## Certified solutions of polynomials with uncertainties

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December 5th, 2007

MACIS 07, Rocquencourt

$$\begin{array}{c|c} \mathbf{1} & \text{I. Introduction} \\ \hline \mathbf{Problem} \\ P(z) = \sum_{i=0}^{d} [a_i] z^i, \\ [a_i] = [\underline{a}_i, \overline{a}_i] \text{ real intervals.} \\ \hline \text{Find} \\ \\ \Omega_{\mathbb{R}} = \{z \in \mathbb{R}, \exists a_i \in [a_i], i = 0 \cdots d, \sum_{i=0}^{d} a_i z^i = 0\}, \\ \Omega_{\mathbb{C}} = \{z \in \mathbb{C}, \exists a_i \in [a_i], i = 0 \cdots d, \sum_{i=0}^{d} a_i z^i = 0\}. \end{array}$$

### 2 I. Introduction

# Outline

- Motivation
- Interval analysis
- Edge theorem
- Exclusion tests and filtering procedures
- Perspectives

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- **3 I**. Introduction
- Motivation

heta, X, Y, determined by solving some real polynomial of degree 6,

$$\sum_{i=0}^{6}a_{i}(l_{1},l_{2},l_{3},A_{1},A_{2},A_{3})\gamma^{i}=0,$$

$$\theta = f_{\theta}(\gamma), \quad X = f_X(l_1, l_2, l_3, \gamma), \quad Y = f_Y(l_1, l_2, l_3, \gamma)$$

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### 4 I. Introduction

- Direct problem :  $l_i \in [\underline{l}_i, \overline{l}_i]$ , determine the workspace  $W_{l_1, l_2, l_3}$  of the robot ?
- Design problem : Determine intervals  $[\underline{l}_i,\overline{l}_i]$  such that  $W_{l_1,l_2,l_3}$  contains a given workspace  $W_i$

#### Solving the problem :

Ranges for  $l_i$  possibly large to be taken into account.

All possible uncertainties (hopefully small) to be taken into account

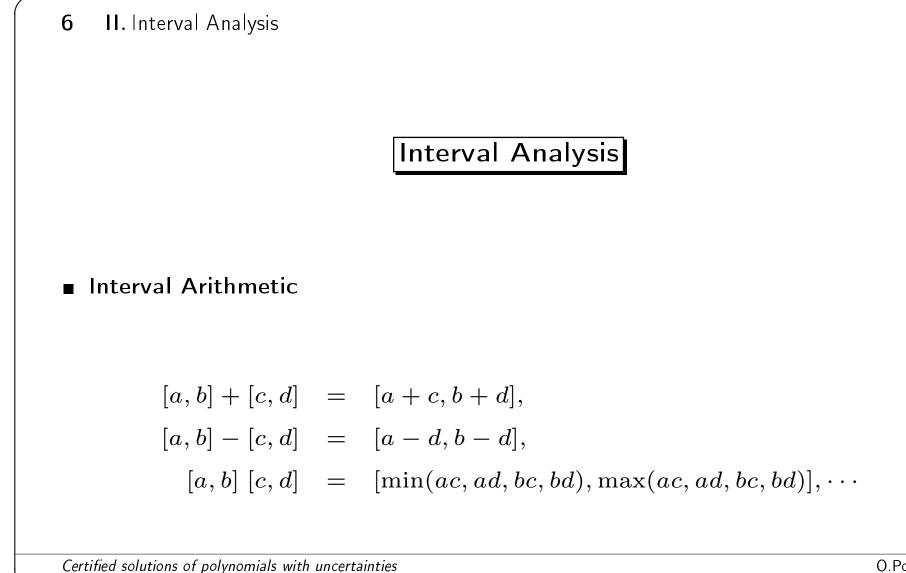
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### 5 I. Introduction Certification :

For some applications (e.g. medical) there is a strong need of **certification** 

# $\rightarrow$ Interval Analysis techniques

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- 7 II. Interval Analysis
- Evaluation

$$f(x) = x^3 + x^2 + x, \quad \{f(x), x \in [X] = [-1, 1]\} = [-1, 3]$$

$$[X]^{3} + [X]^{2} + [X] = [-1, 1] + [0, 1] + [-1, 1] = [-2, 3]$$

Overestimation

$$[X]^{2}([X] + 1) + [X]) = [-1, 3]$$

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8 II. Interval Analysis

### Bisection Algorithm

- exclusion tests,
- Filtering procedures,
- Uniqueness tests.

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- 9 II. Interval Analysis
- Filtering (2B) : Example

Solve  $1 + x + x^2 + x^3 = 0$  in [-1, 1]

$$\begin{array}{rcl} x & = & -1 - x^2 - x^3 \\ [-1,1] & & -1 - [-1,1]^2 - [-1,1]^3 = [-3,0], \end{array}$$

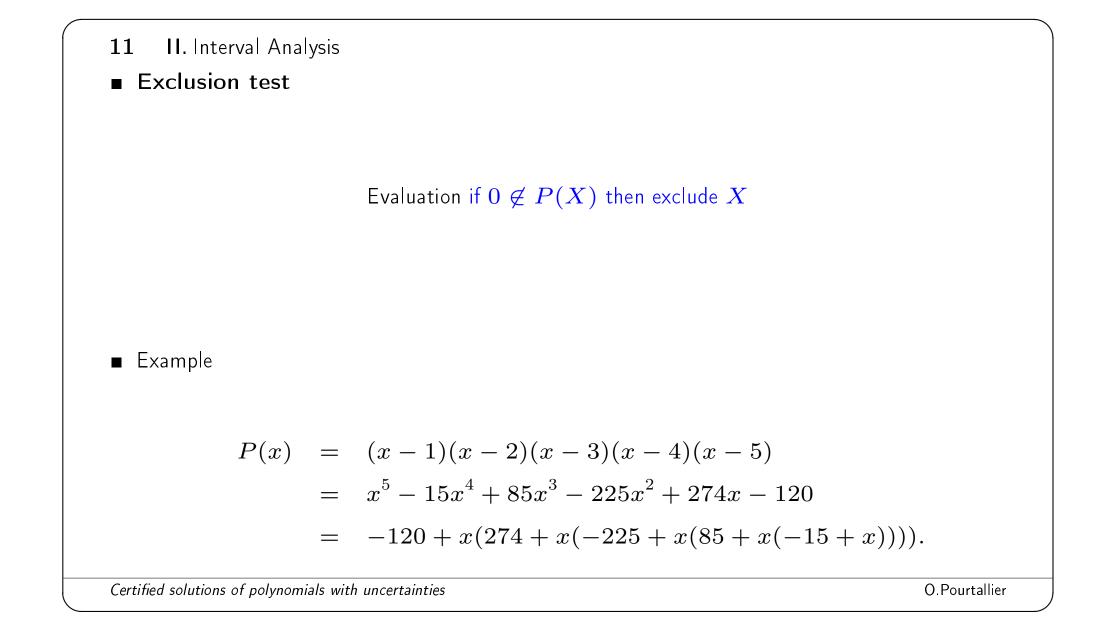
Necessarily  $x \in [-1, 0]$ 

$$x = -x^{2}(x+1) - 1$$
  
-1,1] -[-1,1]<sup>2</sup>([-1,1]+1) - 1 = [-1,0],

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10 II. Interval Analysis Necessarily x = -1 !

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$$X = [4.9, 4.99]$$

$$(X^{5} - 15X^{4} + 85X^{3} - 225X^{2} + 274X - 120) = [-855.229710, 853.060670]$$

$$(-120 + x(274 + x(-225 + x(85 + x(-15 + x))))) = [-133.5775355, 132.1841260]$$
We cannot conclude
$$X = [4.99990, 4.99999]$$

$$(X^{5} - 15X^{4} + 85X^{3} - 225X^{2} + 274X - 120) = [-0.879868..0.877236]$$

$$(-120 + x(274 + x(-225 + x(85 + x(-15 + x))))) = [-0.1373968..0.1347577)]$$
We cannot conclude
$$We cannot conclude$$

**13** II. Interval Analysis

 $\longrightarrow$  need to use adapted exclusion tests

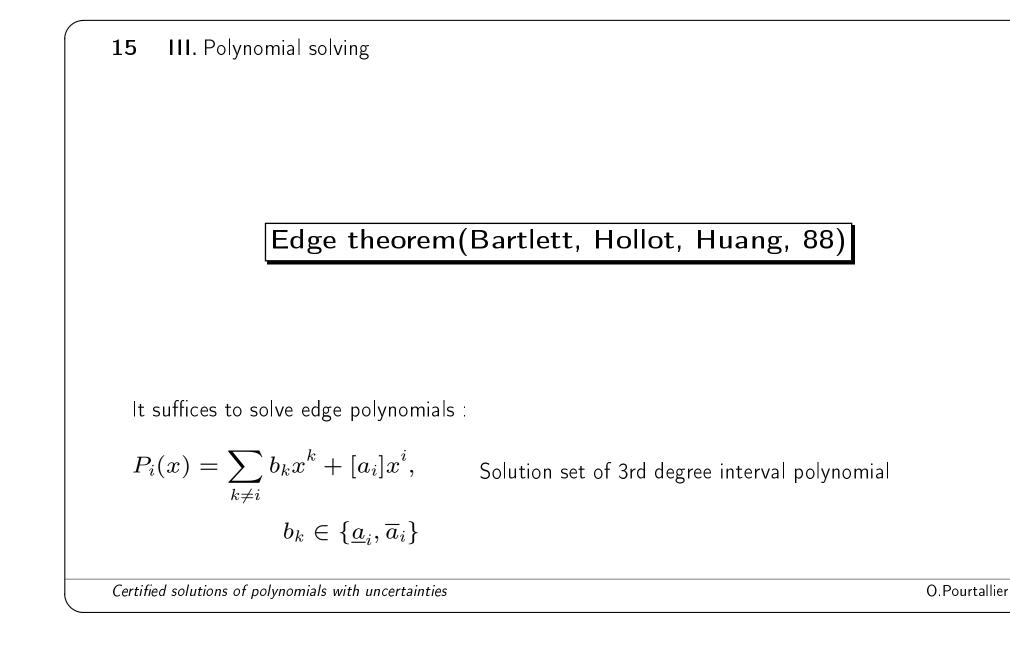
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## use of polynomials with 1 interval coefficient

- If the number of interval coefficient is more than 1, the complex solution set has an non empty interior set.
- Exclusion test always fail in the interior of solution test (  $\rightarrow$  cut).
- $\blacksquare \longrightarrow$  Interior tests or Determine the boundary of solution set
  - $\longrightarrow$  Edge theorem

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- **16** III. Polynomial solving
- 1-D uncertainty. The solution is a 0-D space. no need to have interior test.
- $d.2^d$  polynomials with one interval coef to consider.
- The solutions of a part of the previous polynomials are in the interior of the solution set.
- How to select the polynomials whith solutions on the boundary of the solution set ?

- **17** III. Polynomial solving
- Edge theorem revised

Polar representation.

$$N = lcm(2, 3, \cdots, d) \quad I_k = [k\frac{2\pi}{N}, (k+1)\frac{2\pi}{N}]$$

We can determine 2(d+1) edge polynomials necessary to determine the solution set boundary in  $\mathbb{R}^+ \times I_k$ .

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- -2(d+1) edge polynomials to consider in each angular sector,
- $-lcm(2,3,\cdots,d)$  angular sectors to consider....
- -2(d+1) is not the minimal number of polynomials to consider. We may improve further edge theorem.

Allows to solve only low degree uncertain polynomials.

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**19** III. Polynomial solving

Edge theorem for real solutions

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20 III. Polynomial solving

$$x \ge 0, \quad P(x) = [P_{-}^{+}(x), P_{+}^{+}(x)]$$
  
 $x \le 0, \quad P(x) = [P_{-}^{-}(x), P_{+}^{-}(x)]$ 

$$P_{-}^{+}(x) = \underline{a}_{0} + \underline{a}_{1}x + \underline{a}_{2}x^{2} \cdots + \underline{a}_{d}x^{d},$$
$$P_{+}^{+}(x) = \overline{a}_{0} + \overline{a}_{1}x + \overline{a}_{2}x^{2} \cdots + \overline{a}_{d}x^{d},$$

$$P_{-}^{-}(x) = \underline{a}_{0} + \overline{a}_{1}x + \underline{a}_{2}x^{2} + \overline{a}_{3}x^{3} \cdots ,$$
$$P_{+}^{-}(x) = \overline{a}_{0} + \underline{a}_{1}x + \overline{a}_{2}x^{2} + \underline{a}_{3}x^{3} \cdots ,$$

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21 III. Polynomial solving

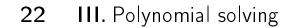
Problem to solve :
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Certified real solutions of non interval polynomials :

- positive zeros of 
$$P_{-}^{+}$$
 and  $P_{+}^{+}$ 

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- negative zeros of P_{-}^{-} and P_{+}^{-}.
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### Complex solutions for one interval coefficient polynomials

- Exclusion tests :
- Weyl exclusion test :

$$P(z) = \sum_{i=0}^{d} a_i z^i = \sum_{i=0}^{d} b_i (z - z_0)^i,$$

If  $|b_0| > \sum_{i=1}^d |b_i| \delta^i$  then no solution in  $B(z_0,\delta)$  .

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**23** III. Polynomial solving

– Yakoubsohn test :

$$M_{z_0}(\delta) = |P(z_0)| - \sum_k \frac{|p^k(x)|}{k} \delta^k.$$

If  $M_{z_0}(\delta)>0$ , then no solution in  $B(z_0,\delta)$ .

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24	III. Poly	nomial	solving
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## Real solutions for one interval coefficient polynomials

- Exclusion tests:
- Weyl
- Yakoubsohn
  - Filtering procedures
- 2-B,
- Interval Newton algorithm

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- **25** III. Polynomial solving
- Uniqueness tests:
- based on Rouche theorem,
- Krawczyk operator,
- Kantorovitch theorem.

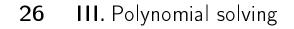
### Deflation

Once a solution is known, division by  $(z-z_0)$ .

#### First numerical results

Find the 19 solutions of Wilkinson 19.

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Perspectives

- There is probably a room for improving Edge theorem for complex solutions
- Implementation for complex solutions,
- Consider correlated Interval coefficients.

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