Chapter 7

ELECTRICITY PRICES IN A GAME THEORY CONTEXT

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Abstract We consider a model of an electricity market in which $\mathcal{S}$ suppliers offer electricity: each supplier S_i offers a maximum quantity q_i at a fixed price p_i. The response of the market to these offers is the quantities bought from the suppliers. The objective of the market is to satisfy its demand at minimal price.

We investigate two cases. In the first case, each of the suppliers strives to maximize its market share on the market; in the second case each supplier strives to maximize its profit.

We show that in both cases some Nash equilibrium exists. Nevertheless a close analysis of the equilibrium for profit maximization shows that it is not realistic. This raises the difficulty to predict the behavior of a market where the suppliers are known to be mainly interested by profit maximization.

1. Introduction

Since the deregulation process of electricity exchanges has been initiated in European countries, many different market structures have appeared (*see e.g. Stoft (2002)*). Among them are the so called *day ahead markets* where suppliers face a decision process that relies on a centralized auction mechanism. It consists in submitting bids, more or less complicated, depending on the design of the *day ahead market* (power pools, power exchanges, ...). The problem is the determination of the quantity and price that will win the process of selection on the market. Our aim in this paper is to describe the behavior of the participants (suppliers) through a static game approach. We consider a market where $\mathcal{S}$ suppliers are involved. Each supplier offers on the market a maximal quantity of electricity, q, that it is ready to deliver at a fixed price p. The response of the market to these offers is the quantities bought from each supplier. The objective of the market is to satisfy its demand at minimal price.

Closely related papers are Supatchiat, Zhang and Birge (2001) and Madrigal and Quintana (2001). They also consider optimal bids on electricity markets. Nevertheless, in Supatchiat, Zhang and Birge (2001), the authors take the quantity of electricity proposed on the market as exogenous, whereas here we consider the quantity as part of the bid. In Madrigal and Quintana (2001), the authors do not consider exactly the same kind of market mechanism, in particular they consider open bids and fix the market clearing price as the highest price among the accepted bids. They consider fixed demand but also stochastic demand.

The paper is organized as follows. The model is described in Section 2, together with the proposed solution concept. In Section 3 we consider the case where the suppliers strive to maximize their market share, while in Section 4 we analyze the case where the goal is profit maximization. We conclude in Section 5 with some comparison remarks on the two criteria used, and some possible directions for future work.

2. Problem statement

2.1 The agents and their choices

We consider a single market, that has an inelastic demand for d units of electricity that is provided by $\mathcal{S}$ local suppliers called S_j, $j = 1, 2, \ldots, \mathcal{S}$.

2.1.1 The suppliers. Each supplier S_j sends an offer to the market that consists in a price function $p_j(\cdot)$, that associates to any quantity of electricity q, the unit price $p_j(q)$ at which it is ready to sell this quantity.

We shall use the following special form of the price function:

DEFINITION 7.1 *For supplier S_j, a* quantity-price *strategy, referred to as the pair (q_j, p_j), is a price function $p_j(\cdot)$ defined by*

$$p_j(q) = \begin{cases} p_j \geq 0, & \text{for } q \leq q_j, \\ +\infty, & \text{for } q > q_j. \end{cases} \tag{7.1}$$

q_j is the maximal quantity S_j offers to sell at the finite price p_j. For higher quantities the price becomes infinity.

Note that we use the same notation, (p_j) for the price function and for the fixed price. This should not cause any confusion.

2.1.2 The market. The market collects the offers made by the suppliers, i.e., the price functions $p_1(\cdot), p_2(\cdot), \dots, p_{\mathcal{S}}(\cdot)$, and has to choose the quantities $\overline{q}_j$ to buy from each supplier S_j, $j = 1, \dots, \mathcal{S}$. The unit price paid to S_j is $p_j(\overline{q}_j)$.

We suppose that an admissible choice of the market is such that the demand is fully satisfied at finite price, i.e., such that,

$$\sum_{j=1}^{\mathcal{S}} \overline{q}_j = d, \ \overline{q}_j \geq 0, \text{ and } p_j(\overline{q}_j) < +\infty, \ \forall j. \tag{7.2}$$

When the set of admissible choices is empty, i.e., when the demand cannot be satisfied at finite cost (for example when the demand is too large with respect to some finite production capacity), then the market buys the maximal quantity of electricity it can at finite price, though the full demand is not satisfied.

2.2 Evaluation functions and objective

2.2.1 The market. We suppose that the objective of the market is to choose an admissible strategy (i.e., satisfying (7.2)), $(\overline{q}_1, \dots, \overline{q}_{\mathcal{S}})$ in response to the offers $p_1(\cdot), \dots, p_{\mathcal{S}}(\cdot)$ of the suppliers, so as to minimize the total cost.

More precisely the market problem is:

$$\min_{\{\overline{q}_j\}_{j=1\cdots\mathcal{S}}} \varphi_M(p_1(\cdot), \dots, p_{\mathcal{S}}(\cdot), \overline{q}_1, \dots, \overline{q}_{\mathcal{S}}), \tag{7.3}$$

with

$$\varphi_M(p_1(\cdot), \dots, p_{\mathcal{S}}(\cdot), \overline{q}_1, \dots, \overline{q}_{\mathcal{S}}) \stackrel{\text{def}}{=} \sum_{j=1}^{\mathcal{S}} p_j(\overline{q}_j)\overline{q}_j, \tag{7.4}$$

subject to constraints (7.2).

2.2.2 The suppliers. The two criteria, profit and market share, will be studied for the suppliers:

- **The profit** — When the market buys quantities $\overline{q}_j$, $j = 1, \dots, \mathcal{S}$, supplier S_j's profit to be maximized is

  $$\varphi_{S_j}(p_1(\cdot), \dots, p_{\mathcal{S}}(\cdot), \overline{q}_j) \stackrel{\text{def}}{=} p_j(\overline{q}_j)\overline{q}_j - C_j(\overline{q}_j), \tag{7.5}$$

  where $C_j(\cdot)$ is the production cost function.

ASSUMPTION 7.1 *We suppose that, for each supplier S_j, the production cost $C_j(\cdot)$ is a piecewise $\mathcal{C}^1$ and convex function.*

When C_j is not differentiable we define the marginal cost $C'(q)$ as $\lim_{\epsilon \to 0^+} \frac{dC_j}{dq}(q - \epsilon)$.
Because of the assumption made on C_j, the marginal cost C_j' is monotonic and nondecreasing. In particular it can be a piecewise constant increasing function.

A typical special case in electricity production is when the marginal costs are piecewise constant. It corresponds to the fact that the producers starts producing in its cheapest production facility. If the market asks more electricity, the producers start up the one but cheapest production facility, etc.

- **The market share** — for supplier S_j, $\overline{q}_j$ is the quantity bought from him by the market, i.e., we define this criterion as

$$\varphi_{S_j}(p_1(\cdot), \dots, p_{\mathcal{S}}(\cdot), \overline{q}_j) \stackrel{\text{def}}{=} \overline{q}_j. \tag{7.6}$$

For this criterion, it is necessary to introduce a price constraint. As a matter of fact, the obvious, but unrealistic, solution without price constraint would be to set the price to zero whatever the quantity bought is.

We need a constraint such that, for example, the profit is non-negative, or such that the unit price is always above the marginal cost, C_j'.

For the sake of generality we suppose the existence of a minimal unit price function $\mathcal{L}_j$ for each supplier. Supplier S_j is not allowed to sell the quantity q at a unit price lower than $\mathcal{L}_j(q)$.

A natural choice for $\mathcal{L}_j$ is C_j', which expresses the usual constraint that the unit price is above the marginal cost.

2.3 Equilibria

From a game theoretical point of view, a two time step problem with $\mathcal{S} + 1$ players will be formulated. At a first time step the suppliers announce their offers (the price functions) to the market, and at the second time step the market reacts to these offers by choosing the quantities $\overline{q}_j$ of electricity to buy from each supplier. Each player strives to optimize (i.e., maximize for the suppliers, minimize for the market) his own criterion function (φ_{S_j}, $j = 1, \dots, \mathcal{S}$, φ_M) by properly choosing his own decision variable(s). The numerical outcome of each criterion function will in general depend on all decision variables involved. In contrast to conventional optimization problems, in which there is only one decision maker, and where the word "optimum" has an unambiguous meaning, the notion of "optimality" in games is open to discussion and must be defined properly. Various notions of "optimality" exist (see Başar and Olsder (1999)).

Here the structure of the problem leads us to use a combined *Nash Stackelberg* equilibrium. Please note that the "leaders", i.e., suppliers, choose and announce functions $p_j(\cdot)$. In Başar and Olsder (1999) the corresponding equilibrium is referred to as *inverse Stackelberg*.

More precisely, define $\{\overline{q}_j(p_1(\cdot),\ldots,p_{\mathcal{S}}(\cdot)), j=1,\ldots,\mathcal{S}\}$, the best response of the market to the offers $(p_1(\cdot),\ldots,p_{\mathcal{S}}(\cdot))$ of the suppliers, i.e., a solution of the problem ((7.2)-(7.3)). The choices $(\{p_j^*(\cdot)\},\{\overline{q}_j^*\},\ j=1,\ldots,\mathcal{S})$ will be said optimal if the following holds true,

$$\overline{q}_j^* \stackrel{\text{def}}{=} \overline{q}_j^*(p_1^*(\cdot),\ldots,p_{\mathcal{S}}^*(\cdot)), \tag{7.7}$$

For every supplier S_j, $j=1,\ldots,\mathcal{S}$ and any admissible price function $\tilde{p}_j(\cdot)$ we have

$$\varphi_{S_j}(p_1^*(\cdot),\ldots,p_{\mathcal{S}}^*(\cdot),\overline{q}_j^*) \geq \varphi_{S_j}(p_1^*(\cdot),\ldots,\tilde{p}_j(\cdot),\ldots,p_{\mathcal{S}}^*(\cdot),\tilde{q}_j), \tag{7.8}$$

where

$$\tilde{q}_j \stackrel{\text{def}}{=} \overline{q}_j^*(p_1^*(\cdot),\ldots,\tilde{p}_j(\cdot),\ldots,p_{\mathcal{S}}^*(\cdot)). \tag{7.9}$$

The Nash equilibrium Equation (7.8) tells us that supplier S_j cannot increase its outcome by deviating unilaterally from its equilibrium choice $(p_j^*(\cdot))$. Note that in the second term of Equation (7.8), the action of the market is given by (7.9): if S_j deviates from $p_j^*(\cdot)$ by offering the price function $\tilde{p}_j(\cdot)$, the market reacts by buying from S_j the quantity $\tilde{q}_j$ instead of $\overline{q}_j^*$.

REMARK 7.1 As already noticed the minimization problem ((7.2)–(7.3)) defining the behavior of the market may not have any solution. In that case the market reacts by buying the maximal quantity of electricity it can at finite price.

At the other extreme, it may have infinitely many solutions (for example when several suppliers use the same price function). In that case $\overline{q}_j^*(\cdot)$ is not uniquely defined by Equation (7.7), nor consequently is the Nash equilibrium defined by Equation (7.8).

We would need an additional rule that says how the market reacts when its minimization problem has several (possibly infinitely many) solutions. Such an additional rule could be, for example, that the market first buys from supplier S_1 then from supplier S_2, etc. or that the market prefers the offers with larger quantities, etc. Nevertheless, it is not necessary to make this additional rule explicit in this paper. So we do assume that there is an additional rule, known by all the suppliers that insures that the reaction of the market is unique.

3. Suppliers maximize market share

In this section we analyze the case where the suppliers strive to maximize their market shares by appropriately choosing the price functions $p_j(\cdot)$ at which they offer their electricity on the market. We restrict our attention to price functions $p_j(\cdot)$ given in Definition 7.1 and referred to as the *quantity-price* pair (q_j, p_j).

For supplier S_j we denote $\mathcal{L}_j(\cdot)$ its minimal unit price function that we suppose nondecreasing with respect to the quantity sold. Classically this minimal unit price function may represent the marginal production cost.

Using a *quantity-price* pair (q_j, p_j) for each supplier, the market problem (7.3) can be written as

$$\begin{aligned} &\min_{\{\overline{q}_j, j=1,\ldots,\mathcal{S}\}} \sum_{j=1}^{\mathcal{S}} p_j \overline{q}_j, \\ \text{under} \qquad &0 \le \overline{q}_j \le q_j, \ \sum_{j=1}^{\mathcal{S}} \overline{q}_j = d. \end{aligned}$$

To define a unique reaction of the market we use Remark 7.1, when Problem (7.9) does not have any solution (i.e., when $\sum_{j=1}^{\mathcal{S}} q_j < d$) or at the other extreme when Problem (7.9) has possibly infinitely many solutions.

Hence we can define the evaluation function of the suppliers by

$$J_{S_j}((q_1,p_1),\ldots,(q_{\mathcal{S}},p_{\mathcal{S}})) \stackrel{\text{def}}{=} \varphi_{S_j}(p_1(\cdot),\ldots,p_{\mathcal{S}}(\cdot),\overline{q}_j^*(p_1(\cdot),\ldots,p_{\mathcal{S}}(\cdot)),$$

where the price function $p_j(\cdot)$ is the pair (q_j, p_j) and $\overline{q}_j^*(p_1(\cdot),\ldots,p_{\mathcal{S}}(\cdot))$ is the unique optimal reaction of the market.

Now the Nash Stackelberg solution can be simply expressed as a Nash solution, i.e., find $\mathbf{u}^* \stackrel{\text{def}}{=} (u_1^*,\ldots,u_{\mathcal{S}}^*)$, $u_j^* \stackrel{\text{def}}{=} (q_j^*, p_j^*)$, so that for any supplier S_j and any pair $\tilde{u}_j = (\tilde{q}_j, \tilde{p}_j)$ we have

$$J_{S_j}(\mathbf{u}^*) \ge J_{S_j}(\mathbf{u}_{-j}^*, \tilde{u}_j), \tag{7.10}$$

where $(\mathbf{u}_{-j}^*, \tilde{u}_j)$ denotes the vector $(u_1^*,\ldots,u_{j-1}^*,\tilde{u}_j,u_{j+1}^*,\ldots,u_{\mathcal{S}}^*)$.

Assumption 7.2 *We suppose that there exist quantities Q_j for $j = 1, \ldots, \mathcal{S}$, such that*

$$\sum_{j=1}^{\mathcal{S}} Q_j \geq d, \tag{7.11}$$

and such that the minimal price functions $\mathcal{L}_j$ are defined for the set $[0, Q_j]$ to $\mathbb{R}^+$, with finite values for any q in $[0, Q_j]$. The quantities Q_j represent the maximal quantities of electricity supplier S_j can offer to the market. It may reflect maximal production capacity for producers or more generally any other constraints such that transportation constraints.

Remark 7.2 The condition (7.11) insures that shortage can be avoided even if this implies high, but finite, prices.

We consider successively in the next subsections the cases where the minimal price functions $\mathcal{L}_j$ are continuous (Subsection 3.1) or discontinuous (Subsection 3.2). This last case is the most important from the application point of view, since we often take $\mathcal{L}_j = C_j'$ which is not in general continuous.

3.1 Continuous strictly increasing minimal price

We suppose the following assumption holds,

Assumption 7.3 *For any supplier S_j, $j \in \{1, \ldots, \mathcal{S}\}$ the minimal price function $\mathcal{L}_j$ is continuous and strictly increasing from $[0, Q_j]$ to $\mathbb{R}^+$.*

Proposition 7.1

1. *Suppose that Assumption 7.3 holds. Then any strategy profile $\mathbf{u}^* = (u_1^*, u_2^*, \ldots, u_{\mathcal{S}}^*)$ with $u_j^* = (q_j^*, p^*)$ such that*

$$\begin{array}{l} \mathcal{L}_j(q_j^*) = p^*, \ \ \forall j \in \{1, \ldots, \mathcal{S}\} \text{ such that } q_j^* > 0 \\ \mathcal{L}_j(0) \geq p^*, \ \ \forall j \in \{1, \ldots, \mathcal{S}\} \text{ such that } q_j^* = 0 \\ \sum_{j \in \{1,2 \cdots \mathcal{S}\}} q_j^* = d, \end{array} \tag{7.12}$$

   is a Nash equilibrium.

2. *Suppose furthermore that Assumption 7.2 holds, then the equilibrium exists and is unique.*

We omit the proof of this proposition which is in the same vein as the proof of the following Proposition 7.2. Nevertheless it can be found in Bossy et al. (2004).

3.2 Discontinuous nondecreasing minimal price

We now address the problem where the minimal price functions $\mathcal{L}_j$ are not necessarily continuous and not necessarily strictly increasing. Nevertheless we assume that they are non decreasing. We set the following assumption,

ASSUMPTION 7.4 *We suppose that the minimal price functions $\mathcal{L}_j$ are nondecreasing, piecewise continuous, and that $\lim\limits_{y \to x^-} \mathcal{L}_j(y) = \mathcal{L}(x)$ for any $x \geq 0$.*

Replacing Assumption 7.3 by Assumption 7.4, there may not be any strategy profile or at the other extreme there may be possibly infinitely many strategy profiles that satisfy Equations (7.12). Proposition 7.1 fail to characterize the Nash equilibria.

For any $p \geq 0$, we define $\rho_j(p)$, the maximal quantity supplier S_j can offer at price p, i.e.,

$$\rho_j(p) = \begin{cases} \max\{q \geq 0, \mathcal{L}_j(q) \leq p\}, \text{ if } j \text{ is such that } \mathcal{L}_j(0) \leq p, \\ 0, \text{ otherwise } . \end{cases} \tag{7.13}$$

Hence $\rho_j(p)$ is only determined by the structure of the minimal price function $\mathcal{L}_j$. In particular it is not dependent on any choice of the suppliers.

As a consequence of Assumption 7.4, $\rho_j(p)$ increases with p, and for any $p \geq 0$, $\lim_{y \to p^+} \rho_j(y) = \rho_j(p)$.

Denote by $O(\cdot)$ the function from $\mathbb{R}^+$ to $\mathbb{R}^+$ defined by

$$O(p) = \sum_{j=1}^{\mathcal{S}} \rho_j(p). \tag{7.14}$$

$O(p)$ is the maximal total offer that can be achieved at price p by the suppliers respecting the price constraints. The function O is possibly discontinuous, non decreasing (but not necessarily strictly increasing) and satisfies $\lim_{y \to p^+} O(y) = O(p)$. Assumption 7.2 implies that

$$O(\sup_j \mathcal{L}_j(Q_j)) \;\geq\; \sum_{j=1}^{\mathcal{S}} \rho_j(\mathcal{L}_j(Q_j)) \;\geq\; \sum_{j=1}^{\mathcal{S}} Q_j \;\geq\; d,$$

hence there exists a unique $p^* \leq \sup_j \mathcal{L}_j(Q_j) < +\infty$ such that

$$\begin{aligned} &O(p^*) = \sum_{j=1}^{\mathcal{S}} \rho_j(p^*) \geq d, \\ &\forall \epsilon > 0, \quad O(p^* - \epsilon) < d. \end{aligned} \tag{7.15}$$

The price p^* represents the minimal price at which the demand could be fully satisfied taking into account the minimal price constraint.

ASSUMPTION 7.5 *For p^* defined by (7.15), one of the following two condition holds:*

1. *We suppose that there exists a unique $\bar{j} \in \{1, \ldots, \mathcal{S}\}$ such that $\mathcal{L}_{\bar{j}}^{-1}(p^*) \neq \emptyset$, where $\mathcal{L}_j^{-1}(p) \stackrel{\text{def}}{=} \{q \in [0, d],\ \ \mathcal{L}_j(q) = p\}$. In particular, there exists a unique $\bar{j} \in \{1, \ldots, \mathcal{S}\}$ such that $\mathcal{L}_{\bar{j}}(\rho_{\bar{j}}(p^*)) = p^*$, and such that for $j \neq \bar{j}$, we have $\mathcal{L}_j(\rho_j(p^*)) < p^*$.*

2. *At price p^* the maximal total quantity suppliers are allowed to propose is exactly d, i.e., $\sum_{j=1}^{\mathcal{S}} \rho_j(p^*) = d$.*

PROPOSITION 7.2 *Suppose Assumptions 7.4 and 7.5 hold. Consider the strategy profile $\mathbf{u}^* = (u_1^*, \ldots, u_{\mathcal{S}}^*)$, $u_j^* = (q_j^*, p^*)$ such that,*

- *p^* is defined by Equation (7.15),*
- *for $j \neq \bar{j}$,* i.e. *such that $\mathcal{L}_j(\rho_j(p^*)) < p^*$ (see Assumption 7.5), we have $q_j^* = \rho_j(p^*)$ and $p_j^* \in [\mathcal{L}_j(q_j^*), p^*[$,*
- *for $j = \bar{j}$,* i.e. *such that $\mathcal{L}_{\bar{j}}(\rho_{\bar{j}}(p^*)) = p^*$ (see Assumption 7.5), we have $q_{\bar{j}}^* \in [\min((d - \sum_{k \neq \bar{j}} q_k^*)\ ,\ \rho_{\bar{j}}(p^*))\ ,\ \rho_{\bar{j}}(p^*)]$, and $p_{\bar{j}}^* \in [p^*, \overline{p}[$, where $\overline{p}$ is defined by*

$$\overline{p} \stackrel{\text{def}}{=} \min\{\mathcal{L}_k(q_k^{*+}),\ k \neq \bar{j}\} \tag{7.16}$$

then, $\mathbf{u}^*$ is a Nash equilibrium.

REMARK 7.3 There exists an infinite number of strategy profiles that satisfy the conditions of Proposition 7.2 (the prices p_j^* are defined as elements of some intervals). Nevertheless, we can observe that there is no need for any coordination among the suppliers to get a Nash equilibrium. Each supplier can choose independently a strategy as described in Proposition 7.2, the resulting strategy profile is a Nash equilibrium. Note that this property does not hold in general for non-zero sum games (see the classical "battle of the sexes" game Luce and Raifa (1957)). We can also observe that for each supplier the outcome is the same whatever the Nash equilibrium set. In that sense we can say that all these Nash equilibria are equivalent.

A reasonable manner to select a particular Nash equilibrium is to suppose that the suppliers may strive for the maximization of their profits as an auxiliary criteria. More precisely, among the equilibria with market share maximization as criteria, they choose the equilibrium that brings them the maximal income. Because the equilibria we have found are independent, it is possible for each supplier to choose its preferred equilibrium. More precisely, with this auxiliary criterion, the equilibrium selected will be,

$$\begin{aligned} &q_j^* = \rho_j(p^*),\ p_j^* = p^* - \epsilon, \quad \text{for } j \neq \bar{j} \text{ (i.e. such that } \mathcal{L}_j(\rho_j(p^*)) < p^*), \\ &q_{\bar{j}}^* = \rho_{\bar{j}}(p^*),\ p_{\bar{j}}^* = \overline{p} - \epsilon, \end{aligned}$$

where ϵ can be defined as the smallest monetary unit.

Remark 7.4 Assumption 7.5 is necessary for the solution of the market problem (7.9) to have a unique solution for the strategies described in Proposition 7.2, which are consequently well defined.

If Assumption 7.5 does not hold, we would need to make the additional decision rule of the market explicit (see Remark 7.1). This is shown in the following example (Figure 7.1), with $\mathcal{S} = 2$. The Nash equilibrium may depend upon the additional decision rule of the market. In Figure 7.1, we have $\mathcal{L}_1(\rho_1(p^*)) = \mathcal{L}_2(\rho_2(p^*)) = p^*$ and $\rho_1(p^*) + \rho_2(p^*) > d$, where p^* is the price defined at (7.15). This means that Assumption 7.5 does not hold. Suppose the additional decision rule of the market

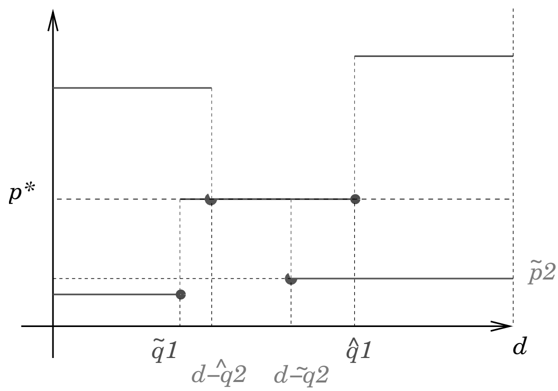

Figure 7.1. Example

is to give the preference to supplier S_1, i.e., for a pair of strategies $((q_1, p), (q_2, p))$ such that $q_1 + q_2 > d$ the market reacts by buying the respective quantities q_1 and $d - q_1$ respectively to supplier S_1 and to supplier S_2. The Nash equilibria for market share maximization are,

$$u_1^* = (q_1^* \in [d - \tilde{q}_2, \hat{q}_1], p^*), \quad u_2^* = (\tilde{q}_2, p_2^* \in [\tilde{p}_2, p^*[),$$

where $\hat{q}_i = \rho_i(p^*)$, $\tilde{q}_i = \rho_i(p^* - \epsilon)$, and $\tilde{p}_2 = \mathcal{L}_2(\tilde{q}_2)$.

Suppose now the additional decision rule of the market is a preference for supplier S_2. The previous pair of strategies is not a Nash equilibrium any more. Indeed, supplier S_2 can increase its offer, at price p^*, to the quantity $\hat{q}_2$. The equilibrium in that case is

$$u_1^* = (q_1^* \in [d - \hat{q}_2, \hat{q}_1], p^*), \quad u_2^* = (\hat{q}_2, p^*).$$

REMARK 7.5 In Proposition 7.2 we see that at equilibrium, the maximal price $\bar{p}$ that can be proposed is given by (7.16). A sufficient condition for that price to be finite is that for any $j \in \{1, 2, \ldots, \mathcal{S}\}$ we have,

$$\sum_{k \neq j} Q_k > d. \tag{7.17}$$

Equation (7.17) means that with the withdrawal of an individual supplier, the demand can still be satisfied. This will insure that none of the suppliers can create a fictive shortage and then increase unlimitedly the price of electricity.

Proof of Proposition 7.2. We have to prove that for supplier S_j there is no profitable deviation of strategy, *i.e.* for any $u_j \neq u_j^*$, we have $J_{P_j}(\mathbf{u}_{-j}^*, u_j^*) \geq J_{P_j}(\mathbf{u}_{-j}^*, u_j)$.

- Suppose first that $j \notin \mathcal{S}(p^*)$ so that $\mathcal{L}_j(\rho_j(p^*)) < p^*$. Since for the proposed Nash strategy $u_j^* = (q_j^*, p_j^*)$, we have $p_j^* < p^*$, the total quantity proposed by S_j is bought by the market ($\overline{q}_j = q_j^*$). Hence $J_j(\mathbf{u}^*) = q_j^*$.
  - If the deviation $u_j = (q_j, p_j)$ is such that $q_j \leq q_j^*$, then clearly $J_{P_j}(\mathbf{u}^*) = q_j^* \geq q_j \geq J_{P_j}(\mathbf{u}_{-j}^*, u_j)$, whatever the price p_j is.
  - If the deviation $u_j = (q_j, p_j)$ is such that $q_j > \rho_j(p^*)$ then necessarily, by the minimal price constraint, Assumption 7.5 and the definition of $q_j^* = \rho_j(p^*)$, we have

    $$p_j \geq \mathcal{L}_j(q_j) \geq \mathcal{L}_j(q_j^{*\,+}) > \sup_{k \neq j} p_k^*.$$

    Hence now supplier S_j is the supplier with the highest price. Consequently the market first buys from the other suppliers and satisfies the demand, when necessary, with the electricity produced by supplier S_j (instead of the supplier $S_{\bar{j}}$). Hence the market share of S_j cannot increase with this deviation.
- Suppose now that $j = \bar{j}$, i.e., we have, $\mathcal{L}_{\bar{j}}(\rho_{\bar{j}}(p^*)) = p^*$.

– If the first item of Assumption 7.5 holds, then at the proposed Nash equilibrium, supplier $S_{\bar{j}}$ is the supplier that meets the demand since it proposes the highest price.

  Hence if supplier $S_{\bar{j}}$ wants to increases its market share, it has to sell a quantity $\tilde{q}_{\bar{j}} \geq d - \sum_{k \neq \bar{j}} q_k^*$. But we have,

$$\mathcal{L}_{\bar{j}}(\tilde{q}_{\bar{j}}) \geq \mathcal{L}_{\bar{j}}(d - \sum_{k \neq \bar{j}} q_k^*) = p^* > \max_{k \neq \bar{j}} p_k^*.$$

  This proves that the quantity $\tilde{q}_{\bar{j}}$ cannot be offered at a price such that the market would buy it.

– If the second item of Assumption 7.5 holds, then the proposition states that the quantity proposed, and bought by the market is $\rho_{\bar{j}}$. An increase in the quantity proposed would imply a higher price, which would not imply a higher quantity proposed by the market since now the supplier would have the highest price.

□

Now we suppose that Assumption 7.5 does not hold. So for the price p^* defined by (7.15) we have more than one supplier S_j such that $\mathcal{L}_j(\rho_j(p^*)) = p^*$.

As shown in the example of Remark 7.4 (see Figure 7.1), the Nash equilibria may depend upon the reaction of the market when two suppliers, S_i and S_j, have the same price $p_i = p_j = p^*$. It is clear that for a supplier S_j in such a way that $\mathcal{L}_j(\rho_j(p^*)) = p^*$, two possibilities may occur at equilibrium. Either, for some supplier S_j that fixes its price to $p_j = p^*$, the market reacts in such a way that $\overline{q}_j < \rho_j(p^* - \epsilon)$, in which case at equilibrium we will have $p_j^* = p^* - \epsilon$, or the market reacts such that $\overline{q}_j \geq \rho_j(p^* - \epsilon)$, and in that case we will have $p_j^* = p^*$.

Although the existence of Nash equilibria seems clear for any possible reaction of the market, we restrict our attention to the case where the market reacts by choosing quantities $(\overline{q}_j)_{j=1,\ldots \mathcal{S}}$ that are monotonically nondecreasing with respect to the quantity q_j proposed by each supplier S_j. More precisely we have the following assumption,

ASSUMPTION 7.6 *Let* $\mathbf{u} = (\mathbf{u_1}, \ldots, \mathbf{u}_{\mathcal{S}})$ *be a strategy profile of the suppliers with* $u_i = (q_i, p)$ *for* $i \in \{1, \ldots, \mathtt{k}\}$. *Suppose the market has to use its additional rule to decide how to share the quantity* $\tilde{d} \leq d$ *among suppliers* S_1 *to* $S_{\mathtt{k}}$ *(the quantity* $d - \tilde{d}$ *has already been bought from suppliers with a price lower than* p*).*

Let $i \in \{1, \ldots, \mathtt{k}\}$, *we define the function that associates* $\overline{q}_i$ *to any* $q_i \geq 0$, *where* $\overline{q}_i$ *is the i-th component of the reaction* $(\overline{q}_1, \ldots, \overline{q}_{\mathcal{S}})$ *of the* *market to the strategy profile* $(\mathbf{u}_{-i}, (q_i, p))$. *We suppose that this function is not decreasing with respect to the quantity* q_i.

The meaning of this assumption is that the market does not penalize an "over-offer" of a supplier. For fixed strategies $\mathbf{u}_{-i}$ of all the suppliers but S_i, if supplier S_i, such that $p_i = p$ increases its quantity q_i, then the quantity bought by the market from S_i cannot decrease. It can increase or stay constant. In particular, it encompasses the case where the market has a preference order between the suppliers (for example, it first buys from supplier S_{j_1}, then supplier S_{j_2} etc), or when the market buys some fixed proportion from each supplier. It does not encompass the case where the market prefers the smallest offer.

PROPOSITION 7.3 *Suppose Assumption 7.5 does not hold while Assumption 7.6 does. Let the strategy profile* $((q_1^*, p_1^*), \ldots, (q_{\mathcal{S}}^*, p_{\mathcal{S}}^*))$ *be defined by*

- *If* S_j *is such that* $\mathcal{L}_j(\rho_j(p^*)) < p^*$ *then*

$$p_j^* = p^* - \epsilon, \quad q_j^* = \rho_j(p^* - \epsilon).$$

- *If* S_j *is such that* $\mathcal{L}_j(\rho_j(p^*)) = p^*$, *then either*

$$p_j^* = p^*, \quad q_j^* = \rho_j(p^*), \tag{7.18}$$

when the reaction of the market is such that

$$\overline{q}_j \geq \rho_j(p^* - \epsilon), \quad \forall \epsilon > 0,$$

or

$$p_j^* = p^* - \epsilon, \quad q_j^* = \rho_j(p^* - \epsilon). \tag{7.19}$$

when the new reaction of the market, for a deviation $p_j = p^*$ *would be such that* $\overline{q}_j < \rho_j(p^* - \epsilon)$.

This strategy profile is a Nash equilibrium.

Proof. The proof follows directly from the discussion made before the proposition, and from the proof of Proposition 7.2. □

Example. We consider a market with 5 suppliers and a demand d equal to 10. We suppose that the minimal price functions $\mathcal{L}_j$ of suppliers are increasing staircase functions, given in the following table (the notation $(]a, b]; c)$ indicates that the value of the function in the interval $]a, b]$ is c),

supplier 1	$([0,1];10), (]1,3];15), (]3,4];25), (]4,10];50)$
supplier 2	$([0,5];20), (]5,6];23), (]6,7];40), (]7,10];70)$
supplier 3	$([0,2];15), (]2,6];25), (]6,7];30), (]7,10];50)$
supplier 4	$([0,1];10), (]1,4];15), (]4,5];20), (]5,10];50)$
supplier 5	$([0,4];30), (]4,8];90), (]8,10];100)$

We display in the following table the values for $\rho_j(p)$ and $O(p)$ respectively defined by equations (7.13) and (7.14).

p	$\rho_1(p)$	$\rho_2(p)$	$\rho_3(p)$	$\rho_4(p)$	$\rho_5(p)$	$O(p)$
$p \in [0,10[$	0	0	0	0	0	0
$p \in [10,15[$	1	0	0	1	0	2
$p \in [15,20[$	3	0	2	4	0	9
$p \in [20,23[$	3	5	2	5	0	15

The previous table shows that for a price p in $[15,20[$, only suppliers S_1, S_3 and S_4 can bring some positive quantity of electricity. The total maximal quantity that can be provided is 9 which is strictly lower than the demand $d = 10$. For a price in $[20,23[$, we see that supplier S_2 can also bring some positive quantity of electricity, the total maximal quantity is then 15 which is higher than the demand. Then we conclude that the price p^* defined by Equation (7.15) is $p^* = 20$. Moreover, $\mathcal{L}_2(\rho_2(p^*)) = \mathcal{L}_4(\rho_4(p^*)) = p^*$ which means that Assumption 7.5 is not satisfied. Notice that for supplier S_5, we have $\mathcal{L}_5(0) = 30 > p^*$. Supplier S_5 will not be able to sell anything to the market, hence, whatever its bid is, we have $\overline{q}_5 = 0$. We suppose that Assumption 7.6 holds. According to Proposition 7.3, we have the following equilibria.

$$u_1^* = (3, p_1^* \in [15,20[), u_3^* = (2, p_3^* \in [15,20[) \text{ and } u_5^* = (p^*, q^*),\ p^* \geq \mathcal{L}_5(q)$$

to which the market reacts by buying the respective quantities $\overline{q}_1(\mathbf{u}^*) = 3$, $\overline{q}_3(\mathbf{u}^*) = 2$ and $\overline{q}_5(\mathbf{u}^*) = 0$. The quantity 5 remains to be shared between S_2 and S_4 according to the additional rule of the market. For example, suppose that the market prefers S_2 to all other suppliers. Then

$$u_2^* = (q_2^* \in [1,5], p_2^* = 20) \text{ and } u_4^* = (4, p_4^* \in [15,20[).$$

to which the market reacts by buying $\overline{q}_2(\mathbf{u}^*) = 1$ and $\overline{q}_4(\mathbf{u}^*) = 4$. If now the market prefers S_4 to any other, then

$$u_2^* = (q_2^* \in [1,5], p_2^* = 20) \text{ and } u_4^* = (5, p_4^* = 20).$$

to which the market reacts by buying $\overline{q}_2(\mathbf{u}^*) = 0$ and $\overline{q}_4(\mathbf{u}^*) = 5$.

4. Suppliers maximize profit

In this section, the objective of the suppliers is to maximize their profit, *i.e.* for a strategy profile $\mathbf{u} = (u_1, \ldots, u_{\mathcal{S}})$, $u_j = (q_j, p_j)$, their evaluation functions are

$$J_{S_j}(\mathbf{u}) = p_j \overline{q}_j - C_j(\overline{q}_j), \tag{7.20}$$

where $C_j(\cdot)$ denotes supplier S_j's production cost function, and $\overline{q}_j$ is the optimal reaction of the market, *i.e.* the solution of Problem (7.9) together with an additional decision rule, known by all the suppliers, in case of nonunique solutions (see Remark 7.1). As before, we do not need to make this rule explicit.

In contrast to the market share maximization, we do not need a minimal price functions $\mathcal{L}_j$. Nevertheless we need a maximal unit price $p_{\max}$ under which the suppliers are allowed to sell their electricity. This maximal price can either be finite and fixed by the market or be infinite.

From all the assumptions previously made, we only retain in this section Assumption 7.1.

LEMMA 7.1 *We define, for any finite price $p \geq 0$, $\widehat{Q}_j(p)$ as the set of quantities that maximizes the quantity $qp - C_j(q)$, i.e.,*

$$\widehat{Q}_j(p) \stackrel{\text{def}}{=} \arg \max_{q \in [0,d]} qp - C_j(q),$$

and for infinite price,

$$\widehat{Q}_j(+\infty) = \min(Q_j, d),$$

where Q_j is the maximal production capacity of S_j.

We have for finite p,

$$\begin{aligned} &\widehat{Q}_j(p) = \{0\} \text{ if } C'(0) > p, \\ &\widehat{Q}_j(p) = \{d\} \text{ if } C'(d) < p, \\ &\widehat{Q}_j(p) = \{q, \;\; C_j'(q^-) \leq p \leq C_j'(q^+)\}, \text{ otherwise.} \end{aligned} \tag{7.21}$$

Proof. We prove the last equality of (7.21). For any $q \in \widehat{Q}_j(p)$, we have for any $\epsilon > 0$,

$$pq - C_j(q) \geq p(q+\epsilon) - C_j(q+\epsilon),$$

from which we deduce that

$$\frac{C_j(q+\epsilon) - C_j(q)}{\epsilon} \geq p,$$

and letting ϵ tends to zero, it follows that $C_j'(q^+) \geq p$. The other equality is obtained with negative ϵ.

The first two equalities of (7.21) follow directly from the fact that C' is supposed to be non decreasing. $\square$

Note that if $C'(\cdot)$ is a continuous and non decreasing function, then (7.21) is equivalent to the classical first order condition for the evaluation function of the supplier.

LEMMA 7.2 *The function* $p \to \max_{q\in[0,d]}(qp - C_i(q))$ *is continuous and strictly increasing.*

Proof. We recognize the Legendre-Fenchel transform of the convex function C_j. The continuity follows from classical properties of this transform.

The function is strictly increasing, since for $p > p'$, if we denote by $\tilde{q}$ a quantity in $\arg\max_{q\in[0,d]}(qp' - C_i(q))$, we have

$$\begin{aligned}\max_{q\in[0,d]}(qp - C_i(q)) &\geq \tilde{q}p - C_i(\tilde{q}) \\ &> \tilde{q}p' - C_i(\tilde{q}) = \max_{q\in[0,d]}(qp' - C_i(q)).\end{aligned}$$

$\square$

We now restrict our attention to the two suppliers' case, *i.e.* $\mathcal{S} = 2$.

Our aim is to determine the Nash equilibrium if such an equilibrium exists. Hence we need to find a pair $((q_1^*, p_1^*), (q_2^*, p_2^*))$ such that (q_1^*, p_1^*) is the best strategy of supplier S_1 if supplier S_2 chooses (q_2^*, p_2^*), and conversely, (q_2^*, p_2^*) is the best strategy of supplier S_2 if supplier S_1 chooses (q_1^*, p_1^*). Equivalently we need to find a pair $((q_1^*, p_1^*), (q_2^*, p_2^*))$ such that there is no profitable deviation for any supplier S_i, $i = 1, 2$.

Let us determine the conditions which a pair $((q_1^*, p_1^*), (q_2^*, p_2^*))$ must satisfy in order to be a Nash equilibrium, i.e., no profitable deviation exists for any supplier. We will successively examine the case where we have an excess demand $(q_1^* + q_2^* \leq d)$ and the case where we have an excess supply $(q_1^* + q_2^* > d)$.

Excess demand: $q_1^* + q_2^* \leq d$**.** In that case the market buys all the quantities proposed by the suppliers, i.e., $\overline{q}_i^* = q_i^*$, $i = 1, 2$.

1. Suppose that for at least one supplier, say supplier S_1, we have $p_1^* < p_{\max}$. Then supplier S_1 can increase its profit by increasing its price to $p_{\max}$. Since $q_1^* + q_2^* \leq d$ the reaction of the market to the new pair of strategies $((q_1^*, p_{\max}), (q_2^*, p_2^*))$ is still q_1^*, q_2^*. Hence the new profit of S_1 is now $q_1^* p_{\max} - C_1(q_1^*) > q_1^* p_1^* - C_1(q_1^*)$.

We have exhibited a profitable deviation, $(q_1^*, p_{\max})$ for supplier S_1. This proves that a pair of strategies such that $q_1^* + q_2^* \leq d$ with at least one price $p_i^* < p_{\max}$ cannot be a Nash equilibrium.

2. Suppose that $p_1^* = p_2^* = p_{\max}$, and that there exists at least one supplier, say supplier S_1, such that $q_1^* = \overline{q}_1^* \notin \widehat{Q}_1(p_{\max})$, i.e., such that the reaction of the market does not maximize S_1's profit (see Lemma 7.1). Consequently, the profit for S_1, associated with the pair $((q_1^*, p_{\max}), (q_2^*, p_{\max}))$ is such that

$$\overline{q}_1^* p_{\max} - C_1(\overline{q}_1^*) < \max_{q \in [0,d]} (q p_{\max} - C_1(q)).$$

Since

$$\lim_{\epsilon \to 0^+} \max_{q \in [0,d]} (q(p_{\max} - \epsilon) - C_1(q)) = \max_{q \in [0,d]} (q p_{\max} - C_1(q)),$$

there exists some $\bar{\epsilon} > 0$ such that

$$\max_{q \in [0,d]} (q(p_{\max} - \bar{\epsilon}) - C_1(q)) > \overline{q}_1^* p_{\max} - C_1(\overline{q}_1^*).$$

This proves that any deviation $(\hat{q}_1, p_{\max} - \bar{\epsilon})$ of supplier S_1, such that $\hat{q}_1 \in \widehat{Q}_1(p_{\max} - \bar{\epsilon})$, is profitable for S_1.

Hence, a pair of strategies such that $q_1^* + q_2^* \leq d$, $p_1^* = p_2^* = p^*$, to which the market reacts with, for at least one supplier, a quantity $\overline{q}_i^* \notin \hat{Q}_i(p_{\max})$ cannot be a Nash equilibrium.

3. Suppose that $p_1^* = p_2^* = p_{\max}$, $q_1^* = \overline{q}_1^* \in \hat{Q}_1(p_{\max})$ and $q_2^* = \overline{q}_2^* \in \hat{Q}_2(p_{\max})$ (i.e., the market reacts optimally for both suppliers).

In that case the pair $((q_1^*, p_1^*), (q_2^*, p_2^*))$ is a Nash equilibrium. As a matter of fact no deviation by changing the quantity can be profitable: since $\overline{q}_i^*$ is optimal for $p_{\max}$, the price cannot be increased, and a decrease of the profit will follow from a decrease of the price of one supplier (Lemma 7.2).

Excess supply: $q_1^* + q_2^* > d$. Two possibilities occur depending on whether the prices $p_j,\ j = 1, 2$ differ or not.

1. **The prices are different**, i.e., $p_1^* < p_2^*$ for example.

In that case the market first buys from the supplier with lower price (hence $\overline{q}_1^* = \inf(q_1^*, d)$), and then completes its demand to the supplier with highest price, S_2 (hence $\overline{q}_2^* = d - \overline{q}_1^*$).

For $\bar{\epsilon} > 0$ such that $p_1^* + \bar{\epsilon} < p_2^*$, we have

$$q_1^* p_1^* - C_1(q_1^*) \leq \max_{q \in [0,d]} \{qp_1^* - C_1(q)\} < \max_{q \in [0,d]} \{q(p_1^* + \bar{\epsilon}) - C_1(q)\}.$$

Hence supplier S_1 is better off increasing its price to $p_1^* + \bar{\epsilon}$ and proposing quantity $\hat{q}_1 \in \widehat{Q}_1(p_1^* + \bar{\epsilon})$. As a matter of fact, since $p_1^* + \bar{\epsilon} < p_2^*$, the reaction of the market will be $\overline{q}_1 = \hat{q}_1$.

So a pair of strategies with $p_1 \neq p_2$ cannot be a Nash equilibrium.

2. **The prices are equal**, i.e., $p_1^* = p_2^* \stackrel{\text{def}}{=} p^*$.

   Now the market faces an optimization problem (7.9) with several solutions. Hence it has to use its additional rule in order to determine the quantities $\overline{q}_1^*, \overline{q}_2^*$ to buy from each supplier in response to their offers q_1^*, q_2^*.

   If this response is not optimal for any supplier, i.e., $\overline{q}_1^* \notin \widehat{Q}_1(p^*)$ and $\overline{q}_2^* \notin \widehat{Q}_2(p^*)$, the same line of reasoning as in Item 2 of the excess demand case proves that the pair $((q_1^*, p_1^*), (q_2^*, p_2^*))$ cannot be a Nash equilibrium.

   Suppose that the reaction of the market is optimal for both suppliers, i.e., $\overline{q}_1^* \in \widehat{Q}_1(p^*)$ and $\overline{q}_2^* \in \widehat{Q}_2(p^*)$. A necessary condition for a supplier, say S_1, to increase its profit is to increase its price and consequently to complete the offer of the other supplier S_2. We have two possibilities,

   (a) If for at least one supplier, say supplier S_1, we have,

   $$(d - q_2^*)p_{\max} - C_1(d - q_2^*) > \max_{q \in [0,d]} \{qp^* - C_1(q)\},$$

   then supplier S_1 is better off increasing its price to $p_{\max}$ and completing the market to sell the quantity $d - q_2^*$.

   (b) Conversely, if none of the suppliers can increase its profit by "completing the offer of the other", i.e., if

   $$(d - q_2^*)p_{\max} - C_1(d - q_2^*) \leq \max_{q \in [0,d]} \{qp^* - C_1(q)\}, \quad (7.22)$$

   $$(d - q_1^*)p_{\max} - C_2(d - q_1^*) \leq \max_{q \in [0,d]} \{qp^* - C_2(q)\}, \quad (7.23)$$

   then the pair $((q_1^*, p_1^*), (q_2^*, p_2^*))$ is a Nash equilibrium.

   As a matter of fact, for one supplier, say S_1, changing only the quantity is not profitable since $q_1^* \in \widehat{Q}_1$, decreasing the price is not profitable because of Lemma 7.2. Inequality (7.22) prevents S_1 from increasing its price.

REMARK 7.6 Note that a sufficient condition for Inequality (7.22) and (7.23) to be true, is that both suppliers choose $q_j^* = d,\ j = 1, 2$.

With this choice each supplier prevents the other supplier from completing its demand with maximal price. This can be interpreted as a wish for the suppliers to obtain a Nash equilibrium. Nevertheless, to do that, the suppliers have to propose to the market a quantity d at price p^* which may be very risky and hence may not be credible. As a matter of fact, suppose S_1 chooses the strategy $q_1 = d, p_1 = p^* < p_{\max}$. If for some reason supplier S_2 proposes a quantity q_2 at a price $p_2 > p_1$, then S_1 has to provide the market with the quantity d at price p_1 since $\overline{q}_1 = d$, which may be disastrous for S_1.

Note also that if $p_{\max}$ is not very high compared with p^* the inequalities (7.22) and (7.23) will not be satisfied. Hence these inequalities could be used for the market to choose a maximal price $p_{\max}$ such that the equilibrium may be possible.

The previous discussion shows that in case of excess supply, the only possibility to have a Nash equilibrium, is that both suppliers propose the same price p^* and quantities $q_1^*,\ q_2^*$, such that, together with an additional rule, the market can choose optimal quantities $\overline{q}_1^*$, $\overline{q}_2^*$ that satisfy its demand and such that $\overline{q}_j^* \in \widehat{Q}_j(p^*)$.

This is clearly not possible for any price p^*. If the price p^* is too small, then the optimal quantity the suppliers can bring to the market is small, and for any $q \in \widehat{Q}_1(p^*) + \widehat{Q}_2(p^*)$, we have $q < d$. If the price p^* is too high, then the optimal quantity the suppliers are willing to bring to the market are large, and for any $q \in \widehat{Q}_1(p) + \widehat{Q}_2(p)$, we have $q > d$. The following Lemma characterizes the possible values of p^* for which it is possible to find q_1 and q_2 that satisfy $q_1 \in \widehat{Q}_1(p^*)$, $q_2 \in \widehat{Q}_2(p^*)$ and $q_1 + q_2 = d$.

Let us first define the function $\mathcal{C}'_j$ from $[0, d]$ to the set of intervals of $\mathbb{R}^+$ as

$$\mathcal{C}'_j(q) = [C'_j(q^-), C'_j(q^+)],$$

for q smaller than the maximal production quantity Q_j, and $\mathcal{C}_j(q) = \emptyset$ for $q > Q_j$. Clearly $\mathcal{C}'_j(q) = \{C'_j(q)\}$ except when C'_j has a discontinuity in q. We now can state the lemma.

LEMMA 7.3 *It is possible to find* q_1, q_2 *such that* $q_1 + q_2 = d$, $q_1 \in \widehat{Q}_1(p)$ *and* $q_2 \in \widehat{Q}_2(p)$ *if and only if*

$$p \in \mathcal{I},$$

where,

$$\mathcal{I} \stackrel{\text{def}}{=} \bigcup_{q\in[0,d]} (\mathcal{C}_1'(q) \cap \mathcal{C}_2'(d-q)),$$

or, equivalently, when $Q_1 + Q_2 \geq d$, $\mathcal{I} \stackrel{\text{def}}{=} [\mathcal{I}^-, \mathcal{I}^+]$, *where*

$$\mathcal{I}^- \stackrel{\text{def}}{=} \min\{p, \ \max(q \in \widehat{Q}_1(p)) + \max(q \in \widehat{Q}_2(p)) \geq d\},$$
$$\mathcal{I}^+ \stackrel{\text{def}}{=} \max\{p, \ \min(q \in \widehat{Q}_1(p)) + \min(q \in \widehat{Q}_2(p)) \leq d\},$$

and $\mathcal{I} = \emptyset$ *when* $Q_1 + Q_2 < d$.

Proof. If $p \in \mathcal{I}$, then there exists $q \in [0, d]$ such that $p \in \mathcal{C}_1'(q)$ and $p \in \mathcal{C}_2'(d-q)$. We take $q_1 = q$, $q_2 = d - q$ and conclude by applying Lemma 7.1.

Conversely, if $p \notin \mathcal{I}$, then it is not possible to find q_1, q_2 such that $q_1 + q_2 = d$, and such that $p \in \mathcal{C}_1'(q_1) \cap \mathcal{C}_2'(q_2)$ i.e., such that according to Lemma 7.1 $q_1 \in \widehat{Q}_1(p)$ and $q_2 \in \widehat{Q}_2(p)$.

Straightforwardly, if $\mathcal{I}^- \leq p \leq \mathcal{I}^+$, then there exists $q_1 \in \widehat{Q}_1(p)$ and $q_2 \in \widehat{Q}_2(p)$ such that $q_1 + q_2 = d$. □

We sum up the previous analysis by the following proposition.

PROPOSITION 7.4 *In a market with maximal price* $p_{\max}$ *with two suppliers, each having to propose quantity and price to the market, and each one wanting to maximize its profit, we have the following Nash equilibrium:*

1. *If* $p_{\max} < \min\{p \in \mathcal{I}\}$-**Excess demand case**, *any strategy profile* $((q_1^*, p_{\max}), (q_2^*, p_{\max}))$, *with* $q_1^* \in \widehat{Q}_1(p_{\max})$ *and* $q_2^* \in \widehat{Q}_2(p_{\max})$ *is a Nash equilibrium. In that case we have* $q_1^* + q_2^* < d$.

2. *If* $p_{\max} = \min\{p \in \mathcal{I}\}$, *any pair* $((q_1^*, p_{\max}), (q_2^*, p_{\max}))$ *where* $q_1^* \in \widehat{Q}_1(p_{\max})$ *and* $q_2^* \in \widehat{Q}_2(p_{\max})$ *is a Nash equilibrium. In that case we may have* $q_1^* + q_2^* \geq d$ *or* $q_1^* + q_2^* < d$.

3. *If* $p_{\max} > \min\{p \in \mathcal{I}\}$- **Excess supply case**, *any pair* $((q_1^*, p^*), (q_2^*, p^*))$, *such that* $p^* \in \mathcal{I}$, $p^* \leq p_{\max}$ *and which induces a reaction* $(\overline{q}_1, \overline{q}_2)$, $\overline{q}_1 \geq q_1^*$, $\overline{q}_2 \geq q_2^*$, *such that*

   (a) $\overline{q}_1 + \overline{q}_2 = d$,

   (b) $\overline{q}_1 \in \widehat{Q}_1(p^*)$, $\overline{q}_2 \in \widehat{Q}_2(p^*)$,

   (c)
$$(d - q_2^*)p_{\max} - C_1(d - q_2^*) \leq \overline{q}_1 p^* - C_1(\overline{q}_1),$$
$$d - q_1^*)p_{\max} - C_2(d - q_1^*) \leq \overline{q}_2 p^* - C_2(\overline{q}_2),$$

   is a Nash equilibrium.

We now want to generalize the previous result to a market with $\mathcal{S} \geq 2$ suppliers.

Let $((q_1^*, p_1^*), \ldots, (q_{\mathcal{S}}^*, p_{\mathcal{S}}^*))$ be a strategy profile, and let $(\overline{q}_1, \ldots, \overline{q}_{\mathcal{S}})$ be the induced reaction of the market. This strategy profile is a Nash equilibrium, if for any two suppliers, S_i, S_j, the pair of strategies $((q_i^*, p_i^*), (q_j^*, p_j^*))$ is a Nash equilibrium for a market with two suppliers (with evaluation function defined by Equation (7.20)) and demand $\tilde{d} = d - \sum_{k \notin \{i,j\}} \overline{q}_k$.

Hence using the above Proposition 7.4, we know that necessarily at equilibrium the prices proposed by the suppliers are equal, and the quantities q_i^* induce a reaction of the market such that $\overline{q}_i \in \widehat{Q}_i(p^*)$.

Let us first extend the previous definition of the set $\mathcal{I}$ by $\mathcal{I} = \emptyset$ if $\sum_{j=1}^{\mathcal{S}} Q_j < d$, and otherwise,

$$\begin{array}{l} \mathcal{I} \stackrel{\text{def}}{=} [\mathcal{I}^-, \mathcal{I}^+], \\ \text{where} \\ \mathcal{I}^- = \min\{p, \ \sum_{j=1}^{\mathcal{S}} \max(q \in \widehat{Q}_j(p)) \geq d\}, \\ \mathcal{I}^+ = \max\{p, \ \sum_{j=1}^{\mathcal{S}} \min(q \in \widehat{Q}_j(p)) \leq d\}. \end{array} \tag{7.24}$$

We have the following

THEOREM 7.1 *Suppose we have $\mathcal{S}$ suppliers on a market with maximal price $p_{\max}$ and demand d.*

1. *If $p_{\max} < \min\{p \in \mathcal{I}\}$*-**Excess demand case**, *any strategy profile $((q_1^*, p_{\max}), \ldots, (q_{\mathcal{S}}^*, p_{\max}))$, with $q_j^* \in \widehat{Q}_j(p_{\max})$, $j = 1, \ldots, \mathcal{S}$, is a Nash equilibrium. In that case we have $\sum_{j=1}^{\mathcal{S}} q_j^* < d$.*
2. *If $p_{\max} = \min\{p \in \mathcal{I}\}$, any strategy profile $((q_1^*, p_{\max}), \ldots, (q_{\mathcal{S}}^*, p_{\max}))$ where $q_j^* \in \widehat{Q}_j(p_{\max})$, $j = 1, \ldots, \mathcal{S}$ is a Nash equilibrium. In that case we may have $\sum_{j=1}^{\mathcal{S}} q_j^* \geq d$ or $\sum_{j=1}^{\mathcal{S}} q_j < d$.*
3. *If $p_{\max} > \min\{p \in \mathcal{I}\}$*- **Excess supply case**, *any strategy profile $((q_1^*, p^*), \ldots, (q_{\mathcal{S}}^*, p^*))$, such that $p^* \in \mathcal{I}$, $p^* \leq p_{\max}$ and which induces a reaction $(\overline{q}_1, \ldots, \overline{q}_{\mathcal{S}})$, $\overline{q}_j \geq q_j^*$, $j = 1, \ldots, \mathcal{S}$, such that*
   - *(a) $\sum_{j=1}^{\mathcal{S}} \overline{q}_j = d$,*
   - *(b) $\overline{q}_j \in \widehat{Q}_j(p^*)$, $j = 1, \ldots, \mathcal{S}$,*
   - *(c) for any $j = 1, \ldots, \mathcal{S}$*

   $$(d - \sum_{k \neq j} q_k^*) p_{\max} - C_j(d - \sum_{k \neq j} q_k^*) \leq \overline{q}_j p^* - C_j(\overline{q}_j), \tag{7.25}$$

   is a Nash equilibrium.

The previous results show that a Nash equilibrium always exists for the case where the profit is used by the suppliers as an evaluation function. For convenience we have supposed the existence of a maximal price $p_{\max}$. On real markets we observe that, usually this maximal price is infinity, since most markets do not impose a maximal price on electricity. Hence the interesting case is the case where $p_{\max} > \min\{p \in \mathcal{I}\}$. The case with $p_{\max} \leq \min\{p \in \mathcal{I}\}$, can be interpreted as a monopolistic situation. The demand is so large compared with the maximal price that each supplier can behave as if it is alone on the market.

When $p_{\max}$ is large enough, Proposition 7.4 and Theorem 7.1 exhibit some conditions for a strategy profile to be a Nash equilibrium. We can make several remarks.

REMARK 7.7 Note that conditions (7.25) are satisfied for $q_j^* = d$. Hence we can conclude that, provided that the market reacts in such a way that $\overline{q}_j \in \widehat{Q}_j$, the strategy profile $((d, p^*), \ldots, (d, p^*))$ is a Nash equilibrium. Nevertheless, this equilibrium is not realistic. As a matter of fact, to implement this equilibrium, the suppliers have to propose to the market a quantity that is higher than the optimal quantity, and which possibly may lead to a negative profit (when $p_{\max}$ is very large). The second aspect that may appear unrealistic is the fact that the suppliers give up their power of decision. As a matter of fact, they announce high quantities, so that (7.25) is satisfied, and let the market choose the appropriate $\overline{q}_j$.

Example. We consider the market, with demand $d = 10$ and maximal price $p_{\max} = +\infty$, with the five suppliers already described page 147. In order to be able to compare the equilibria for both criteria, market share and profit, we suppose that the marginal cost is equal to the minimal price function displayed in the table page 148, i.e., $C_j' = \mathcal{L}_j$.

The following table displays the quantities $o(p) \stackrel{\text{def}}{=} \sum_{j=1}^{5} \min\{q \in \widehat{Q}_j(p)\}$ and $O(p) \stackrel{\text{def}}{=} \sum_{j=1}^{5} \max\{q \in \widehat{Q}_j(p)\}$.

p	$\in [0, 10[$	$= 10$	$\in]10, 15[$	$= 15$	$\in]15, 20[$	$= 20$	$\in]20, 23[$
$o(p)$	0	0	2	2	9	9	15
$O(p)$	0	2	2	9	9	15	15

From the above table we deduce that $\mathcal{I} = \{20\}$, hence the only possible equilibrium price is $p^* = 20$. As a matter of fact, we have $O(20) = 15 > 10 = d$, and for any $p < 20$, $O(p) < 10 = d$, and $o(20) = 9 < 10 = d$, and for any $p > 20$, $o(p) > 10 = d$. Hence $p^* = 20 \in \mathcal{I}$ as defined by Equation (7.24).

Now concerning the quantities, the equilibrium depend upon the additional rule of the market. We suppose that the market chooses $\overline{q}_i \in \widehat{Q}_i,\ \ \forall i \in \{1, \ldots, 5\}$, and then to give preference to S_1, then to S_2, etc.

The equilibrium is $q_1^* \geq 3, \quad q_2^* \geq 0, \quad q_3^* \geq 2, \quad q_4^* \geq 4, \quad q_2^* \geq 0, \quad q_5^* \geq 0$.

The fact that the market wants to choose quantities $\overline{q}_i \in \widehat{Q}_i(20)$ implies that $\overline{q}_1 \in \widehat{Q}_1(20) = 3, \quad \overline{q}_2 \in \widehat{Q}_2(20) = [0, 5], \quad \overline{q}_3 \in \widehat{Q}_3(20) = 2, \quad \overline{q}_4 \in \widehat{Q}_4(20) = [4, 5], \quad \overline{q}_5 = 0$, and the preference for S_2 compared to S_5 implies that $\overline{q}_1 = 3, \quad \overline{q}_2 = 1, \quad \overline{q}_3 = 2, \quad \overline{q}_4 = 4, \quad \overline{q}_5 = 0$.

If the preference would have been S_5 then S_4 then S_3 etc. the equilibrium would have been the same, but we would have had

$$\overline{q}_1 = 3, \quad \overline{q}_2 = 0, \quad \overline{q}_3 = 2, \quad \overline{q}_4 = 5, \quad \overline{q}_5 = 0.$$

5. Conclusion

We have shown in the previous sections that for both criteria, market share and profit maximization, it is possible to find a Nash equilibrium for a number $\mathcal{S}$ of suppliers. It is noticeable that for both cases the equilibrium price involved is the same (i.e., p^* given by Equation (7.14) for market share maximization and $p^* \in \mathcal{I}$ defined by Equation (7.24) for profit maximization), only the quantities proposed by the suppliers differ.

Nevertheless as already discussed in Remark 7.7, for profit maximization, the equilibrium strategies involved are not realistic in the interesting cases ($p_{\max}$ large). This may suggest that on these unregulated markets where suppliers are interested in instantaneous profit maximization, an equilibrium never occurs. Prices may becomes arbitrarily high and anticipation of the market behavior, and particularly market price, basically impossible. An extended discussion on this topic can be found in Bossy et al. (2004).

This paper contains some modeling aspects that could be considered in more detail in future works. A first extension would be to consider more general suppliers. As a matter of fact, in the current paper, the evaluation functions chosen are more suitable for providers. Indeed, for profit maximization, we assumed that we had a production cost only for that part of electricity which is actually sold. This would fit the case where suppliers are producers. They produce only the electricity sold. The evaluation function chosen does not fit the case of traders who may have already bought some electricity and try to sell at best price the maximal electricity they have. The extension of our results in that case should not be difficult.

We supposed that every supplier perfectly knows the evaluation function of the other suppliers, and in particular their marginal costs. In general this is not true. Hence some imperfect information version of the model should probably be investigated.

An other extension worthwhile would be to consider the multi market case, since the suppliers have the possibility to sell their electricity on several markets. This aspect has been briefly discussed in Bossy et al. (2004). It leads to a much more complex model, which in particular involves a two level game where at both levels the agents strive to set a Nash equilibrium.

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References

Başar, T. and Olsder, G.J. (1999). *Dynamic Noncooperative Game Theory*. SIAM, Philadelphia.

Bossy, M., Maïzi, N., Olsder, G.J., Pourtallier, O., and Tanré, E. (2004). Using game theory for the electricity market. *Research report INRIA RR-5274*.

Hämäläinen, R.P. (1996). Agent-based modeling of the electricity distribution system. *Proceedings of the 15th IAESTED International Conference*, pages 344–346, Innsbruck, Austria.

Kempert, C. and Tol, R. (2000). The Liberalisation of the German Electricity Market: Modeling an Oligopolistic Structure by a Computational Game Theoretic Modeling Tool. Working paper, Department of Economics I, Oldenburg University, Oldenburg, Germany.

Luce, D. and Raifa, H. (1957). *Games and Decisions*. Wiley, New York.

Madlener, R. and Kaufmann, M. (2002). Power Exchange Spot Market Trading in Europe: Theoretical Considerations and Empirical Evidence. Contract no. ENSK-CT-2000-00094 of the 5th Framework Programme of the European Community.

Madrigal, M. and Quintana, V.H. (2001). Existence and determination of competitive equilibrium in unit commitment power pool auctions: Price setting and scheduling alternatives. *IEEE Transactions on Power Systems*, 16:380–388.

Murto, P. (2003). On Investment, Uncertainty and Strategic Interaction with Applications to Energy Markets. Ph.D. Thesis, Helsinki University of Technology.

Office of Electricity Regulation (OFFER, now OFGEM). (1998). Pool Price, A consultation by OFFER. Technical Report, Birmingham, U.K.

Ruusunen, J. (1994). Barter contracts in intertemporal energy exchange. *European Journal of Operational research*, 75:589–599.

Stoft, S. (2002). *Power System Economics-Disigning Markets for Electricity*. IEEE Press, John Wiley & Sons, Inc. Publication.

Supatchiat, C., Zhang, R.Q., and Birge, J.R. (2001). Equilibrium values in a competitive power exchange market. *Computational Economics*, 17:93–121.