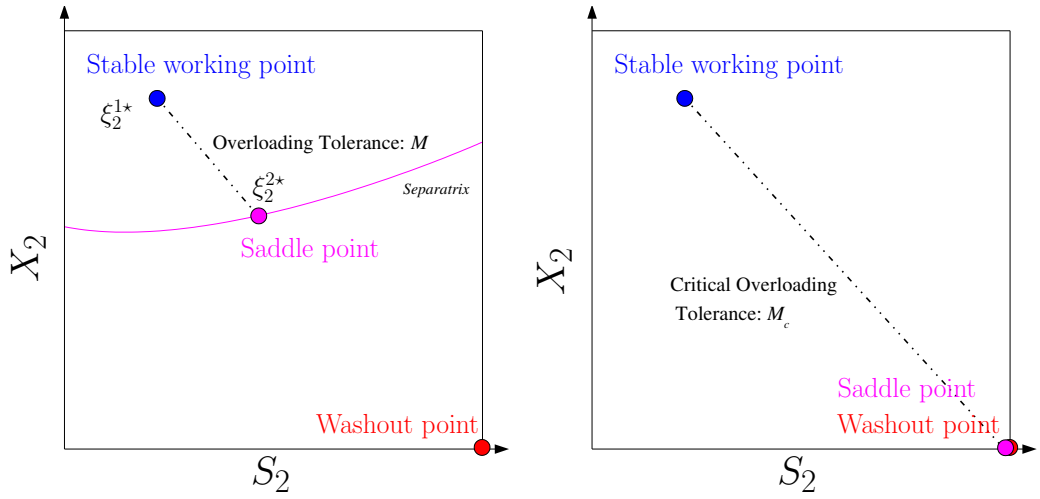

Fig. 4. Definition of a) the Overloading Tolerance b) the Critical Overloading Tolerance in the phase plan.

We also define the ***Critical Overloading Tolerance (COT)*** $\boldsymbol{M_c}$ *, which is the maximum value of the overloading tolerance obtained for* $D = \frac{\mu_2(\tilde{S}_{2in})}{\alpha}$.

The practical stability criterion that we will consider (named ***Relative Overloading Tolerance, ROT****) is then defined as follows:*

$$m(D, \tilde{S}_{2in}) = \begin{cases} 0 & for\ \alpha D > \mu_M \\ \frac{M(D)}{M_c(\tilde{S}_{2in})} & for\ \alpha D \in [\mu_2(\tilde{S}_{2in}), \mu_M] \\ 1 & for\ \alpha D < \mu_2(\tilde{S}_{2in}) \end{cases}$$

The OT given by the distance M between the 2 interior steady states can be computed straightforwardly from equations (7) and (8):

$$M(D) = 2\sqrt{1 + \frac{1}{\alpha^2 {k_3}^2}}\sqrt{\left(\frac{K_{I2}}{2}\left(\frac{\bar{\mu}_2}{\alpha\, D} - 1\right)\right)^2 - K_{I2}K_{S2}}$$

From this relation, we can see that the OT is a strictly decreasing function of the dilution rate and that it is independent from influent composition (S_{1in}, S_{2in}). The COT can be computed as follows:

$$M_c\left(\tilde{S}_{2in}\right) = \sqrt{1 + \frac{1}{\alpha^2 {k_3}^2}}\left(\tilde{S}_{2in} - \frac{K_{I2}\, K_{S2}}{\tilde{S}_{2in}}\right)$$

4.4 Comparison between stability index and relative tolerance

Using model parameters presented in Bernard *et al.* (2001), we have computed the stability index $\mathcal{I}_{\mathcal{S}}$ and the ROT associated with several working conditions (D, S_{1in} and S_{2in}).

As it can be seen on Fig. 5 the ROT represents a good approximation of the stability index $\mathcal{I}_{\mathcal{S}}$ based on the real computation of the attraction basin size.

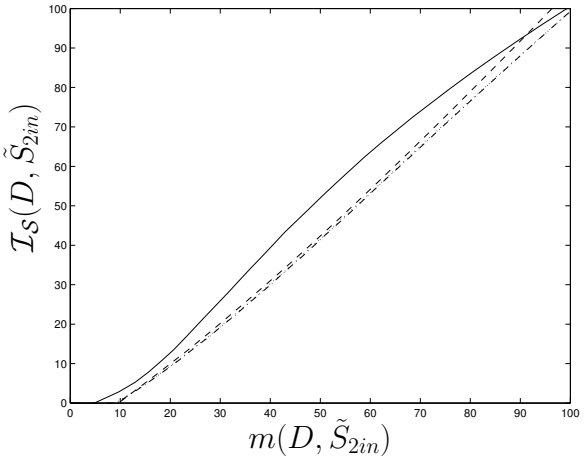

Fig. 5. Relation between the stability index $\mathcal{I}_{\mathcal{S}}(D, \tilde{S}_{2in})$ and the Relative Tolerance Margin $m(D, \tilde{S}_{2in})$ for various couples (S_{1in} in gDCO.L^{-1}, S_{2in} in mmol.L^{-1}): (3,30), (0,25), (15,20), (30,30)

The relative tolerance appears then as a simple but relevant criterion to assess the stability of an anaerobic digester.

From this criterion we define now the **"risk index"** which will on-line indicate to the operator the destabilisation risk he is taking;

$$r = 1 - m(D, \tilde{S}_{2in})$$

In the next section we use this operational criterion to assess the management strategy of a real anaerobic digester.

5 Application to the on-line determination of the destabilisation risk

In this section we apply the proposed index to a real experiment performed at the LBE-INRA in Narbonne, France. The process is an up-flow anaerobic fixed bed reactor with a useful volume of 0.948 m^3. The reactor is highly instrumented and many variables were measured during the experiments (Bernard *et al.*, 2001). More details about the process and evaluation of its on-line instrumentation are available in Steyer *et al.* (2002). The experiments were performed with raw industrial wine distillery vinasses.

The risk index has been computed with parameters of Bernard *et al.* (2001). Nevertheless, in order to favour a prudent strategy, and in the framework of a "worst case analysis" the parameter K_{I2} defining the inhibition level has been multiplied by a security constant δ (we have chosen $\delta = 0.7$). For low values of δ the seeming inhibition constant for the calculus of the risk index is lower; therefore the system would be more easily inhibited and the risk index reacts more rapidly. On the contrary high values of δ lead to a higher seeming inhibition constant and the risk index would be less responsive to small changes in the substrate concentration. The additional parameter δ enables to tune the sensibility of the risk index.

The estimation of the risk index is presented on Fig. 6. It is worth noting that the regimes associated with acid accumulation (*i.e* more than 1g.L^{-1}) are all characterised by a very high risk. More surprising, some a priori less dangerous working modes are indeed also associated to a non zero risk. This is especially the case at day 5 on Fig. 8 where the risk index is maximal while the dilution rate and the VFA level do not foreshadow any specific risk. This amplified sensibility is due to the tuning parameter δ. A very important point is that the risk index increases immediately while it takes time for the VFA to accumulate and even more time to observe a pH decrease which is the usual indicator of the process destabilisation. As such it anticipates a potential process acidification before it can be detected and it becomes to late to avoid a dramatic biomass inhibition.

6 Conclusion

From the analysis of the nonlinear system describing the anaerobic process we have proposed a criterion that assesses the risk associated to an operating strategy. This index is highly correlated to the relative size of the normal working mode attraction basin.

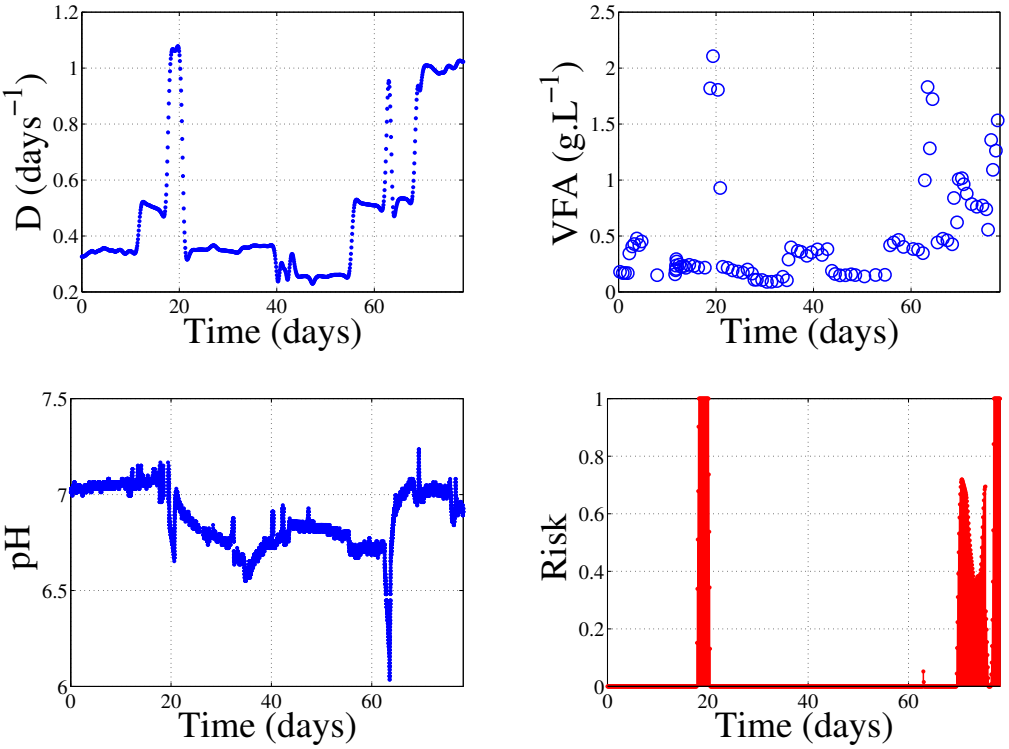

Fig. 6. Dilution rate, measured VFA, pH and computed risk for an experiment performed at INRA LBE.

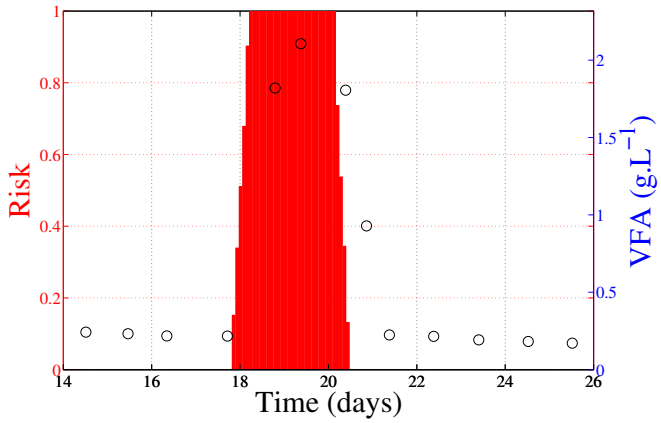

Fig. 7. Risk index and VFA (zoom from Fig. 6c). and d).).

The criterion turns out to be relevant to diagnose an operation strategy since it can predict very early a future accumulation of acids. It can thus be run as an indicator that helps an operator or even diagnoses the strategy of an automatic controller which would not ensure global stability.

As mentionned at the beginning of this paper, several control laws have yet been built to assure the durability of the process stability. A possible application of this stability criterion is the triggering of an automatic robust controller

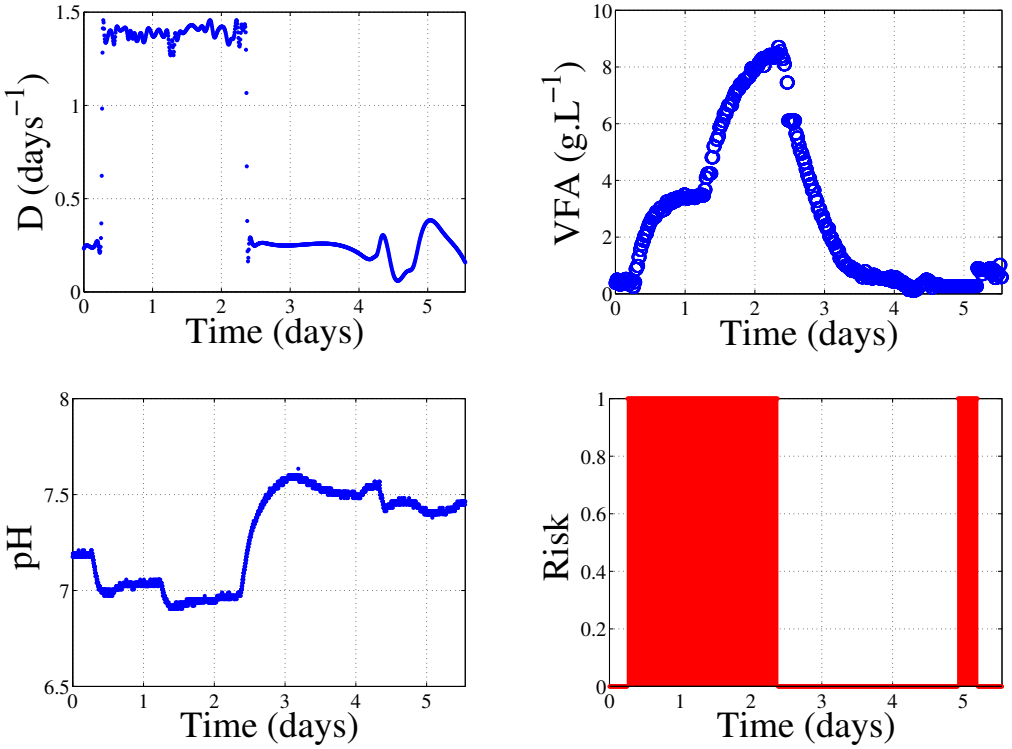

Fig. 8. Dilution rate, measured VFA, pH and computed risk for an overload experiment performed at INRA LBE.

whenever the risk index reaches a predifined level (Bernard *et al.*, 2005).

In order to take into account the biological evolution of the system in the risk index computation the next steps would consist in:

(1) studying the index risk sensitivity to the various parameters,
(2) developing a strategy for the on-line estimation of the most pertinent ones.

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Appendix A

Let us define the generic differential system :

$$\begin{cases} \dot{x} \ = \ f(x,t) \\ x(t_0) \ = \ x_0 \end{cases} \tag{13}$$

with $x(t) \in \mathbb{R}^n$ and $f : \mathbb{R}^n \times \mathbb{R}^+ \ \rightarrow \ \mathbb{R}^n$ a continuous function. A steady-state x^* of system (13) is solution of:

$$0 \ = \ f(x^*, t) \ \ \forall t \geq t_0$$

Definition 4 (Local asymptotic stability) *The steady-state x^* is said to be locally asymptotically stable if for any $\epsilon > 0$,* **there exists** *$\delta > 0$ and there exists T such that*

$$|x_0 - x^*| < \delta \ \Rightarrow \ |x(x_0, t) - x^*| < \epsilon \ \ \forall t \geq T + t_0$$

Definition 5 (Global asymptotic stability) *The steady-state x^* is said to be globally asymptotically stable if for any $\epsilon > 0$ and* **for any** *$\delta > 0$, there exists T such that*

$$|x_0 - x^*| < \delta \ \Rightarrow \ |x(x_0, t) - x^*| < \epsilon \ \ \forall t \geq T + t_0$$

Definition 6 (Local exponential stability) *The steady-state x^* is said to be locally exponentially stable if* **there exists** *$\delta > 0$ and there exist non-negative constants γ_1 and γ_2, such that for all $t_0 > 0$*

$$|x_0 - x^*| < \delta \ \Rightarrow \ |x(x_0, t) - x^*| < \gamma_1 |x_0| e^{-\gamma_2 (t - t_0)} \ \ \forall t \geq t_0$$

Definition 7 (Global exponential stability) *The steady-state x^* is said to be globally exponentially stable if* **for any** *$\delta > 0$ there exist non-negative constants γ_1 and γ_2, such that for all $t_0 > 0$*

$$|x_0 - x^*| < \delta \ \Rightarrow \ |x(x_0, t) - x^*| < \gamma_1 |x_0| e^{-\gamma_2 (t - t_0)} \ \ \forall t \geq t_0$$

Appendix B

The positivity of the variables of this system is trivial since $\mu_1(0) = 0$. To demonstrate the boundedness in a compact set of $\mathbb{R}^2_+$ we consider the quantity $Z = S_1 + k_1 X_1$:

Since $\alpha \in (0\ 1]$:

$$\begin{cases} D(S_{1in} - Z) \leq \dot{Z} \leq \alpha D\left(\frac{S_{in}}{\alpha} - Z\right) \\ \dot{S}_1 \leq D\left(S_{1in} - S_1\right) \end{cases}$$

It follows that Z and S_1 are asymptotically bounded; Since $S_1 \geq 0$ the upper bound for Z is also an upper bound for $k_1 X_1$:

$$\begin{cases} \min\left(Z_0, S_{1in}\right) \leq Z \leq \max\left(Z_0, \frac{S_{1in}}{\alpha}\right) \\ 0 \leq S_1 \leq \max\left(S_{10}, S_{1in}\right) \\ 0 \leq X_1 \leq \frac{\max\left(Z_0, \frac{S_{1in}}{\alpha}\right)}{k_1} \end{cases}$$

The considered system (1,2) has 2 steady states: the trivial washout steady state $X_1^\dagger = 0,\ S_1^\dagger = S_{1in}$ which exists for any D, and another steady state in the positive domain if and only if $\alpha D < \mu_1\left(S_{1in}\right)$, given by:

$$\begin{cases} S_1^\star = \mu_1^{-1}(\alpha D) \\ X_1^\star = \frac{1}{k_1 \alpha}\left(S_{1in} - S_1^\star\right) \end{cases} \tag{14}$$

As $\mu_1(.)$ is monotone system(14) has a unique solution in the positive domain if and only if $\alpha D \leq \mu_1\left(S_{1in}\right)$ (ensuring $S_1^\star \leq S_{1in}$ and thus $X_1^\star \geq 0$).

Application to the specific case of the real example: considering the expression of μ_1 given in (5) we get:

$$(X_1^\star, S_1^\star) = \left(\frac{1}{\alpha k_1}\left(S_{1in} - K_{S1}\frac{\alpha D}{\bar{\mu}_1 - \alpha D}\right), K_{S1}\frac{\alpha D}{\bar{\mu}_1 - \alpha D}\right)$$

In order to assess the stability of $(X_1^\star, S_1^\star)$ and $\left(X_1^\dagger, S_1^\dagger\right)$ we study the Jacobian matrix of System (1,2) at these steady states:

$$\mathcal{J}(X_1, S_1) = \begin{pmatrix} \mu_1(S_1) - \alpha D & X_1 \frac{\partial \mu_1}{\partial S_1}(S_1) \\ -k_1 \mu_1(S_1) & -D - k_1 X_1 \frac{\partial \mu_1}{\partial S_1}(S_1) \end{pmatrix} \tag{15}$$

The computation of the trace and determinant of (15) for both equilibria gives:

- For the useful interior working point (X_1^*, S_1^*):

$$\begin{aligned}\text{trace}(\mathcal{J}) &= -D - k_1 X_1^\star \frac{\partial \mu_1}{\partial S_1}(S_1^\star) \; < 0 \\ \det(\mathcal{J}) &= k_1 \alpha D X_1^\star \frac{\partial \mu_1}{\partial S_1}(S_1^\star) \; > 0\end{aligned}$$

- For the washout steady state $(X_1^\dagger, S_1^\dagger) = (0, S_{1in})$:

$$\begin{aligned}\text{trace}(\mathcal{J}) &= \mu_1(S_{1in}) - (1+\alpha)D \\ \det(\mathcal{J}) &= -D(\mu_1(S_{1in}) - \alpha D) \; < 0\end{aligned}$$

This shows that only the useful working point $(X_1^\star, S_1^\star)$ is an attractor, the washout steady state being a saddle point.

To conclude the proof and determine the global behaviour of (1,2) we change variables (X_1, S_1) to (X_1, Z). With this reformulation the system becomes :

$$\begin{cases} \dot{X}_1 = \mu_1(Z - k_1 X_1) X_1 - \alpha D X_1 \\ \dot{Z} = D\,(S_{in} - Z) + (1-\alpha)\, D k_1 X_1 \end{cases}$$

Then the Jacobian matrix is:

$$\mathcal{J}(X_1, Z) = \begin{pmatrix} \star & \mu_1'\,(Z - k_1 X_1)\, X_1 \\ (1-\alpha)\, D k_1 & \star \end{pmatrix} \tag{16}$$

It follows directly that this system is cooperative (*i.e* the off-diagonal terms of the Jacobian matrix are non-negative). Furthermore the system is asymptotically bounded in a compact included in $\mathbb{R}_+^2$. Hence from *Theorem* 2.2 of Chapter 3 in Smith (1995) for two-dimensional systems, the limit can only be a stable equilibrium point. Since the acidification equilibrium is unstable the system cannot converge towards it.

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