Divisible-Good Auctions with uniform price and non monotonic preference function

Odile Pourtallier, INRIA, Sophia Antipolis, France

Mabel Tidball, INRA, Lameta, Montpellier, France

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Context of the paper

Divisible-good auctions with uniform price

Divisible-good auctions are widely used in practice.

The analysis of divisible-good auctions has gained a lot of interest following the debate on how to auction U.S. Treasury securities.

Uniform-price auctions are also used in other financial markets such as "Initial Public Offering" and share repurchases as well as in nonfinancial markets.

The auctioneer computes the aggregate demand function and the clearing price. He then divides the asset among several bidders.

Context of the paper

- Previous research makes implicit assumptions on the demand function
- Wilson (1979) assumes that agents can only use continuous demand functions. He concludes that there is underpricing: the clearing price is smaller than the expected value of the good.
- The paper of Kremer and Nyborg, 2004, extend the problem to discontinuous non increasing demand functions. They focus on allocation rules that specify the way the asset is divided in cases of excess demand. This may have a dramatic effect on the set of equilibrium prices. In particular, they show that a simple allocation rule (pro-rata) eliminates underpricing.
- All the papers deal with particular assumptions on the demand function and do not give a general procedure to compute equilibria.

The goal of the paper

This is a very preliminary THEORETICAL paper that deals with the case of complete information.

- Evaluation functions may or may not be monotonic.
- Bidders submit a demand that may be discontinuous (left continuous).
- We show how to determine a Nash equilibrium by using dynamic programming.
- We take into account the allocations rules to deal with the problem of excess demand.

The Problem

We consider a divisible good auction market where a known quantity Q of goods is to be sold,

- Each agent i submits a demand schedule $s_i(\cdot)$ to the market
- $s_i(p)$ represents the quantity agent i is willing to buy at price p.
- $\mathbf{s}(\cdot) = (s_1(\cdot), \cdots, s_N(\cdot))$ is a strategy profile.
- For a strategy profile $\mathbf{s}(\cdot)$, the aggregate demand schedule is

$$S(\cdot) = \sum_{i=1}^{N} s_i(\cdot).$$

The market reacts by setting a uniform unit price $\mathbf{p} = p(\mathbf{s}(\cdot))$

$$P_{\mathbf{s}} = p(\mathbf{s}(\cdot)) = \sup\{p \mid S(p) \geq Q\}.$$

Allocation rules

Each agent i gets the amount q_i of good according to its offer

- if $S(P_\mathbf{s}) \leq Q$, then

$$q_i(\mathbf{s}(\cdot)) = s_i(P_s).$$

- if $S(P_\mathbf{s}) > Q$ an allocation rule, denoted f has to be used and

$$q_i(\mathbf{s}(\cdot)) = f(\mathbf{s}(P_s), i).$$

Alloction rule f property :

if $S(P_\mathbf{s}) > Q$, each bidder receives a quantity $q_i(\mathbf{s})$ such that $q_i(\mathbf{s}) \leq s_i(p(\mathbf{s}))$ and $\sum_i q_i(\mathbf{s}) = Q$.

The problem for the agents

For a strategy profile $\mathbf{s}$ the reward for agent i is then given by

$$\psi_i(q_i(\mathbf{s}(\cdot)), p(\mathbf{s}(\cdot))) \triangleq \varphi_i(\mathbf{s}(\cdot)).$$

- This depends upon the choice $s_i(\cdot)$ of the agent, but also upon the choice $s_j(\cdot), j \neq i$ of the other agents.
- The agents strive to maximize their reward.
- We suppose that the agents behave non-cooperatively, hence they will chose their strategy profile so as to satisfy Nash equilibrium condition.

Agents will choose the strategy profile $\mathbf{s}^*(\cdot)$ such that

$$\varphi_i(\mathbf{s}^*(\cdot)) \geq \varphi_i(\mathbf{s}^*_{-i}(\cdot), s_i(\cdot)), \quad \forall s_i(\cdot) \in \mathcal{S}_i.$$

An auxiliary problem: a stopping time game

- To consider a game in a more classical form.
- The solution of our initial game will be grounded on the solution of this new game.

We "reverse" price

$$p \quad \to \quad t = T - p.$$

The auxiliary game

- Each player i is endowed with a real function $g_i(t)$ defined on some interval $[0, T]$.
- At each instant of time, t, the player can choose an action $x_i(t)$ in the set $\{0, 1\}$ that are piecewise right continuous on $[0, T]$ with a finite number of discontinuities.
- Consider $\tilde{s}_i(t) = x_i(t)g_i(t)$, and $\tilde{S}(t) = \sum_{i=1}^{N} \tilde{s}_i(t)$.

Once players have chosen a strategy profile $\mathbf{x}(\cdot)$, the stopping time of the game is,

$$\mathbf{T_s} = T_s(\mathbf{x}(\cdot)) = \inf(t \mid \tilde{S}(t) \geq Q).$$

The reward for each player is

$$\tilde{\varphi}_i(\mathbf{x}) = \tilde{\psi}_i(x_i g_i(\mathbf{T_s}), \mathbf{T_s}), \quad \text{where } \tilde{\psi}_i \text{ is given.}$$

We want to determine Nash Equilibrium of this game.

Solving the auxiliary game in discrete time

Define the set $\mathcal{A} \in [0, T]$ as

$$\mathcal{A} = \{t \mid \sum_i \tilde{s}_i(t) \geq Q.\}$$

$\mathcal{A}$ is the set of time t such that there exist strategy profile $\mathbf{x}$ such that $\mathbf{T_s}^*(\mathbf{x}) = t$. $\mathcal{A}$ is the union of a finite number of intervals.

Denote $\mathcal{T}$ the set of instants $t_k,\ k = 1, \cdots, K$ such that $t_K = T$, $t_1 < t_2 \cdots < t_K$ and such that for $k \neq K$, either

- t_k belongs to the boundary of $\mathcal{A}$,
- there exists a player i such that t_k is a local extreme of the function $\tilde{\psi}_i(g_i(t), t)$,
- there exists a player i such that t_k is a local extreme of the function g_i.

Solving the auxiliary game in discrete time

- Functions $\tilde{\psi}_i(g_i(t), t)$ and $g_i(t)$ are monotonous with respect to t on any interval $[t_j, t_{j+1}]$.
- The game is equivalent to a discrete time dynamic game where at each step k (corresponding to time t_k) each player either prefers to continue the game (until time t_{k+1} and chooses $x_{i,k} = 0$) or prefers to stop (at time t_k, and chooses $x_{i,k} = 1$).
- S_k the state of the game at step k.
  $S_k = 0$ means that the game is not yet finished at time t_k *i.e.* the stopping time is strictly greater that t_k and that the game continue over time step k.

Solving the auxiliary game in discrete time

The state evolves according to the dynamic equation

$$S_0 = 0,$$
$$S_{k+1} = \begin{cases} S_k \text{ if } \sum_{i=1}^{J} x_{i,k} g_i(t_k) < Q \\ 1 \text{ otherwise.} \end{cases}$$

For each player the evaluation function is a final time evaluation function, more precisely,

$$\tilde{\varphi}_i(\mathbf{x}) = \tilde{\psi}_i(x_{il} g_i(t_l), t_l),$$

where l is the stopping step, *i.e.* the first step such that $S_l = 1$.

Solving the auxiliary game in discrete time

Call $\varphi^*_{i,k}$ the Nash equilibrium evaluation level for player i at step k.

$$\tilde{\varphi}^*_{i,K} = \tilde{\psi}_i(g_i(t_K), t_K) = \tilde{\psi}_i(g_i(T), T),$$

and for step k, we set

$$u^*_{i,k} = \begin{cases} 1, & \text{if } \tilde{\psi}_i(g_i(t_k), t_k) \geq \tilde{\varphi}^*_{i,k+1} \\ 0, & \text{otherwise.} \end{cases}$$

$$\tilde{\varphi}^*_{i,k} = \tilde{\varphi}^*_{i,k+1} \mathbb{1}_{\{\sum_i u^*_{i,k} g_i(t_k) < Q\}} + \tilde{\psi}_i(g_i(t_k), t_k) \mathbb{1}_{\{\sum_i u^*_{i,k} g_i(t_k) \geq Q\}}.$$

The strategy profile $\mathbf{u}^* = (u^*_1, \cdots, u^*_N)$ is a Nash equilibrium for the discrete time game and t_l, the stopping time, is the first k such that $S_k = 1$.

This is a consequence of the dynamic programming principle.

The solution of the auxiliary game in continuous time

Define the strategy profile $\mathbf{x}^* = (x_1^*, \cdots x_N^*)$

$$x_i^*(t) = \begin{cases} u_i^*(t), \text{ for } t \in \mathcal{T}. \\ u_i^*(t_k), \text{ for } t \in [t_k, t_{k+1}[\end{cases}$$

Then we have

The strategy profile $\mathbf{x}^*$ is is a Nash equilibrium in the continuous time game.

Link with the initial auction problem

- $g_i(t)$ the Nash equilibrium of $\psi_i(f(\mathbf{g(t)}, \mathbf{i}), t)$,
- Consider a Nash equilibrium of the stopping time problem $(x_i^*(t), t_l)$,

if we call
$$s_i^*(p) = x_i^*(T-p)g_i(T-p),$$
then, the strategy profile $\mathbf{s}^* = (s_1^*, \cdot, s_N^*)$ and $p_s = T - t_l$ is a Nash equilibrium of our auction problem.

Conclusion

- With weak assumptions on the demand function and the preferences of the agents,
- using the discrete time dynamic programming,
- we have found a general procedure to compute an equilibrium in a divisible-good auction with uniform price.

Extensions

- Incomplete information,
- Multiplicity of equilibria,
- Analysis of the influence of different allocation rules,
- How this kind of auctions works in practice. Find examples...
- Discrete- versus continuous-state space. We have described the model in a continuous space. In real life, however, prices and quantities are discrete. There is minimal unit (tick) size for both quantities and prices. Easier to solve using the discrete dynamic programming?
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