Certified solutions of polynomials with uncertainties

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Problem

$$P(z) = \sum_{i=0}^{d} [a_i] z^i,$$

$[a_i] = [\underline{a}_i, \overline{a}_i]$ real intervals.

Find

$$\Omega_{\mathbb{R}} = \{z \in \mathbb{R}, \exists a_i \in [a_i], i = 0 \cdots d, \sum_{i=0}^{d} a_i z^i = 0\},$$

$$\Omega_{\mathbb{C}} = \{z \in \mathbb{C}, \exists a_i \in [a_i], i = 0 \cdots d, \sum_{i=0}^{d} a_i z^i = 0\}.$$

Outline

- Motivation
- Interval analysis
- Edge theorem
- Exclusion tests and filtering procedures
- Perspectives

■ Motivation

θ, X, Y, determined by solving some real polynomial of degree 6,

$$\sum_{i=0}^{6} a_i(l_1, l_2, l_3, A_1, A_2, A_3)\gamma^i = 0,$$

$$\theta = f_\theta(\gamma), \quad X = f_X(l_1, l_2, l_3, \gamma), \quad Y = f_Y(l_1, l_2, l_3, \gamma)$$

- **Direct problem** : $l_i \in [\underline{l}_i, \overline{l}_i]$, determine the workspace W_{l_1,l_2,l_3} of the robot ?
- **Design problem** : Determine intervals $[\underline{l}_i, \overline{l}_i]$ such that W_{l_1,l_2,l_3} contains a given workspace W.

Solving the problem :

Ranges for l_i possibly large to be taken into account.

All possible uncertainties (hopefully small) to be taken into account

Certification :

For some applications (e.g. medical) there is a strong need of **certification**

$\longrightarrow$ Interval Analysis techniques

Interval Analysis

- **Interval Arithmetic**

$$
\begin{aligned}
[a,b]+[c,d] &= [a+c,b+d],\\
[a,b]-[c,d] &= [a-d,b-d],\\
[a,b]\,[c,d] &= [\min(ac,ad,bc,bd),\max(ac,ad,bc,bd)],\cdots
\end{aligned}
$$

■ Evaluation

$$f(x) = x^3 + x^2 + x, \quad \{f(x), x \in [X] = [-1, 1]\} = [-1, 3]$$

$$[X]^3 + [X]^2 + [X] = [-1, 1] + [0, 1] + [-1, 1] = [-2, 3]$$

Overestimation

$$[X]^2([X] + 1) + [X]) = [-1, 3]$$

■ Bisection Algorithm

- exclusion tests,
- Filtering procedures,
- Uniqueness tests.

■ Filtering (2B) : Example

Solve $1 + x + x^2 + x^3 = 0$ in $[-1, 1]$

$$\begin{aligned} x &= -1 - x^2 - x^3 \\ [-1,1] & \quad -1 - [-1,1]^2 - [-1,1]^3 = [-3,0], \end{aligned}$$

Necessarily $x \in [-1, 0]$

$$\begin{aligned} x &= -x^2(x+1) - 1 \\ [-1,1] & \quad -[-1,1]^2([-1,1]+1) - 1 = [-1,0], \end{aligned}$$

Necessarily $x = -1$!

- **Exclusion test**

Evaluation if $0 \notin P(X)$ then exclude X

- Example

$$
\begin{aligned}
P(x) &= (x-1)(x-2)(x-3)(x-4)(x-5) \\
&= x^5 - 15x^4 + 85x^3 - 225x^2 + 274x - 120 \\
&= -120 + x(274 + x(-225 + x(85 + x(-15 + x)))).
\end{aligned}
$$

$$
\begin{aligned}
X &= [4.9, 4.99]\\
(X^5 - 15X^4 + 85X^3 - 225X^2 + 274X - 120) &= [-855.229710, 853.060670]\\
(-120 + x(274 + x(-225 + x(85 + x(-15 + x))))) &= [-133.5775355, 132.1841260]
\end{aligned}
$$

We cannot conclude

$$
\begin{aligned}
X &= [4.99990, 4.99999]\\
(X^5 - 15X^4 + 85X^3 - 225X^2 + 274X - 120) &= [-0.879868..0.877236]\\
(-120 + x(274 + x(-225 + x(85 + x(-15 + x))))) &= [-0.1373968..0.1347577)]
\end{aligned}
$$

We cannot conclude

$\longrightarrow$ **need to use adapted exclusion tests**

$\boxed{\text{use of polynomials with 1 interval coefficient}}$

- **If the number of interval coefficient is more than 1, the complex solution set has an non empty interior set.**
- **Exclusion test always fail in the interior of solution test ($\rightarrow$ cut).**
- $\longrightarrow$ **Interior tests or Determine the boundary of solution set**

$\longrightarrow$ Edge theorem

$\boxed{\text{Edge theorem(Bartlett, Hollot, Huang, 88)}}$

It suffices to solve edge polynomials :

$$P_i(x) = \sum_{k \neq i} b_k x^k + [a_i] x^i,$$

Solution set of 3rd degree interval polynomial

$$b_k \in \{\underline{a}_i, \overline{a}_i\}$$

- 1-D uncertainty. The solution is a 0-D space. no need to have interior test.
- $d.2^d$ polynomials with one interval coef to consider.
- The solutions of a part of the previous polynomials are in the interior of the solution set.
- How to select the polynomials whith solutions on the boundary of the solution set ?

■ Edge theorem revised

Polar representation.

$$N = lcm(2, 3, \cdots, d). \; I_k = [k\frac{2\pi}{N}, (k+1)\frac{2\pi}{N}]$$

We can determine $2(d+1)$ edge polynomials necessary to determine the solution set boundary in $\mathbb{R}^+ \times I_k$.

- $2(d+1)$ edge polynomials to consider in each angular sector,
- $lcm(2, 3, \cdots, d)$ angular sectors to consider....
- $2(d+1)$ is not the minimal number of polynomials to consider. We may improve further edge theorem.

Allows to solve only low degree uncertain polynomials.

■ Edge theorem for real solutions

$$x \geq 0, \quad P(x) = [P_-^+(x), P_+^+(x)]$$

$$x \leq 0, \quad P(x) = [P_-^-(x), P_+^-(x)]$$

$$P_-^+(x) = \underline{a}_0 + \underline{a}_1 x + \underline{a}_2 x^2 \cdots + \underline{a}_d x^d,$$

$$P_+^+(x) = \overline{a}_0 + \overline{a}_1 x + \overline{a}_2 x^2 \cdots + \overline{a}_d x^d,$$

$$P_-^-(x) = \underline{a}_0 + \overline{a}_1 x + \underline{a}_2 x^2 + \overline{a}_3 x^3 \cdots ,$$

$$P_+^-(x) = \overline{a}_0 + \underline{a}_1 x + \overline{a}_2 x^2 + \underline{a}_3 x^3 \cdots ,$$

Problem to solve :

Certified real solutions of non interval polynomials :

- positive zeros of P_-^+ and P_+^+
- negative zeros of P_-^- and P_+^-.

Complex solutions for one interval coefficient polynomials

- **Exclusion tests :**

– **Weyl exclusion test :**

$$P(z) = \sum_{i=0}^{d} a_i z^i = \sum_{i=0}^{d} b_i (z - z_0)^i,$$

If $|b_0| > \sum_{i=1}^{d} |b_i| \delta^i$ then no solution in $B(z_0, \delta)$.

– **Yakoubsohn test :**

$$M_{z_0}(\delta) = |P(z_0)| - \sum_k \frac{|p^k(x)|}{k}\delta^k.$$

If $M_{z_0}(\delta) > 0$, then no solution in $B(z_0, \delta)$.

Real solutions for one interval coefficient polynomials

- **Exclusion tests:**

– Weyl
– Yakoubsohn

- **Filtering procedures**

– 2-B,
– Interval Newton algorithm

- **Uniqueness tests:**

– based on Rouche theorem,

– Krawczyk operator,

– Kantorovitch theorem.

- **Deflation**

Once a solution is known, division by $(z - z_0)$.

- **First numerical results**

Find the 19 solutions of Wilkinson 19.

Perspectives

- There is probably a room for improving Edge theorem for complex solutions
- Implementation for complex solutions,
- Consider correlated Interval coefficients.