Monotony Properties of Connected Visible Graph Searching

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Graph Searching

Goal

In an undirected simple graph, stand

- an invisible omniscient arbitrary fast fugitive;
- a team of searchers;

To find a strategy that catch the fugitive
using the fewest searchers as possible.

Motivations

- network security, speleological rescue,...
- problem related to well known graphs'parameters: treewidth and pathwidth.

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Search Strategy, Parson. [GTC,1978]
Variant of Kirousis and Papadimitriou. [TCS,86]

Sequence of two basic operations,...

1. Place a searcher at a vertex of the graph;
2. Remove a searcher from a vertex of the graph.

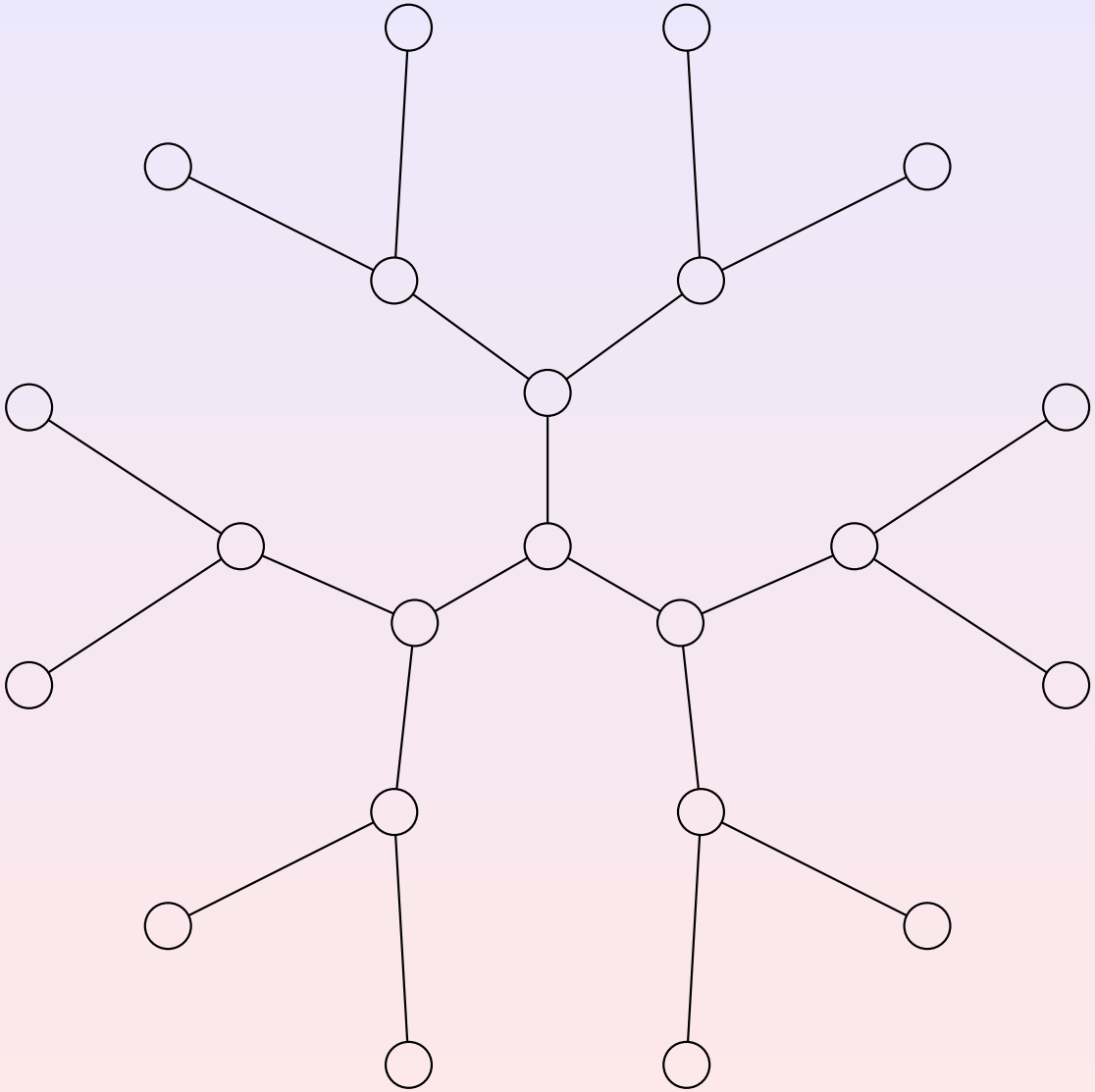
...that must result in catching the fugitive

The fugitive is caugth when it occupies (or crosses) a vertex occupied by a searcher.

We want to minimize the number of searchers.

Let $s(G)$ be the smallest number of searchers needed to catch an invisible fugitive in a graph G.

A simple example



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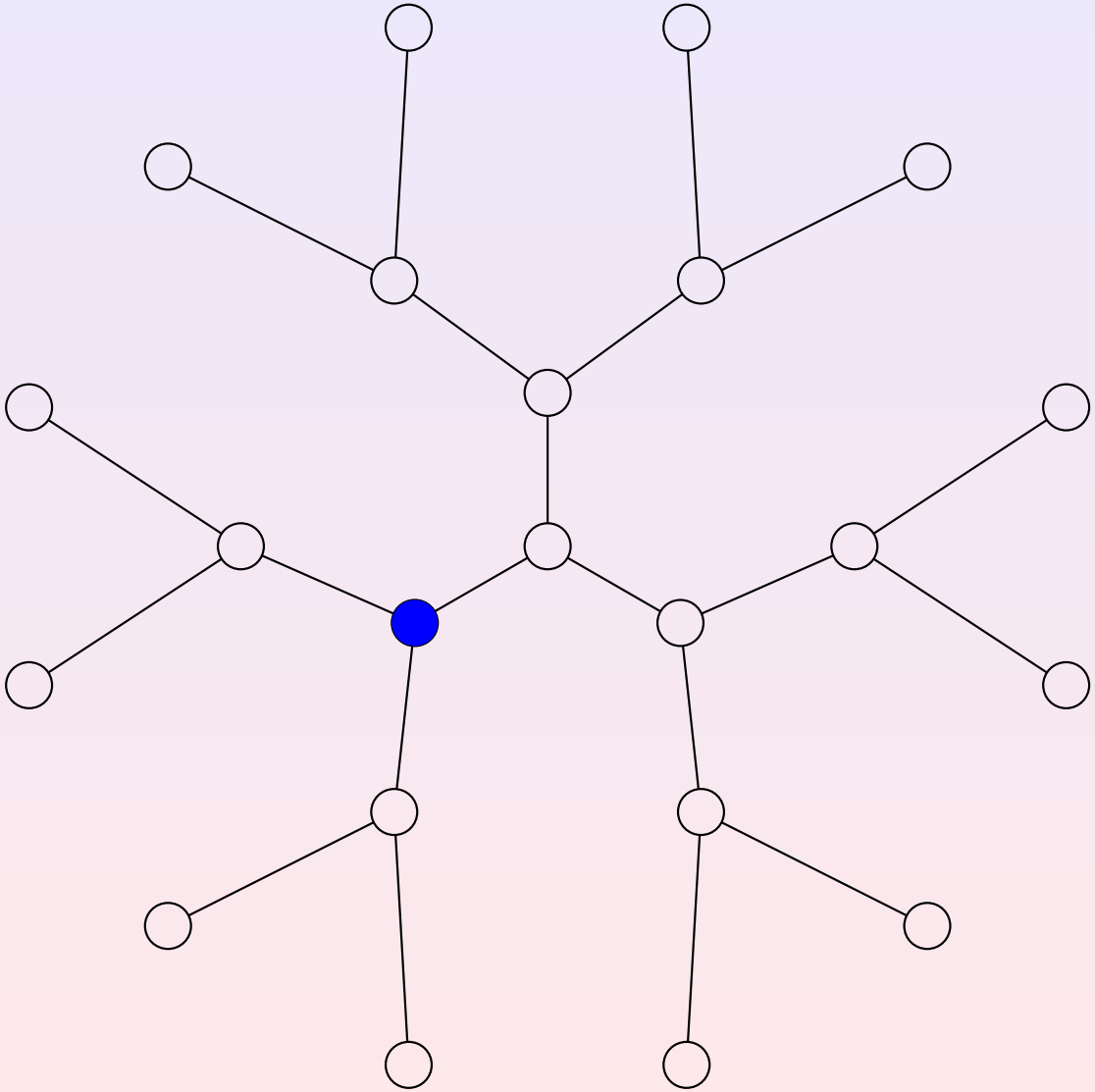

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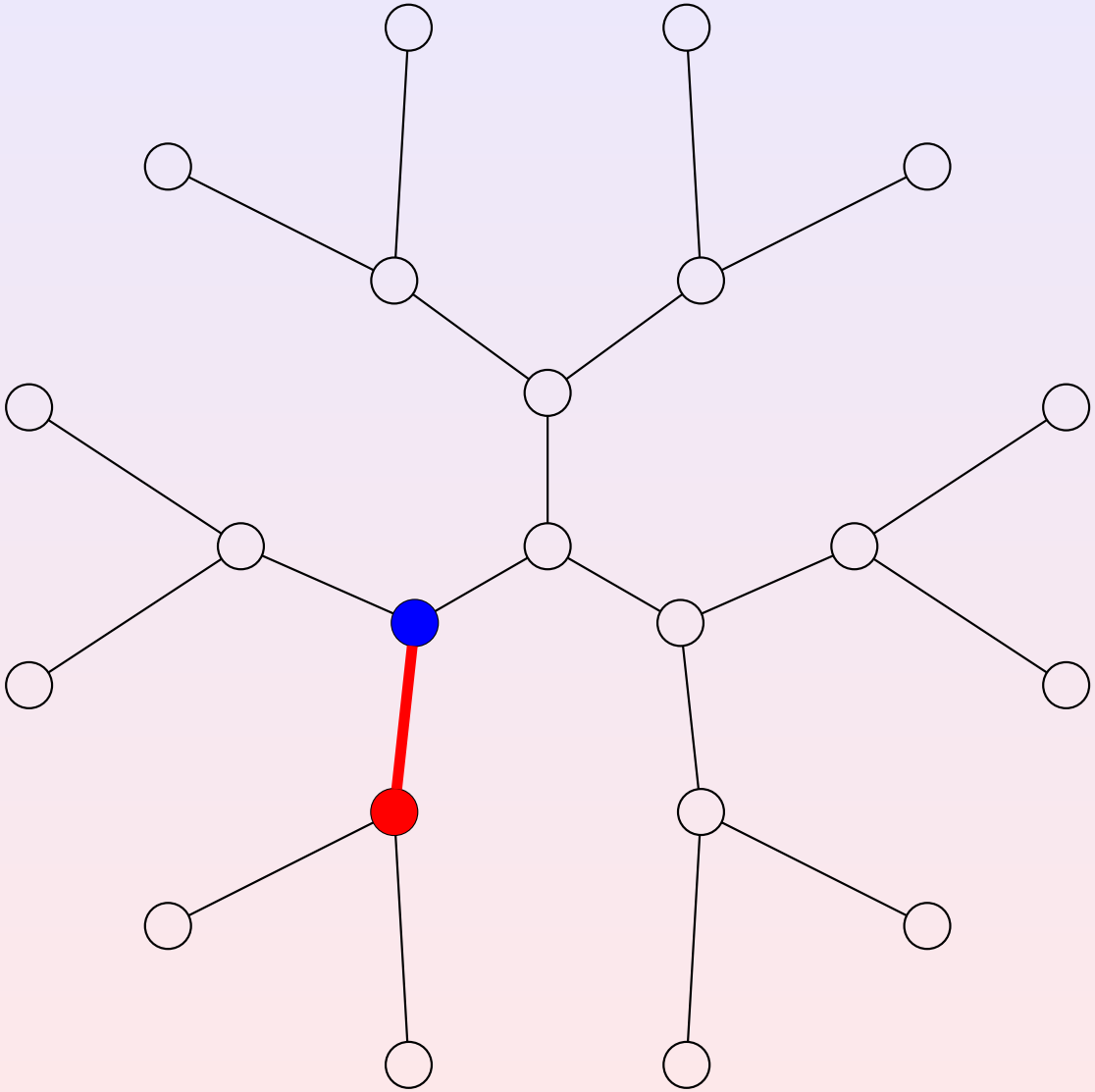

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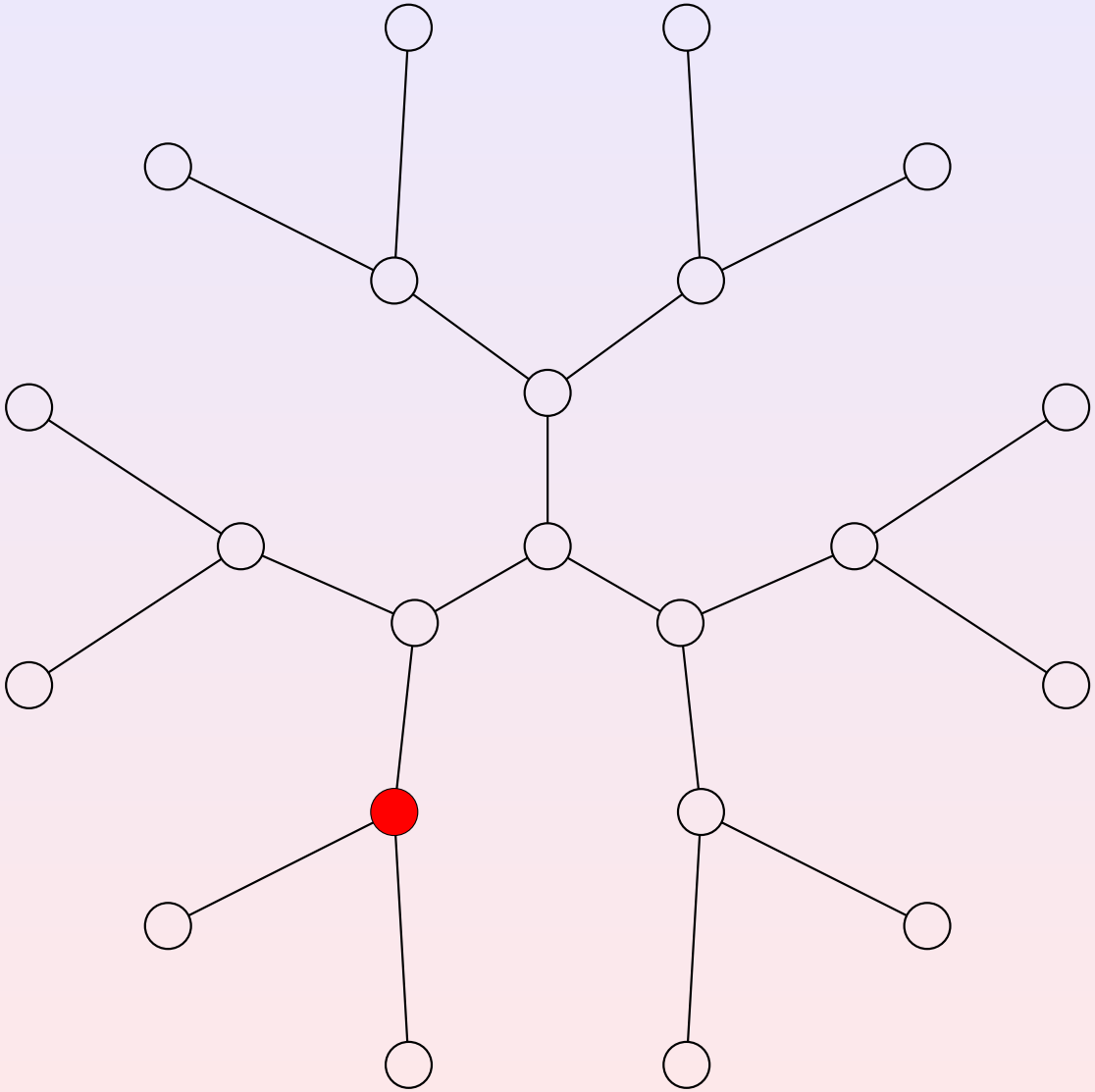

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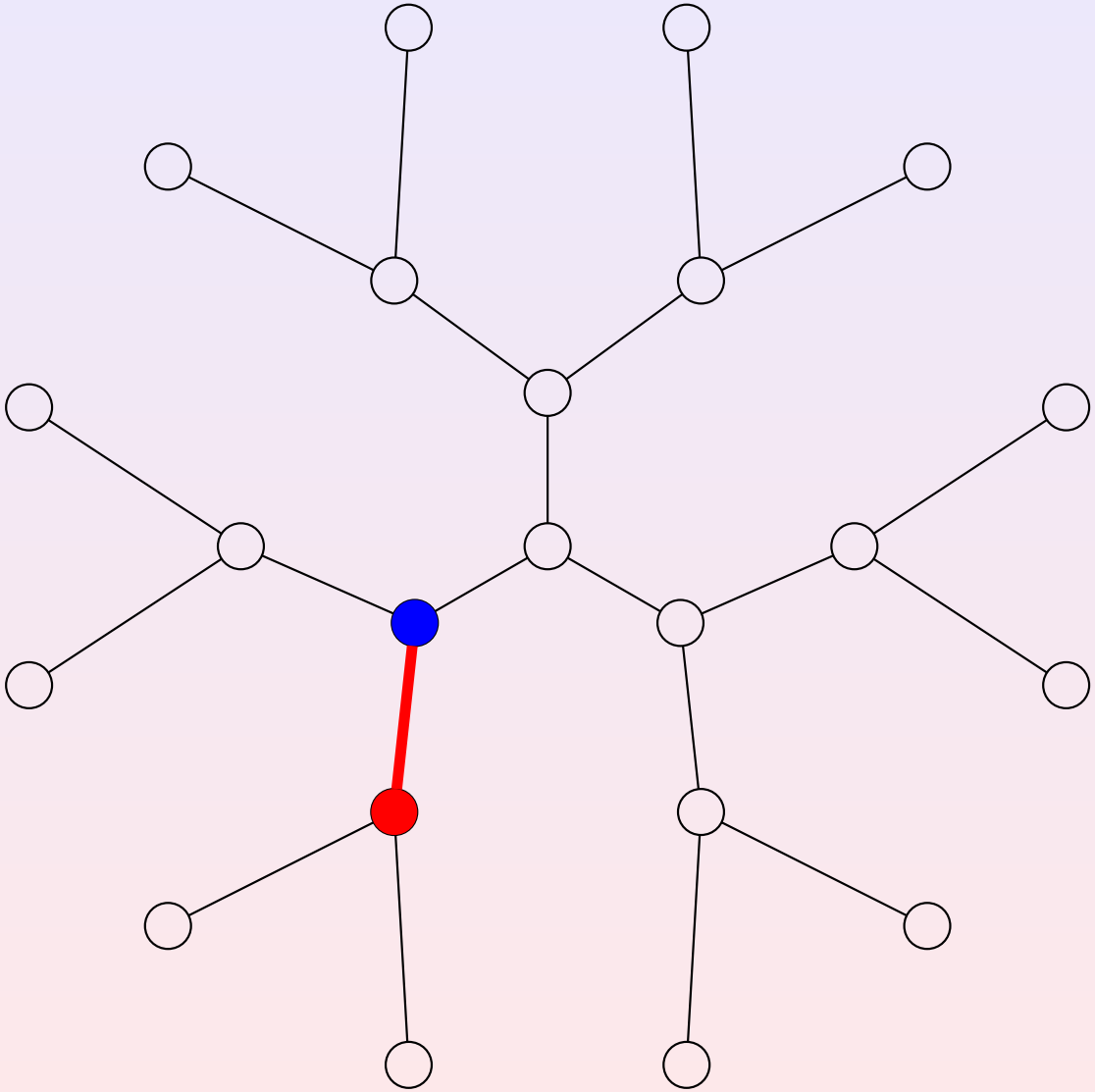

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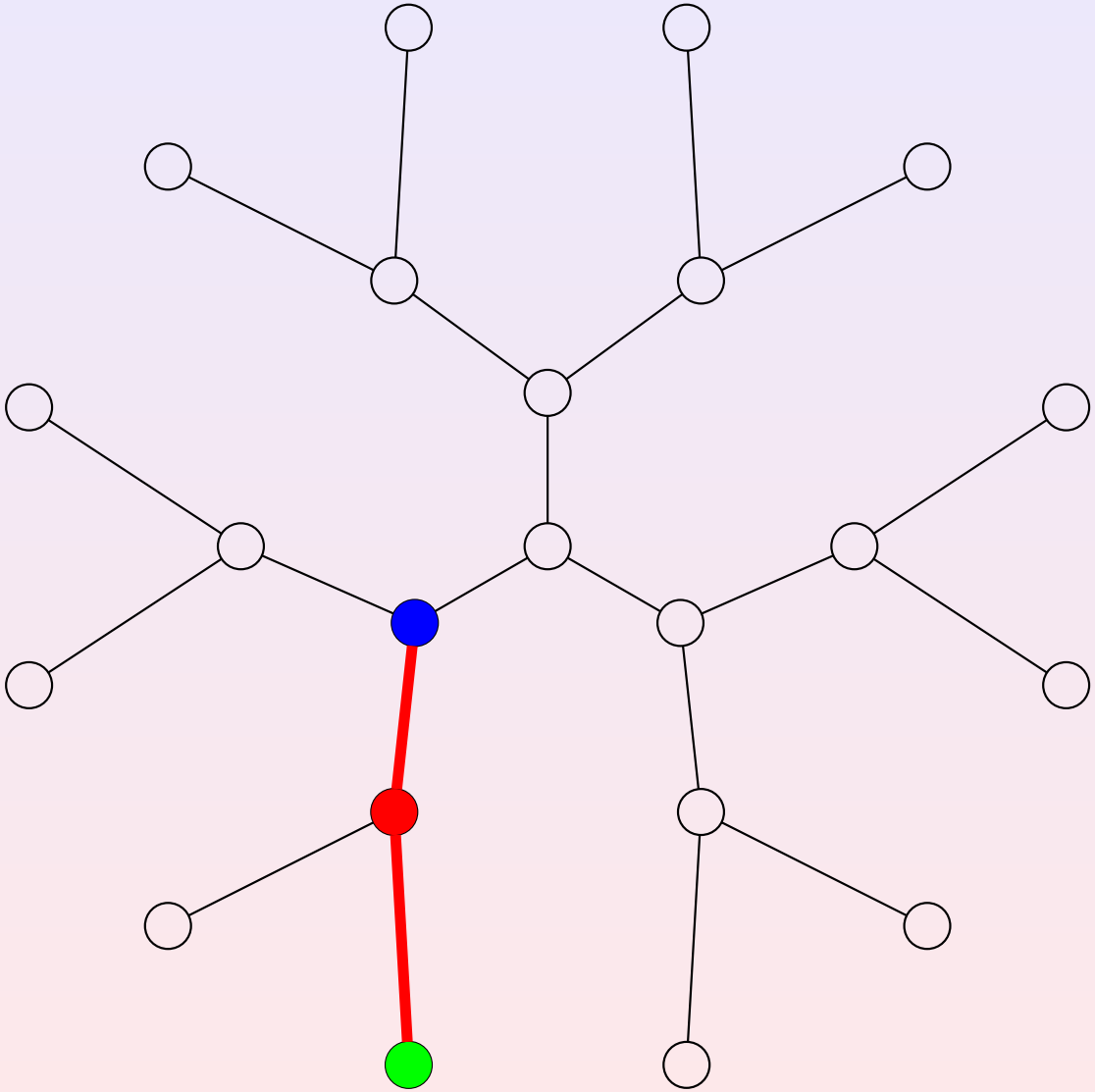

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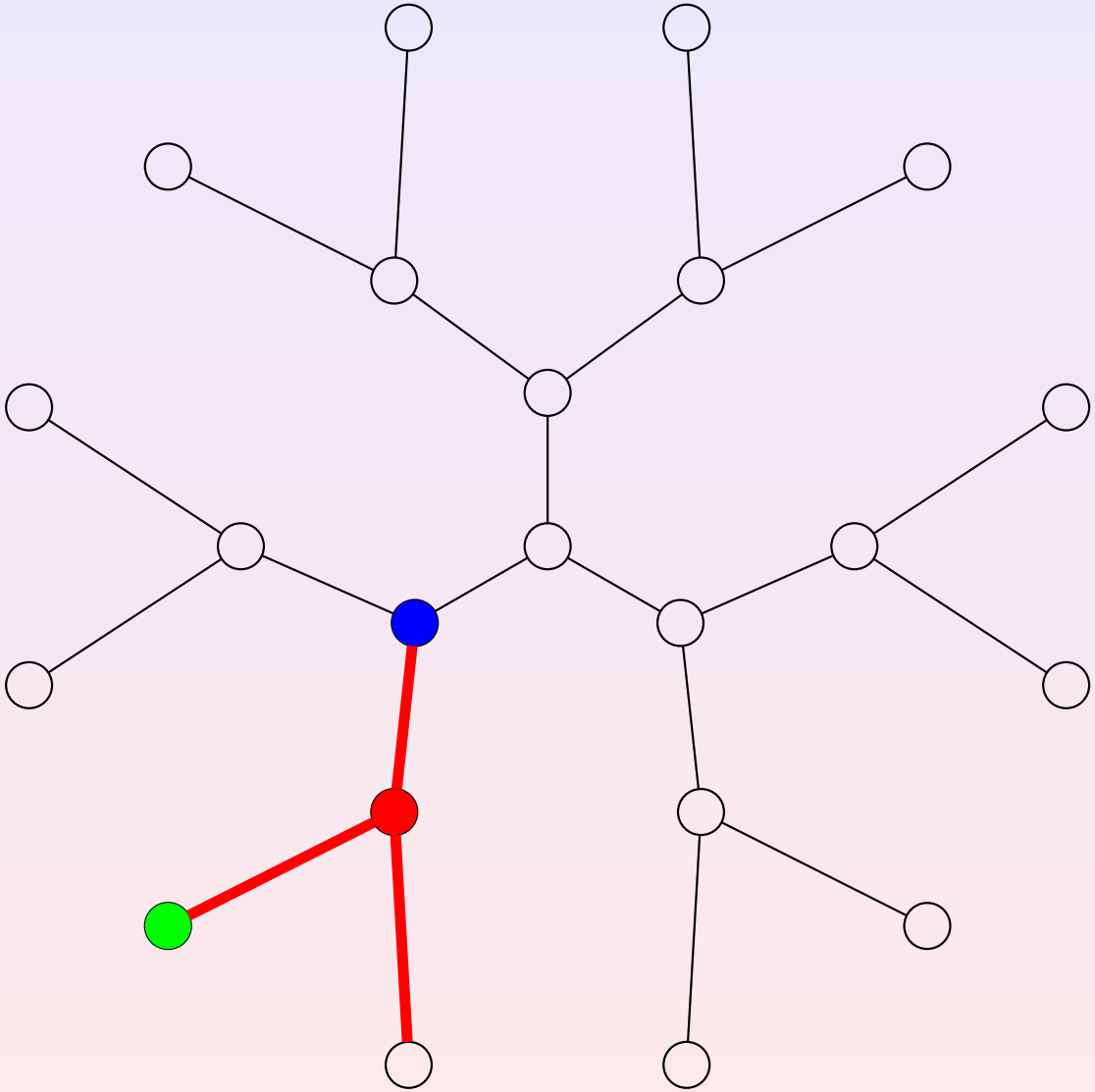

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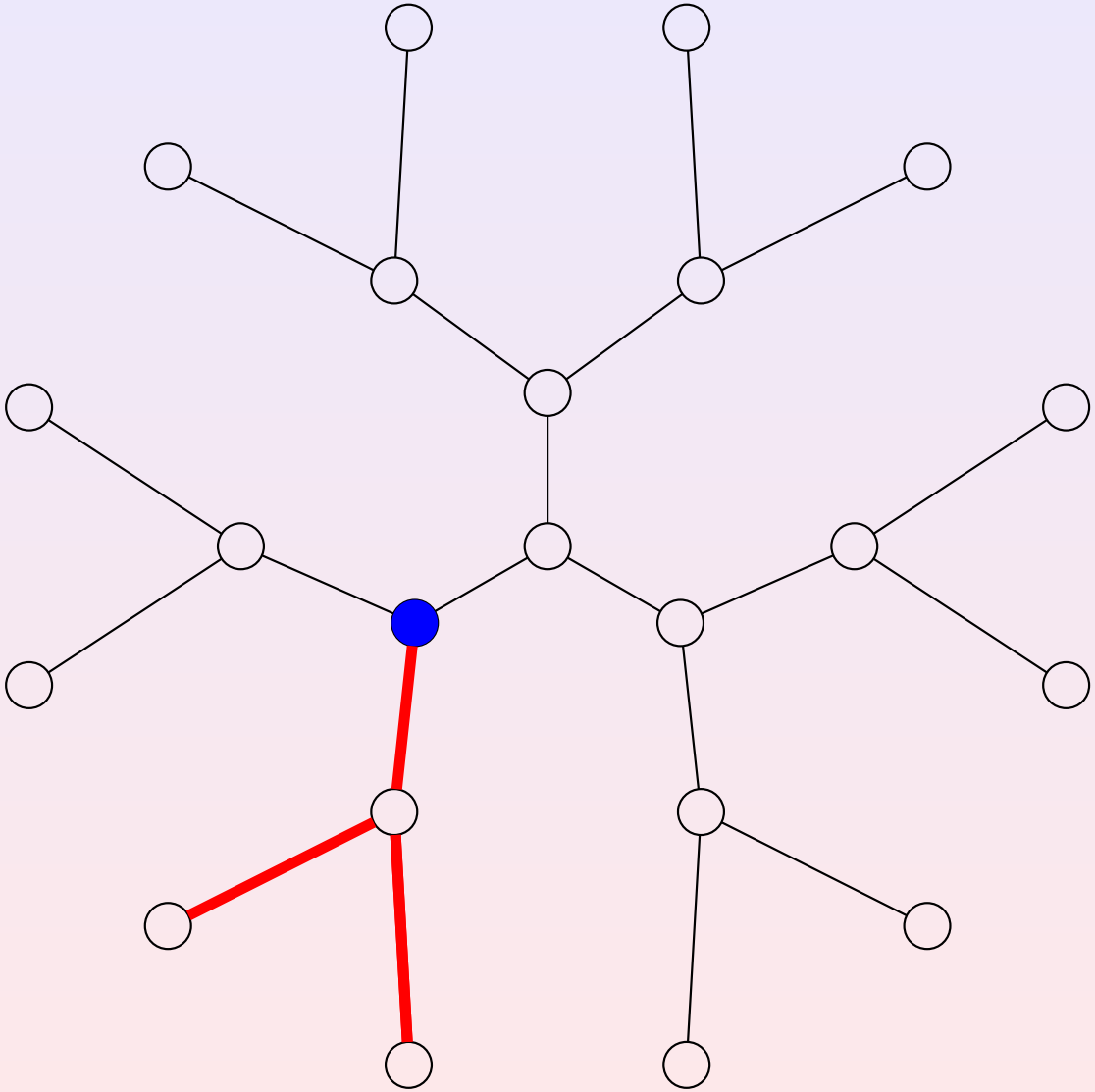

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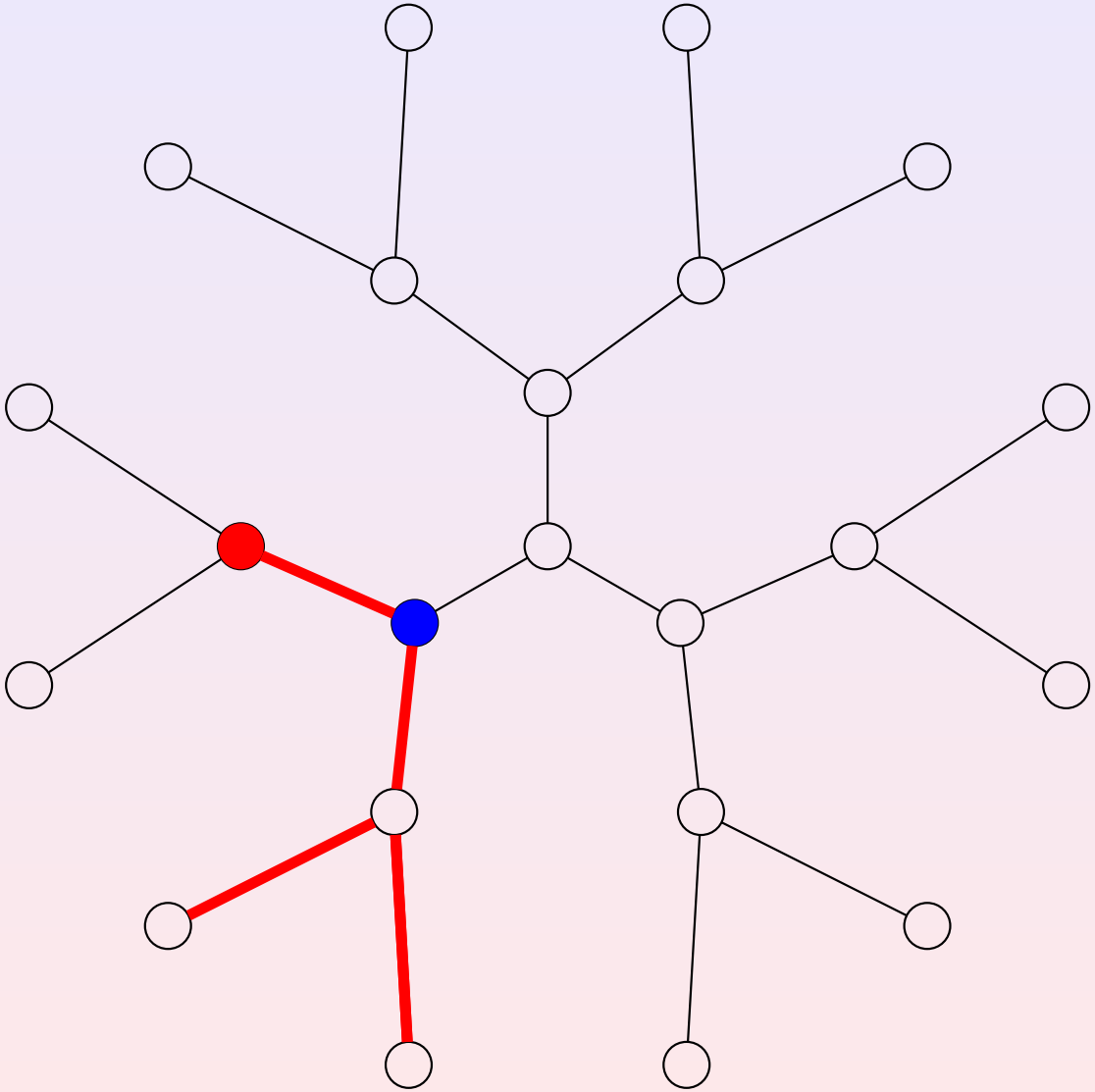

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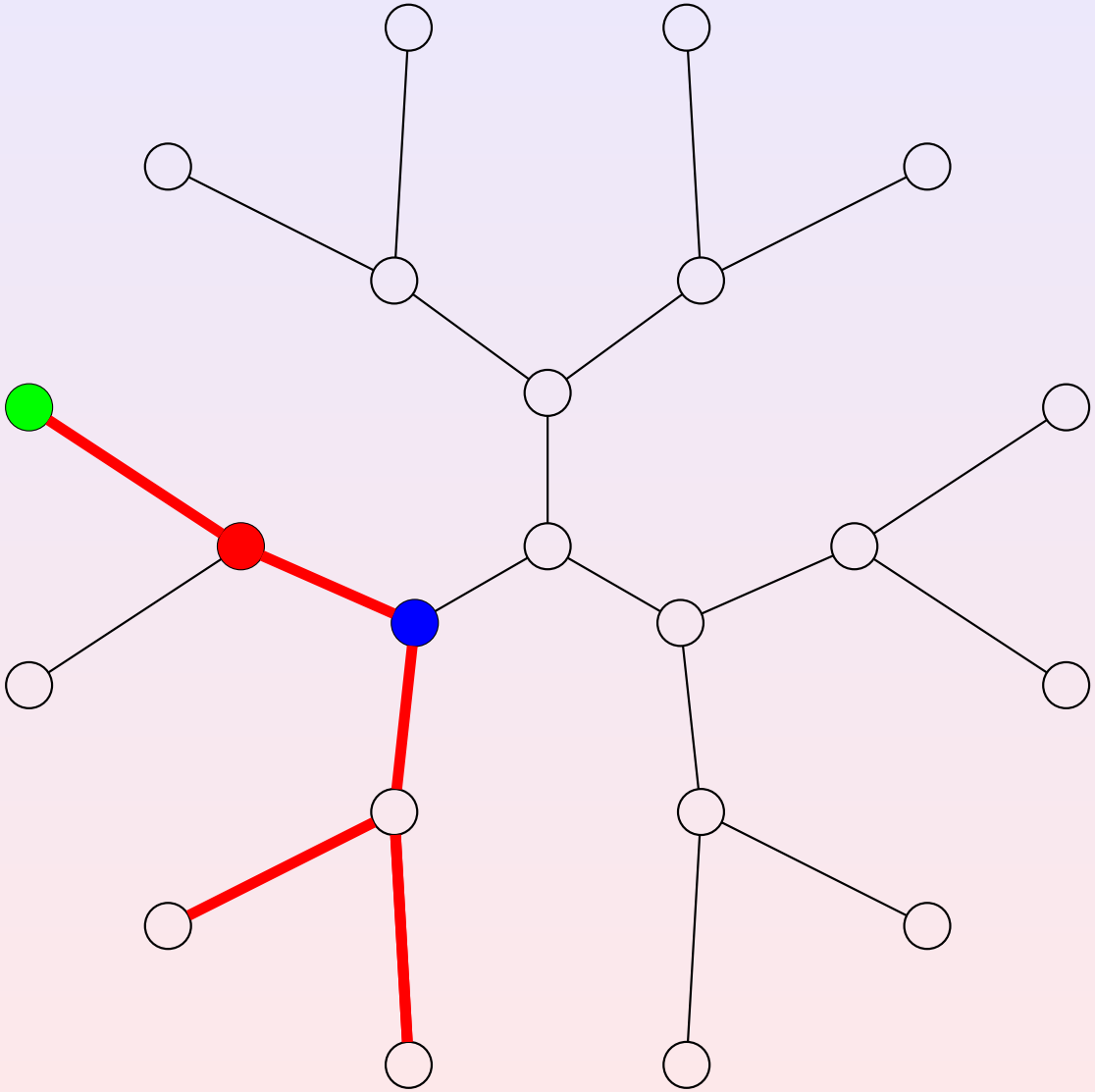

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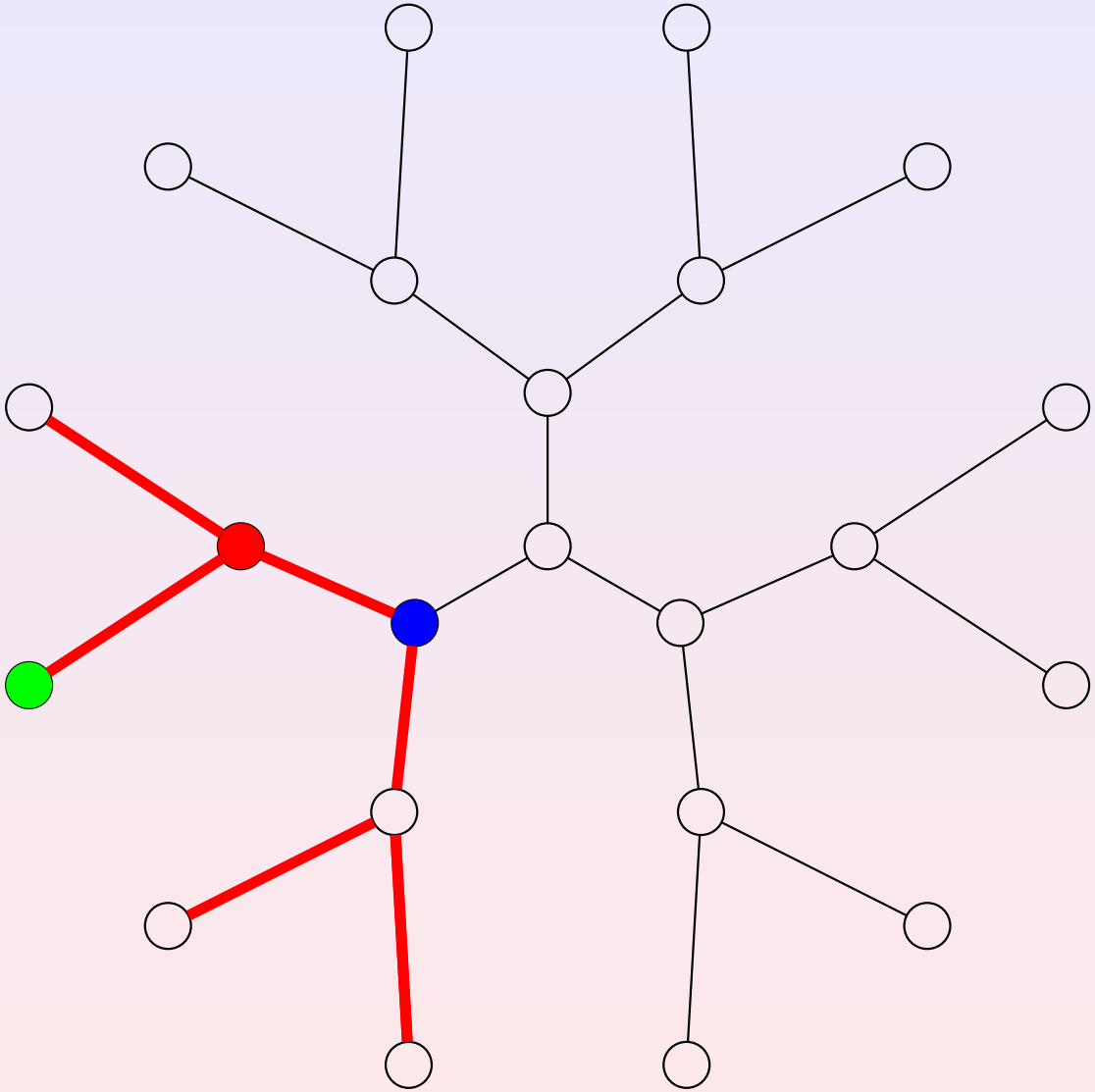

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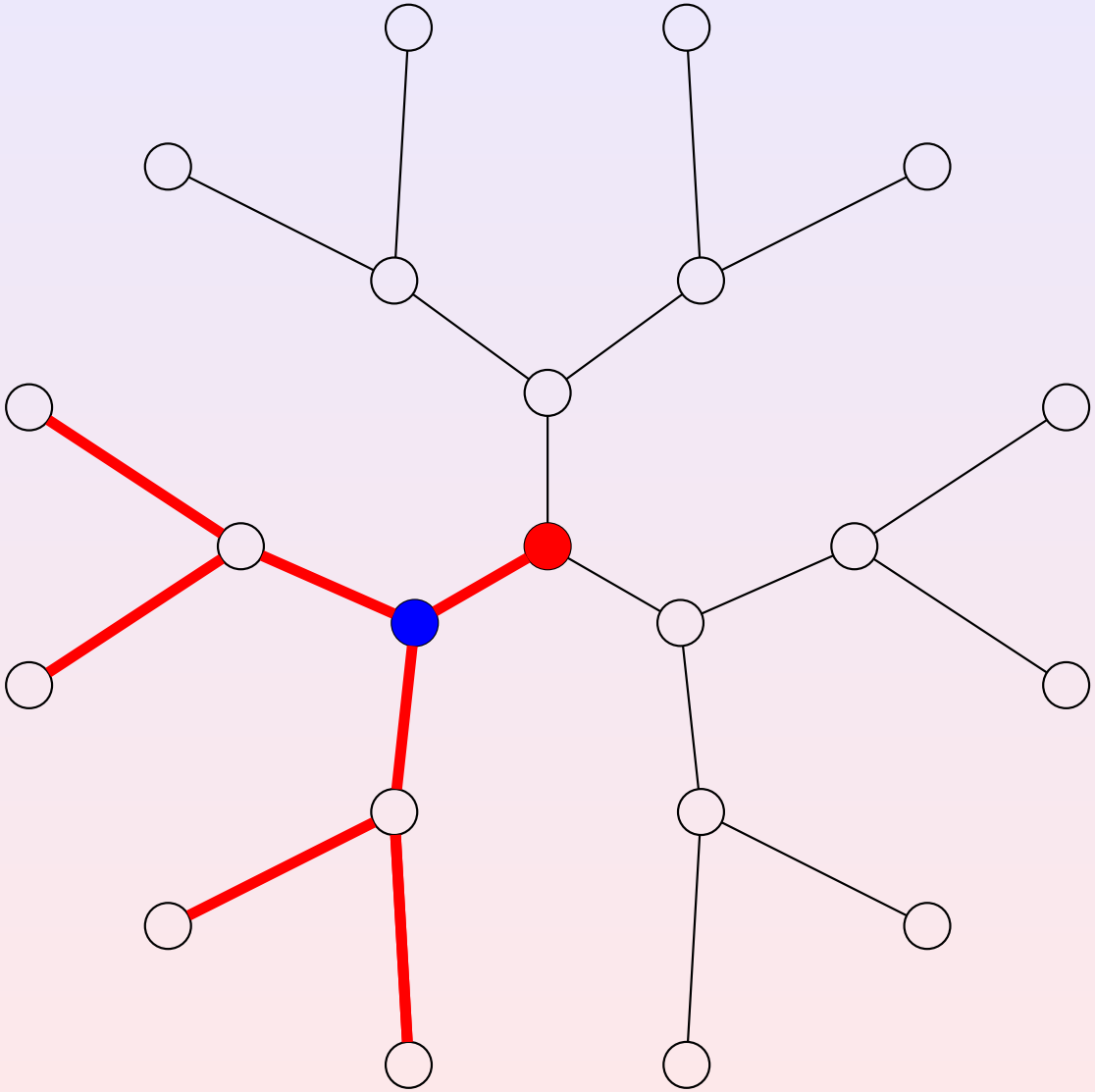

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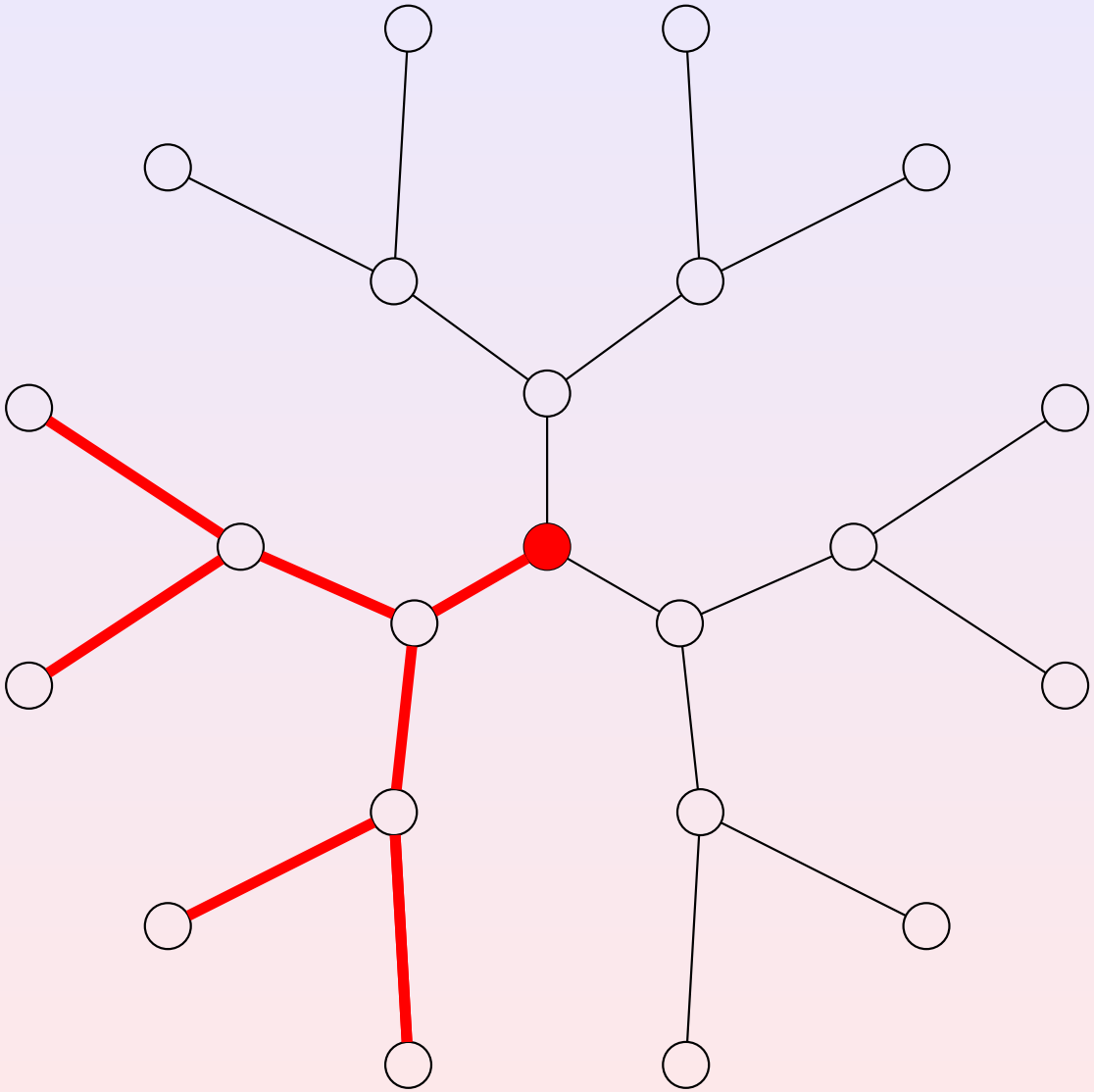

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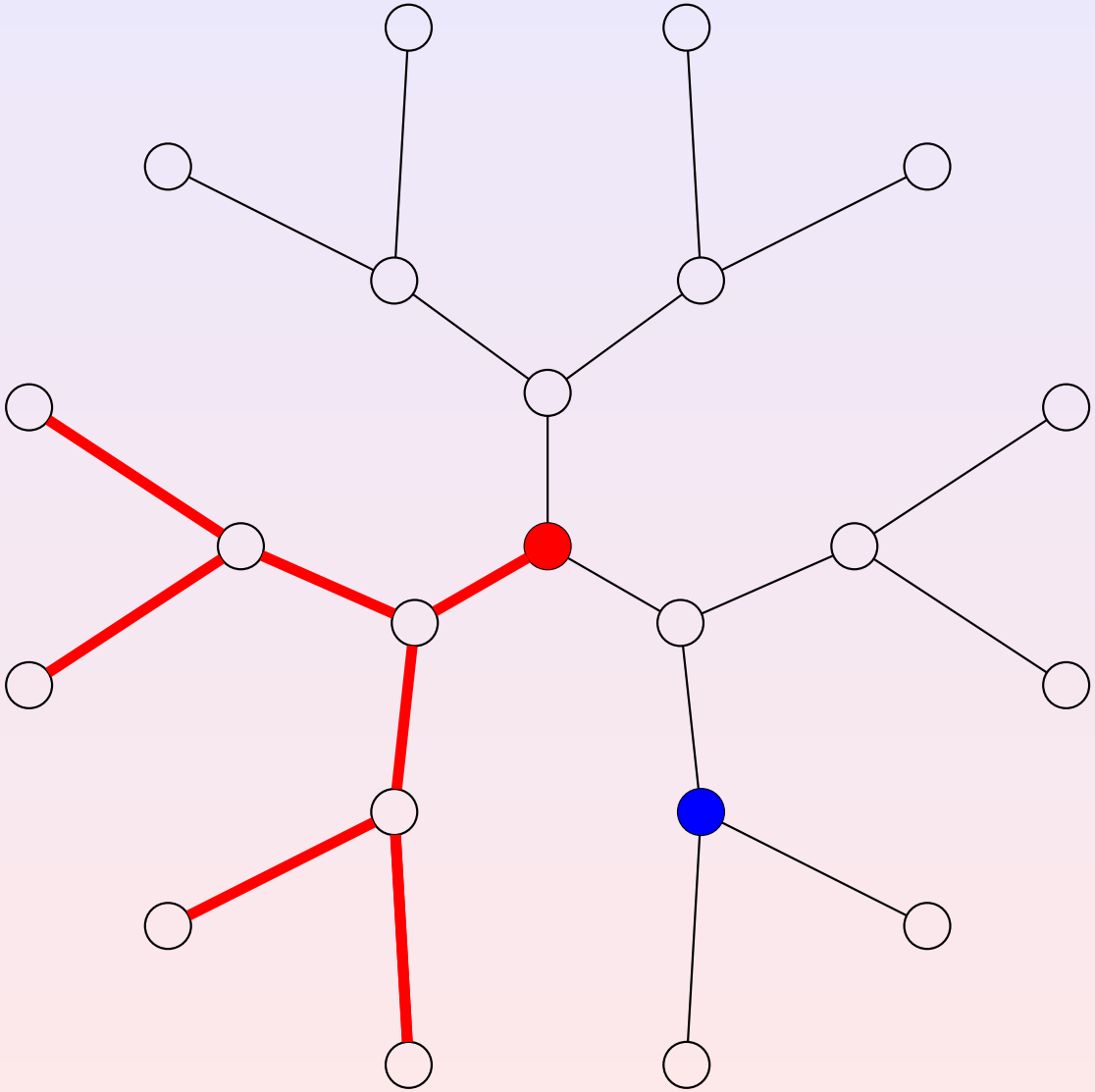

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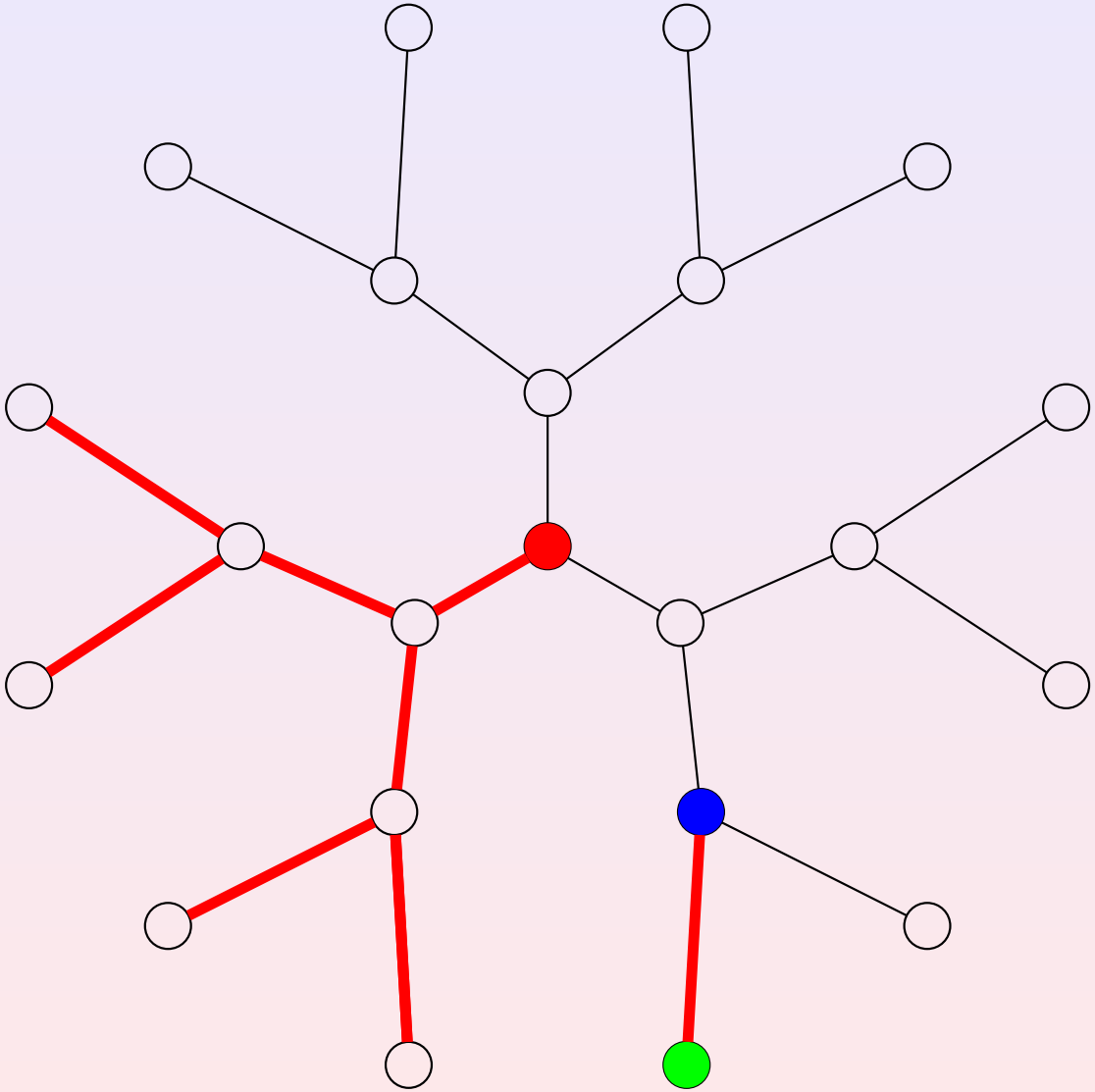

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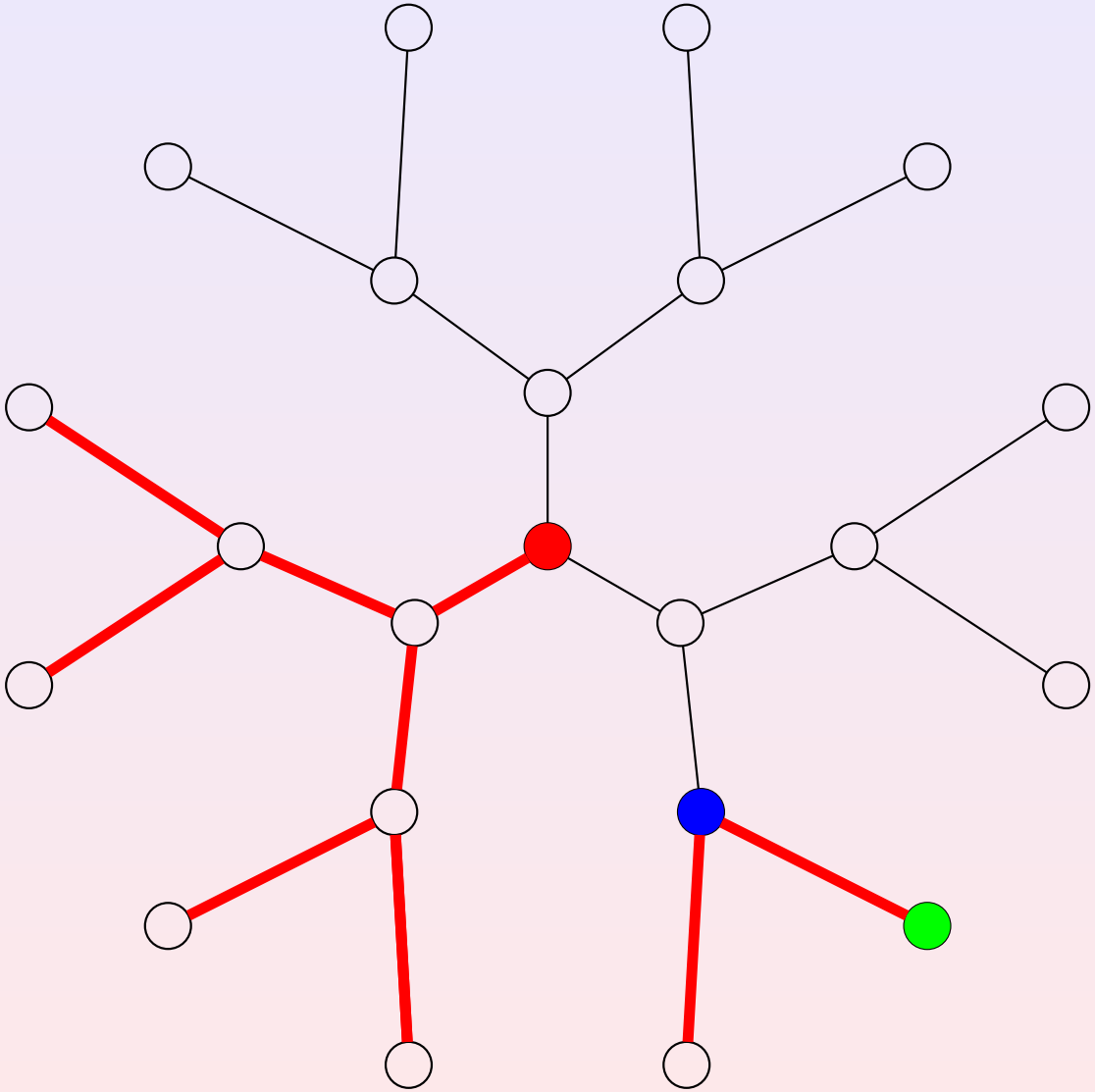

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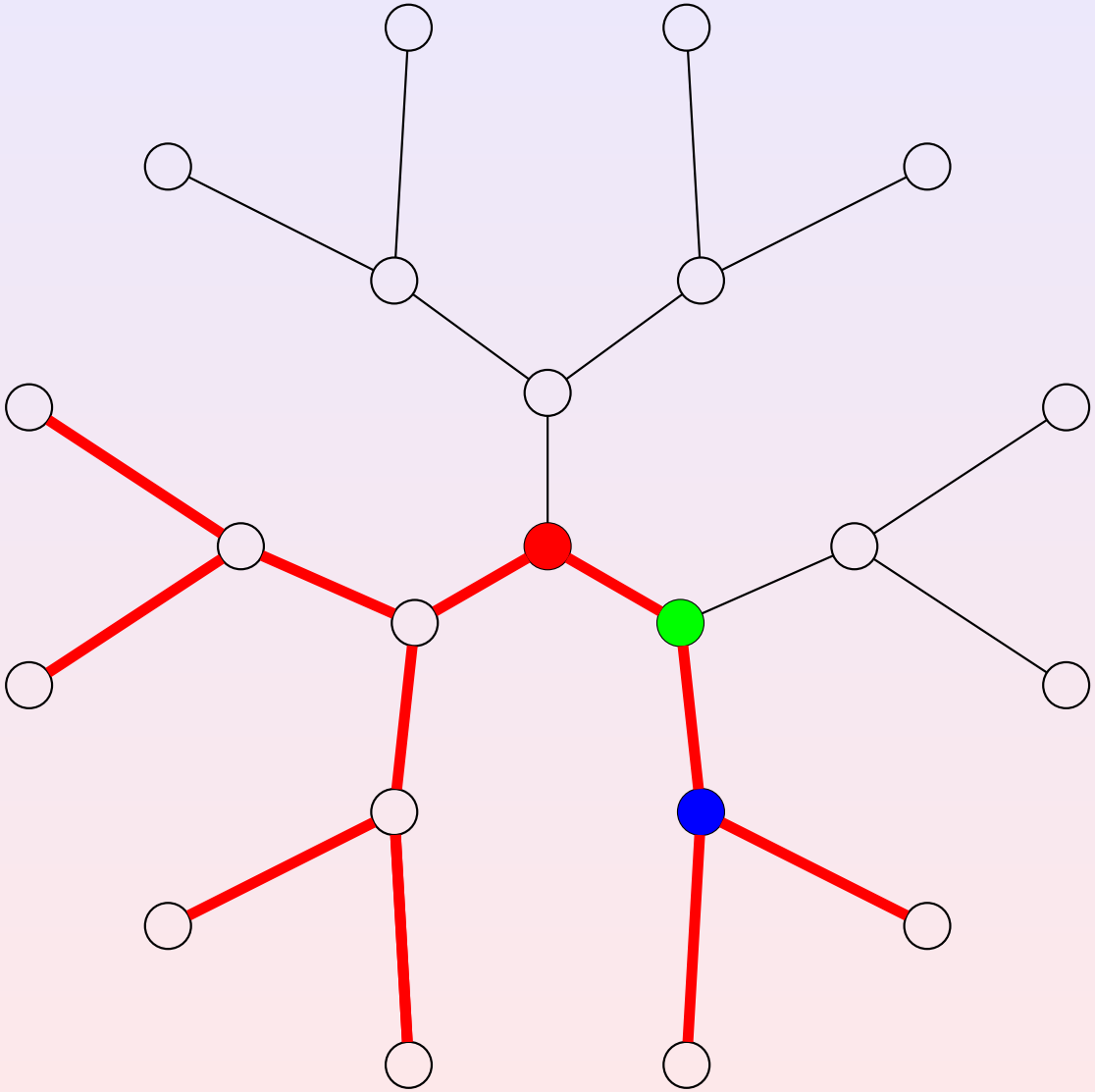

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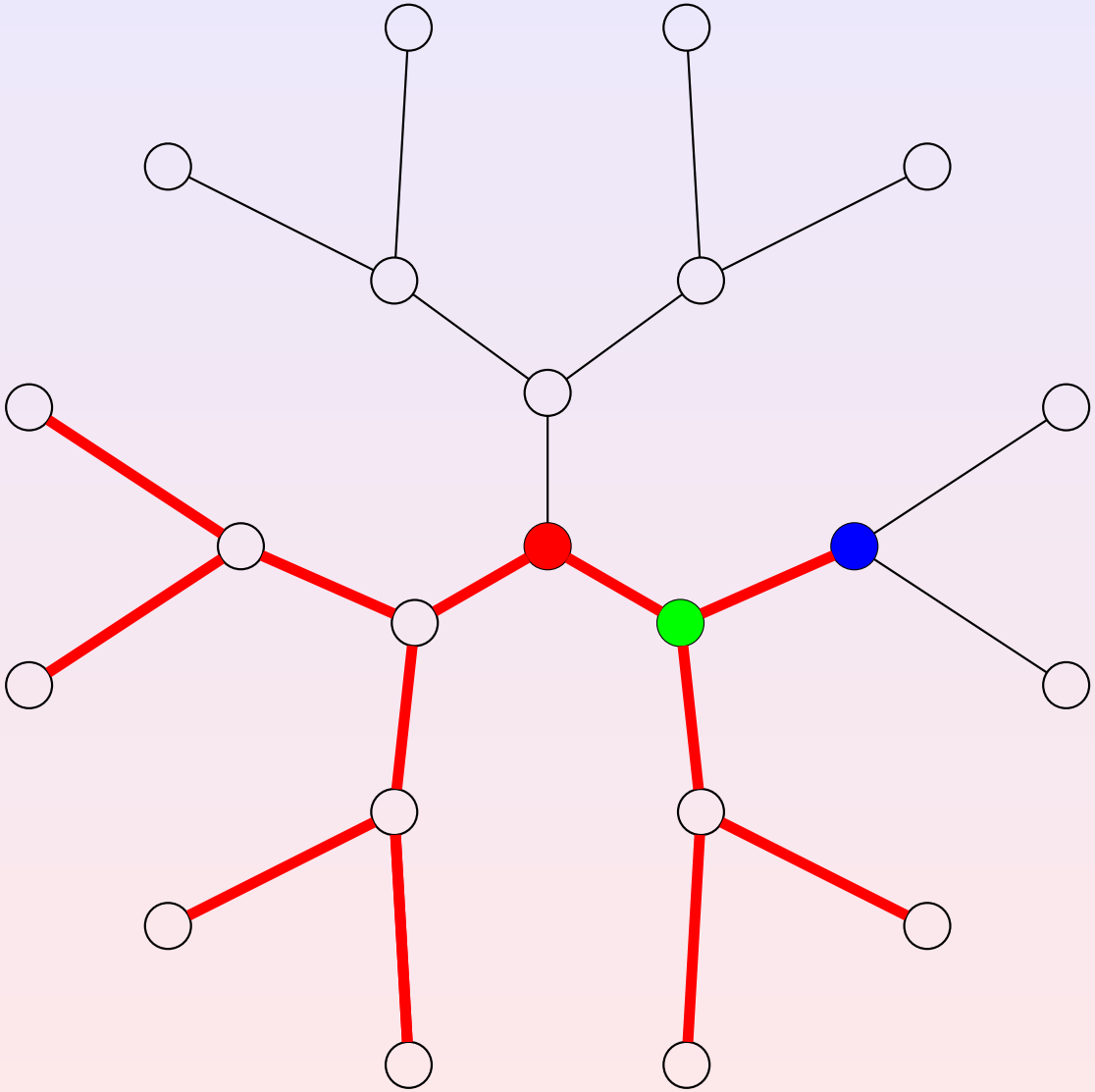

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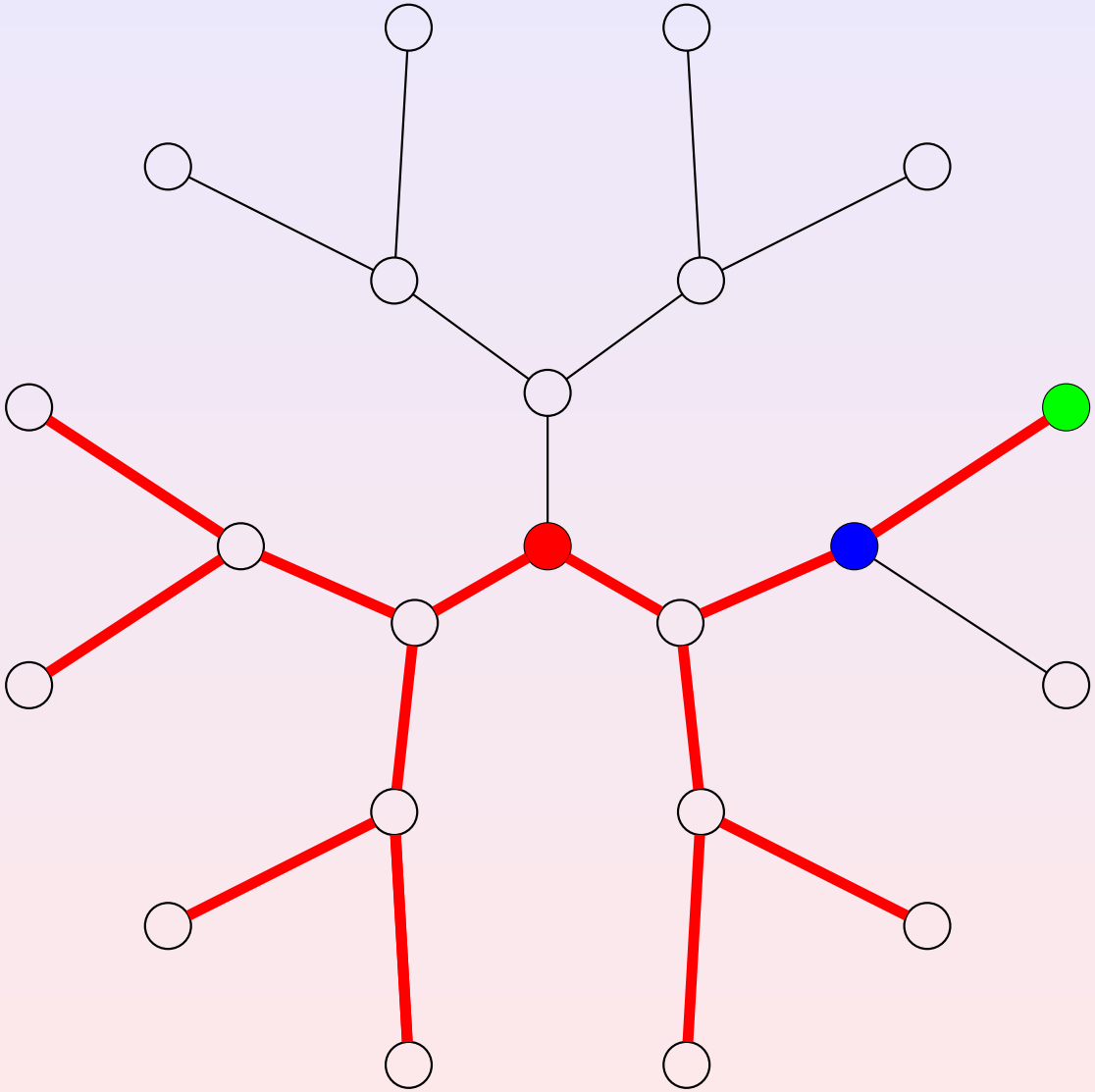

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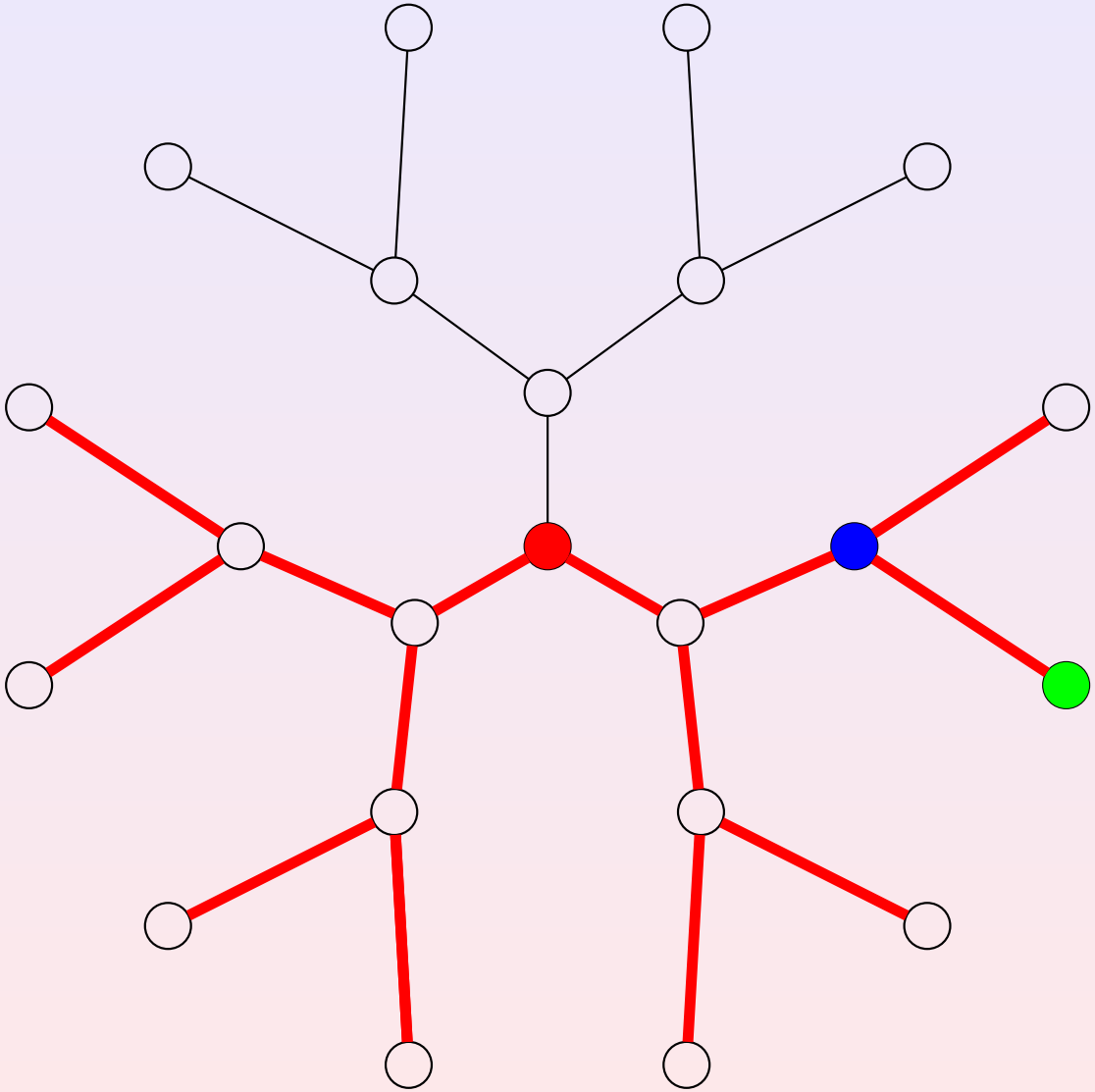

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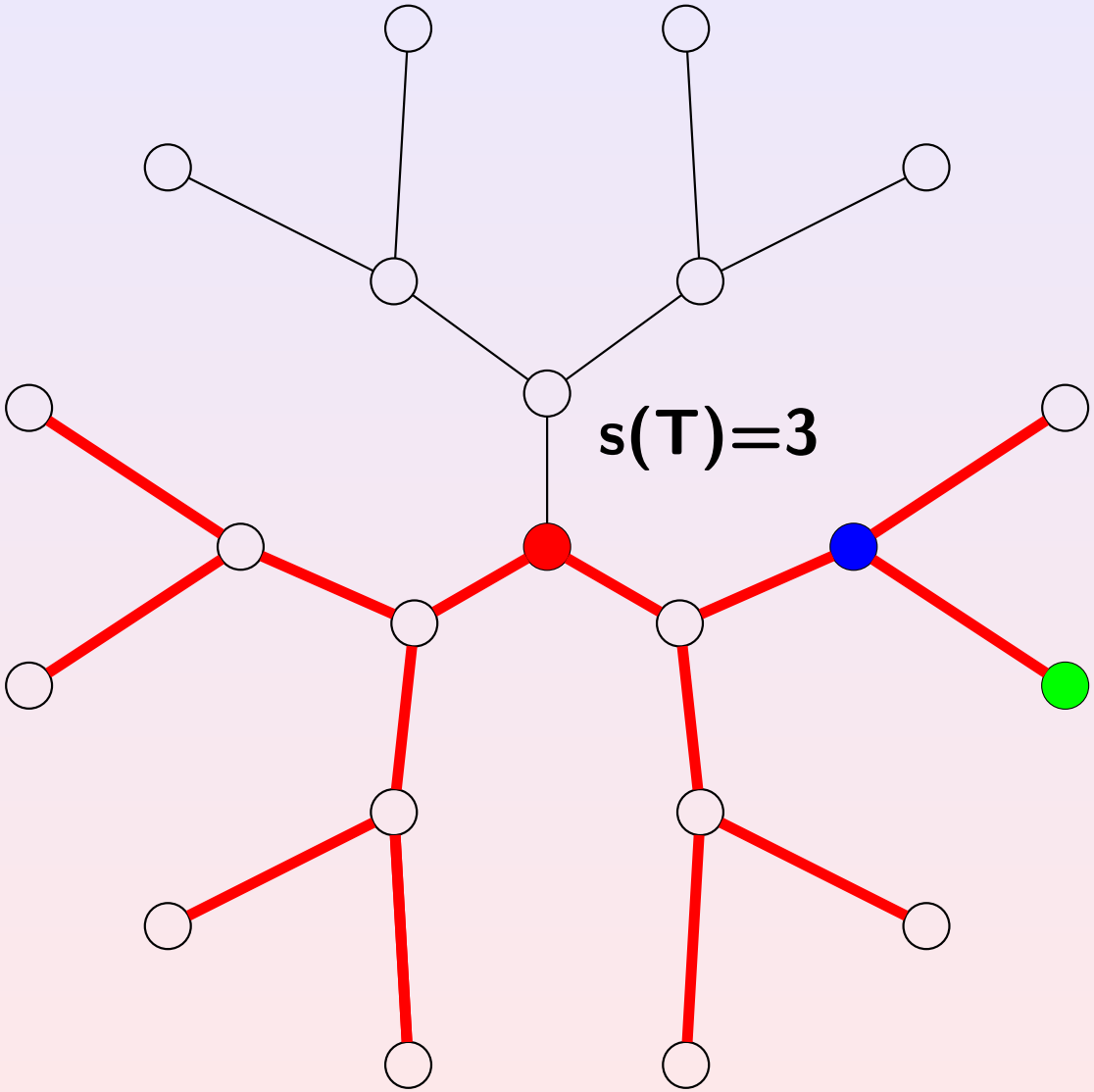
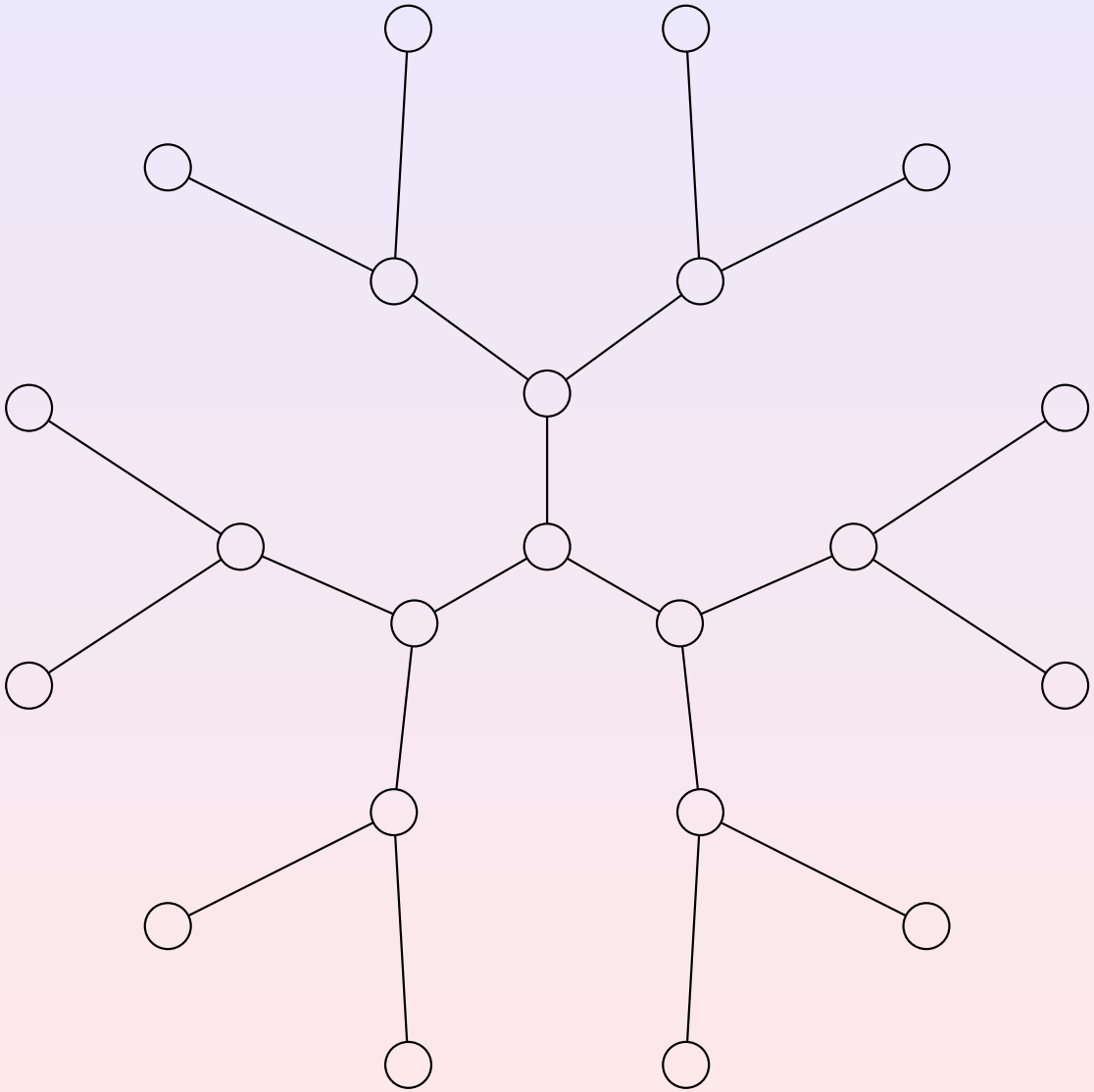


Visible Graph Searching

Visible fugitive

The fugitive is visible if, at every step, searchers know its position, i.e. the connected component of the contaminated part where the fugitive stands.
Let $vs(G)$ be the visible search number of the graph G.

Visible graph searching in a tree



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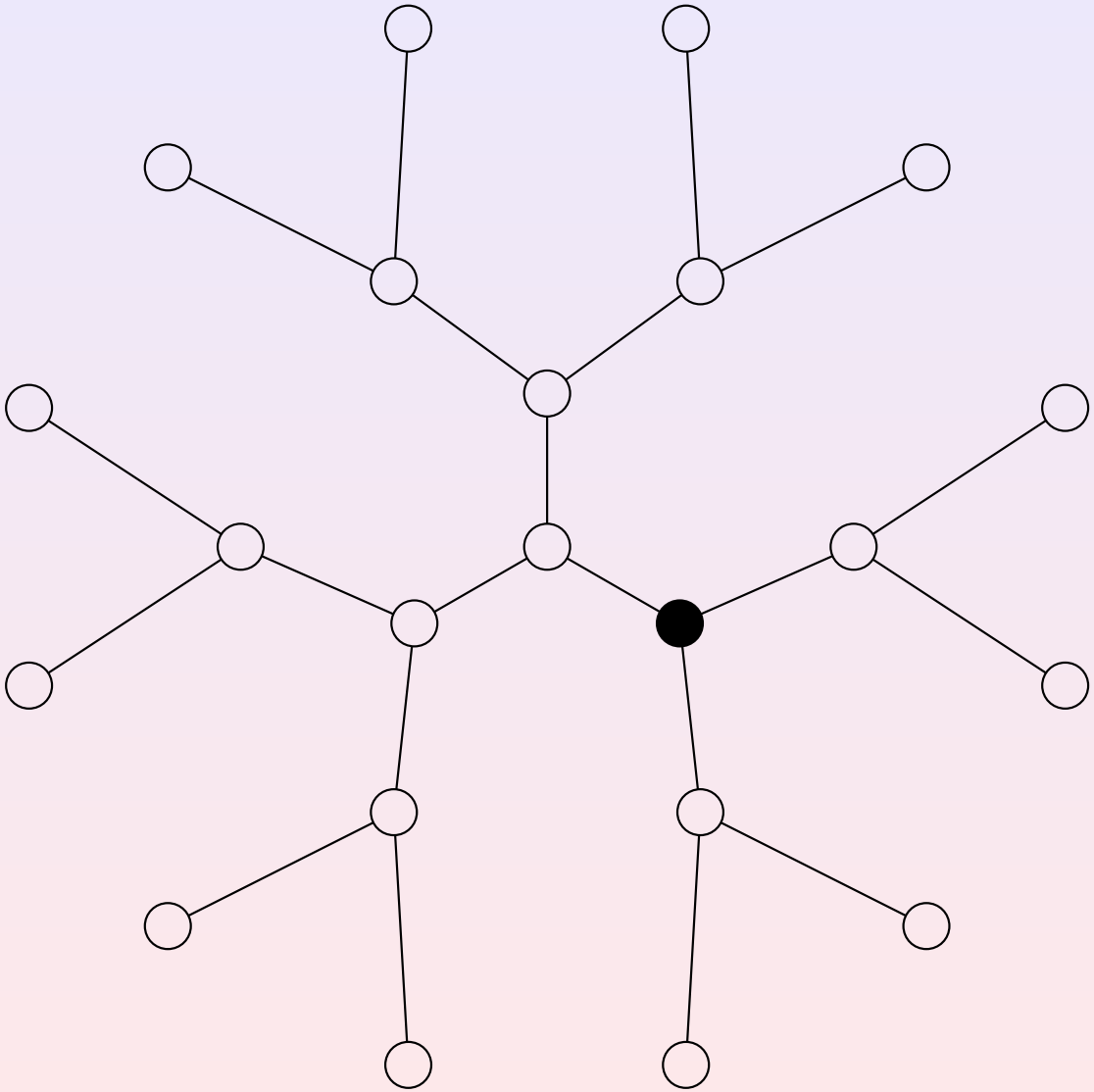

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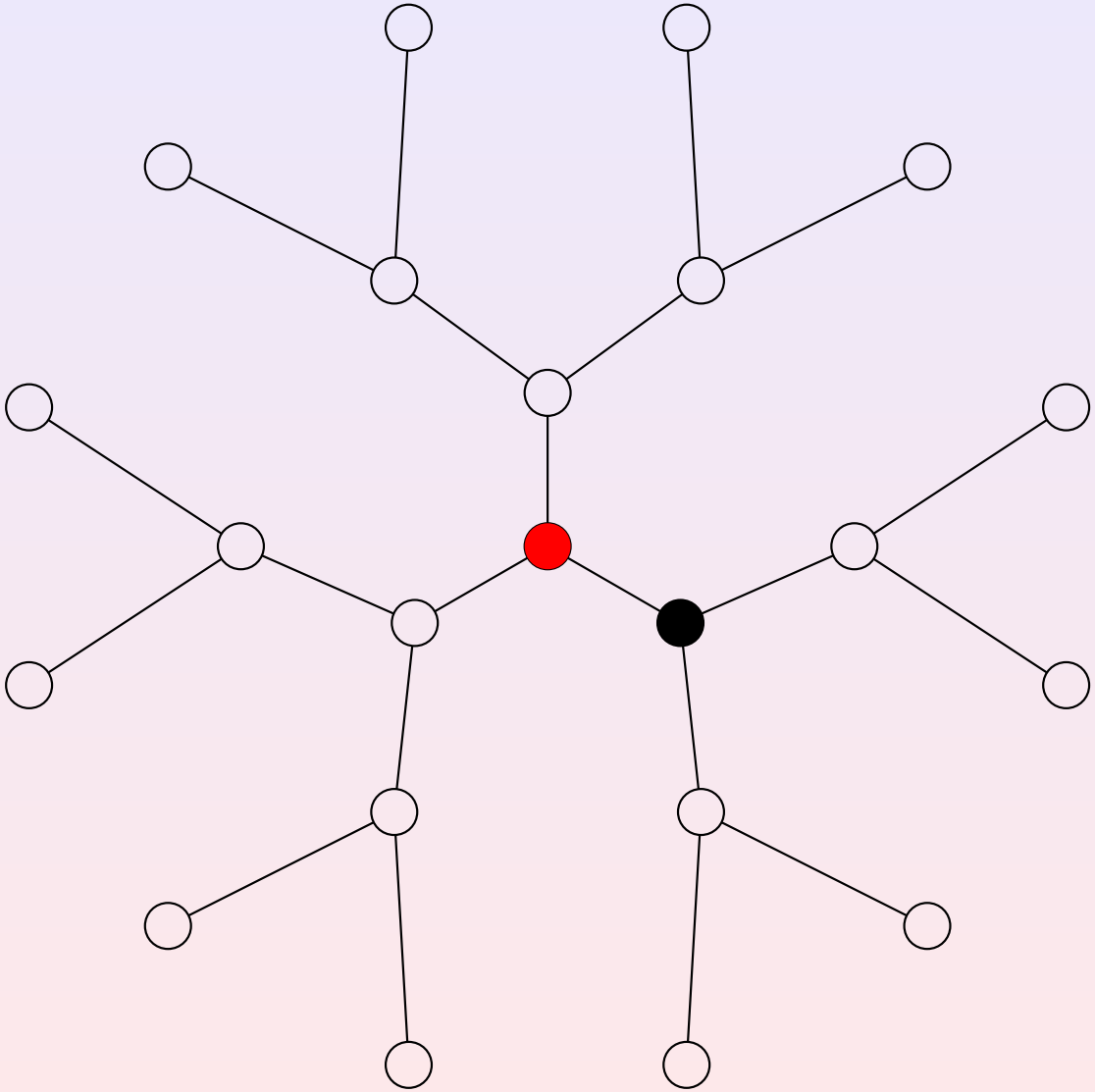

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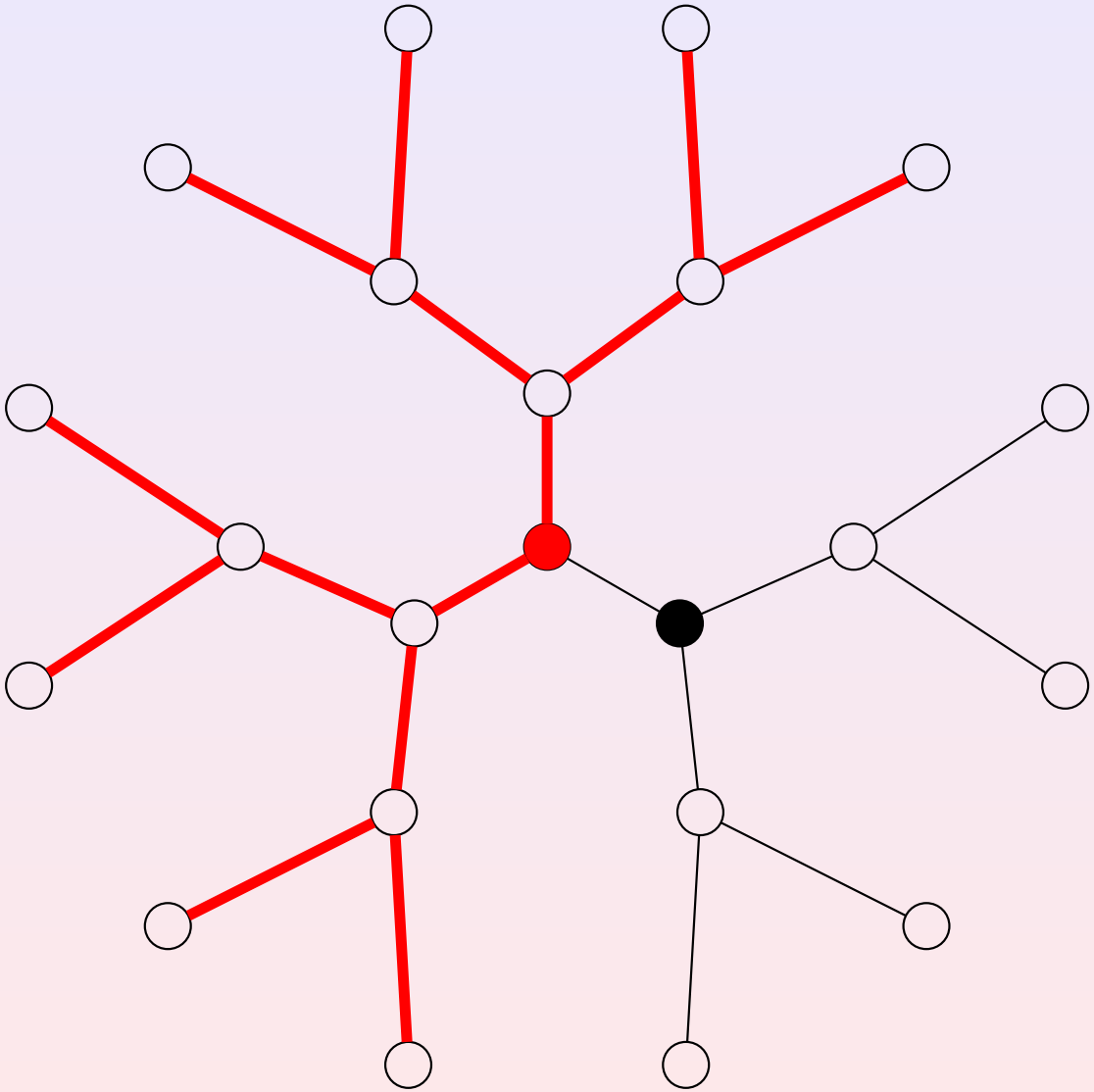

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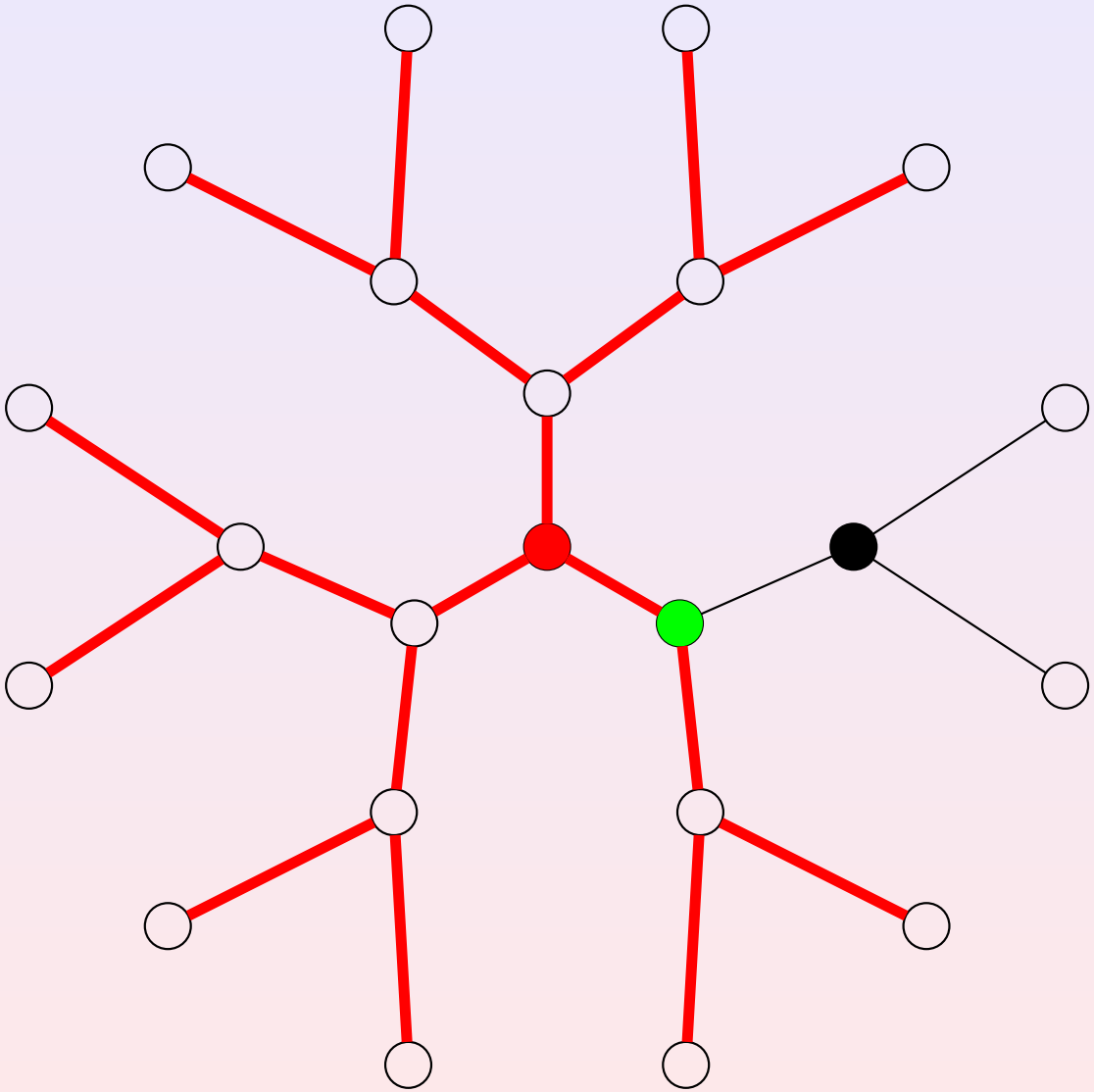

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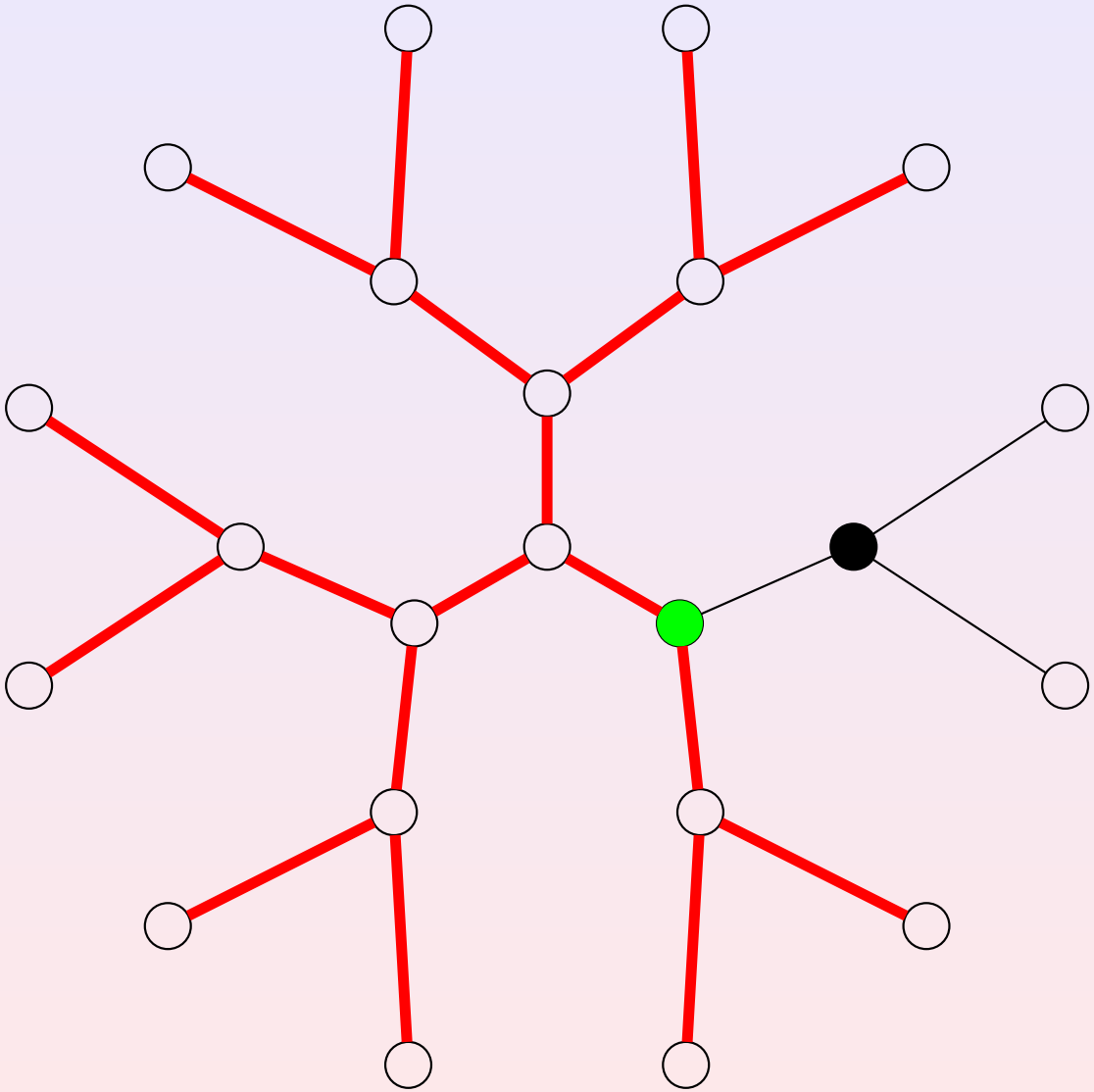

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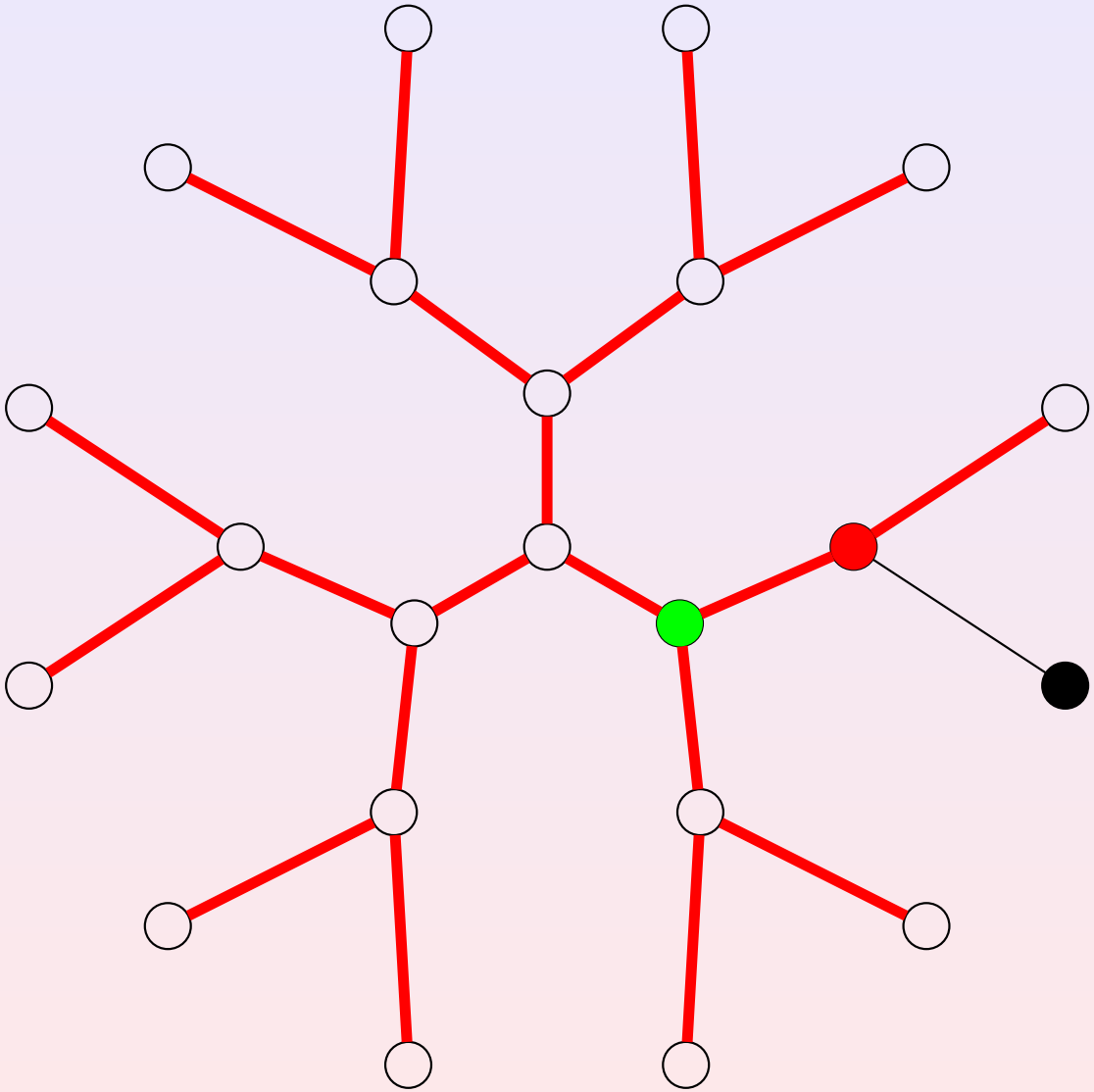
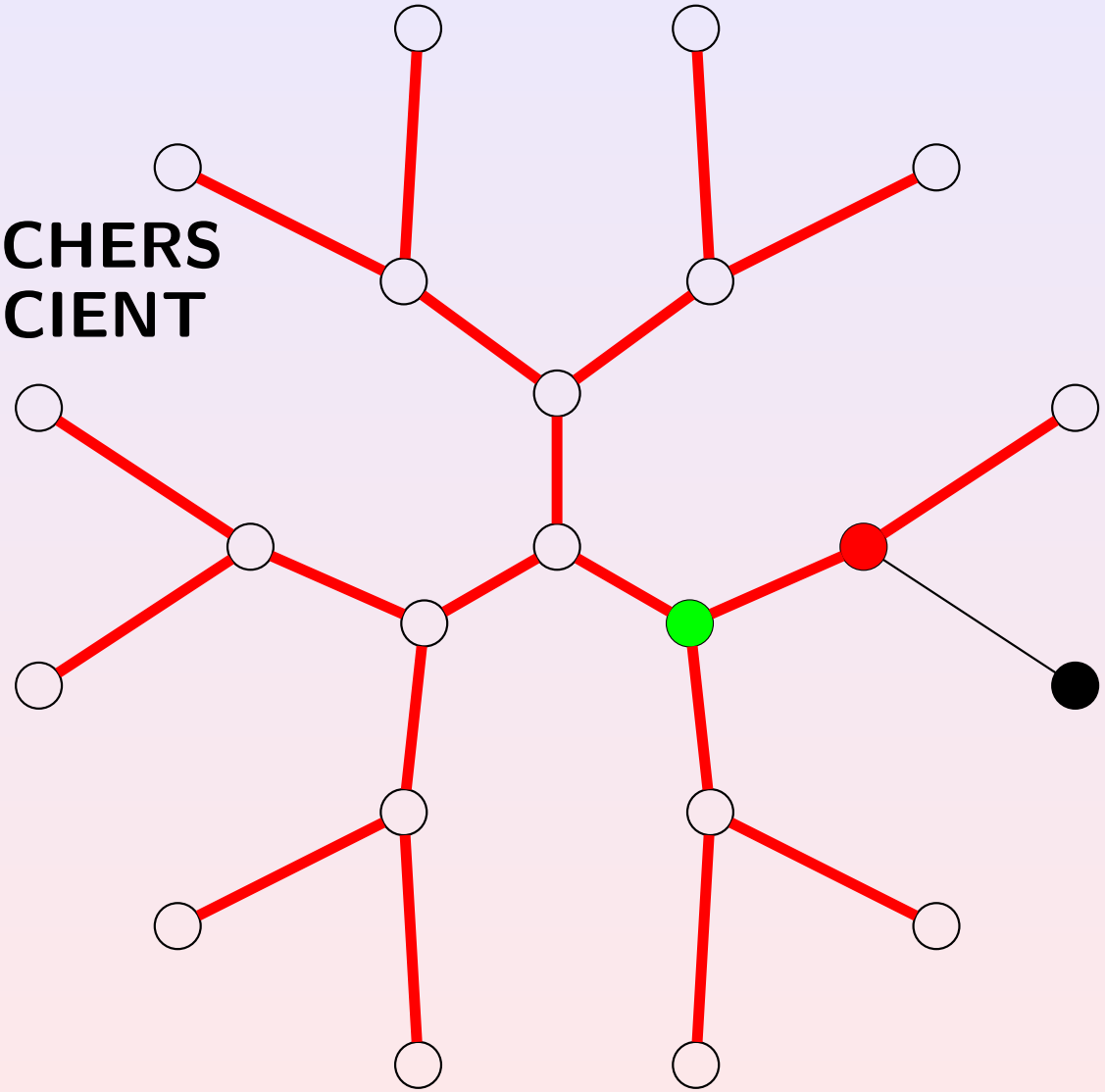


Visible graph searching in a tree

TWO SEARCHERS ARE SUFFICIENT



Monotony and Recontamination

Recontamination

A vertex v is recontaminated during a search strategy S if it has been cleared at a step s of S and the fugitive can access v at step $s' > s$.

Monotony

A search strategy is monotone if no recontamination ever occurs. That is, a cleared vertex remains clear until the end of the strategy.
Let ms(G) be the monotone search number of the graph G.

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Monotony and Recontamination

Does Recontamination help?

Is it more difficult to catch a fugitive in a monotone way? In other words, do we have $\mathbf{ms}(G) > \mathbf{s}(G)$?

Recontamination does not help: $s(G) = ms(G)$

- **Bienstock and Seymour**, J.of Alg., 1991
  Monotonicity in graph searching.
- **LaPaugh**, J.of ACM, 1993
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Case of a visible fugitive: $vs(G) = mvs(G)$

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In other words...

For any graph G, there exists a monotone search strategy using at most $\mathbf{s}(G)$ searchers (resp. $\mathbf{vs}(G)$ searchers in the visible case).

The corresponding decision problems are in NP

A monotone search strategy terminates after a linear number of steps. It gives a certificate checkable in polynomial time.

Another important consequence

We can restrict the analysis to monotone search strategies.

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- Searchers cannot move at will in a real network;
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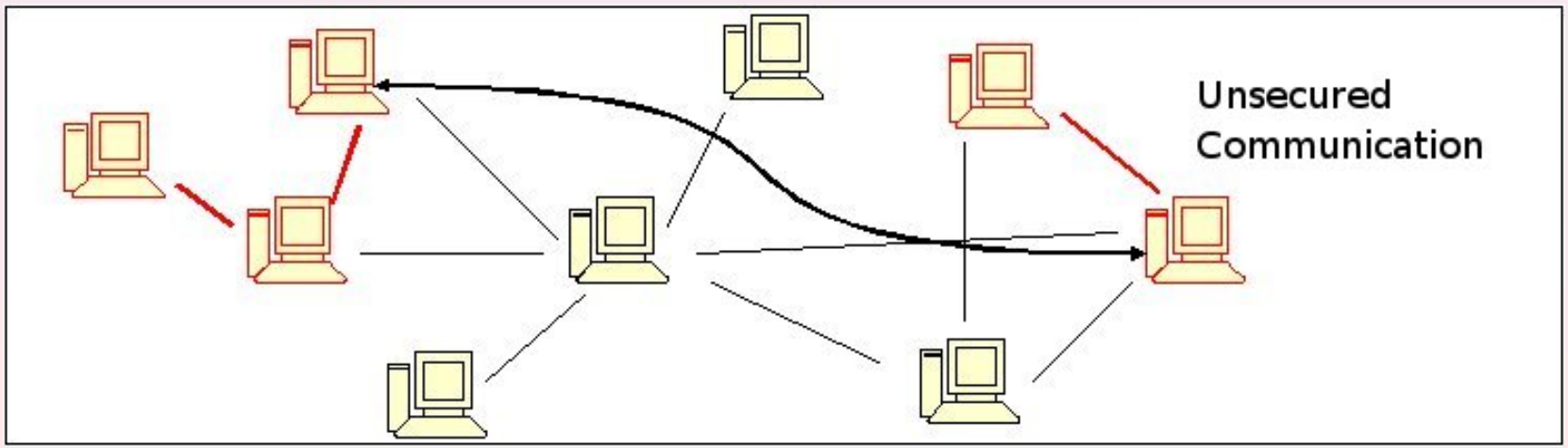

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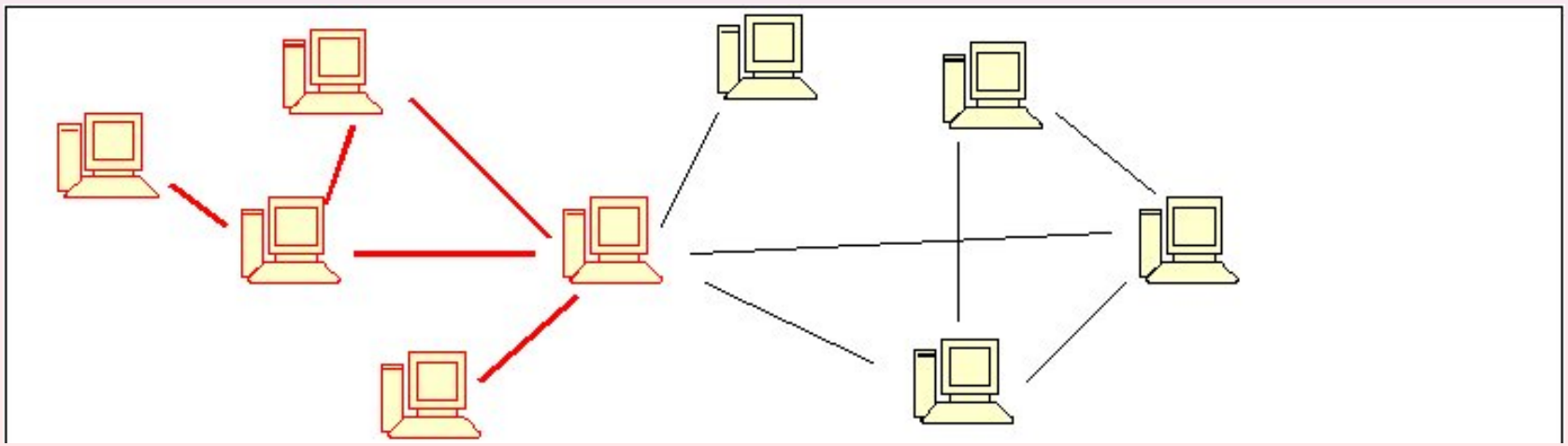
Connected Search Strategy

At any step, the cleared part of the graph must induce a connected subgraph.
Let $cs(G)$ be the connected search number of the graph G.



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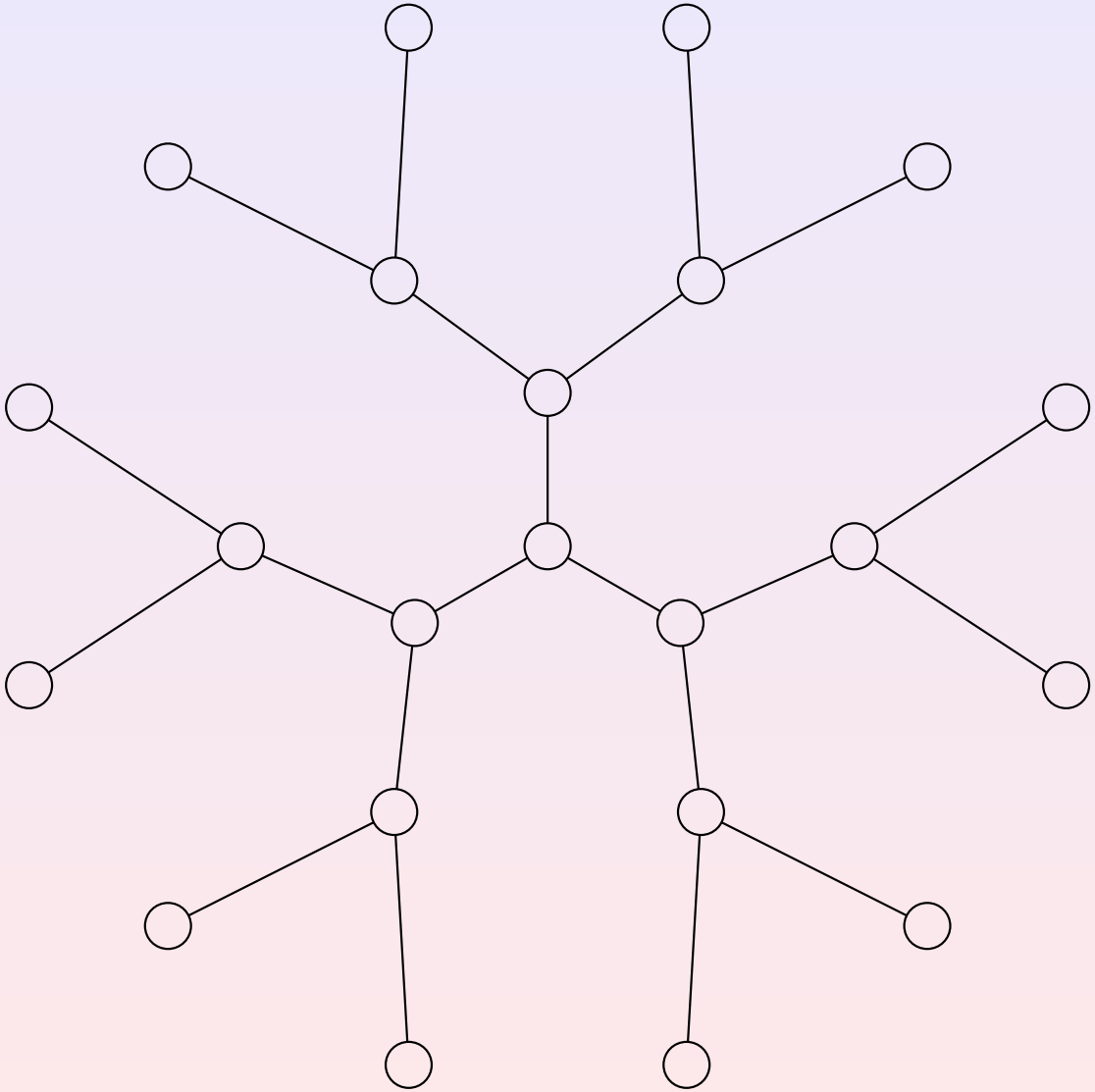

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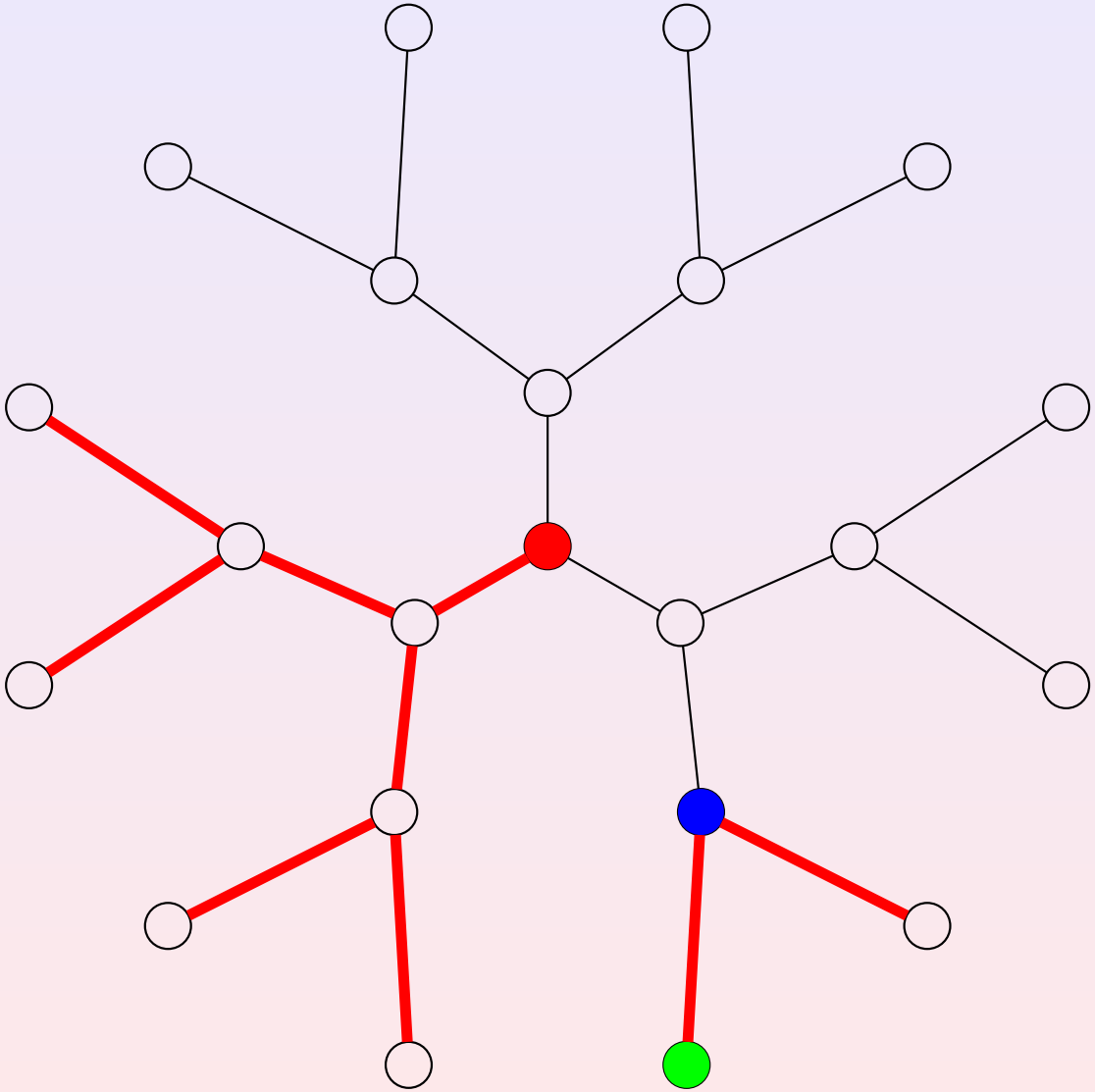

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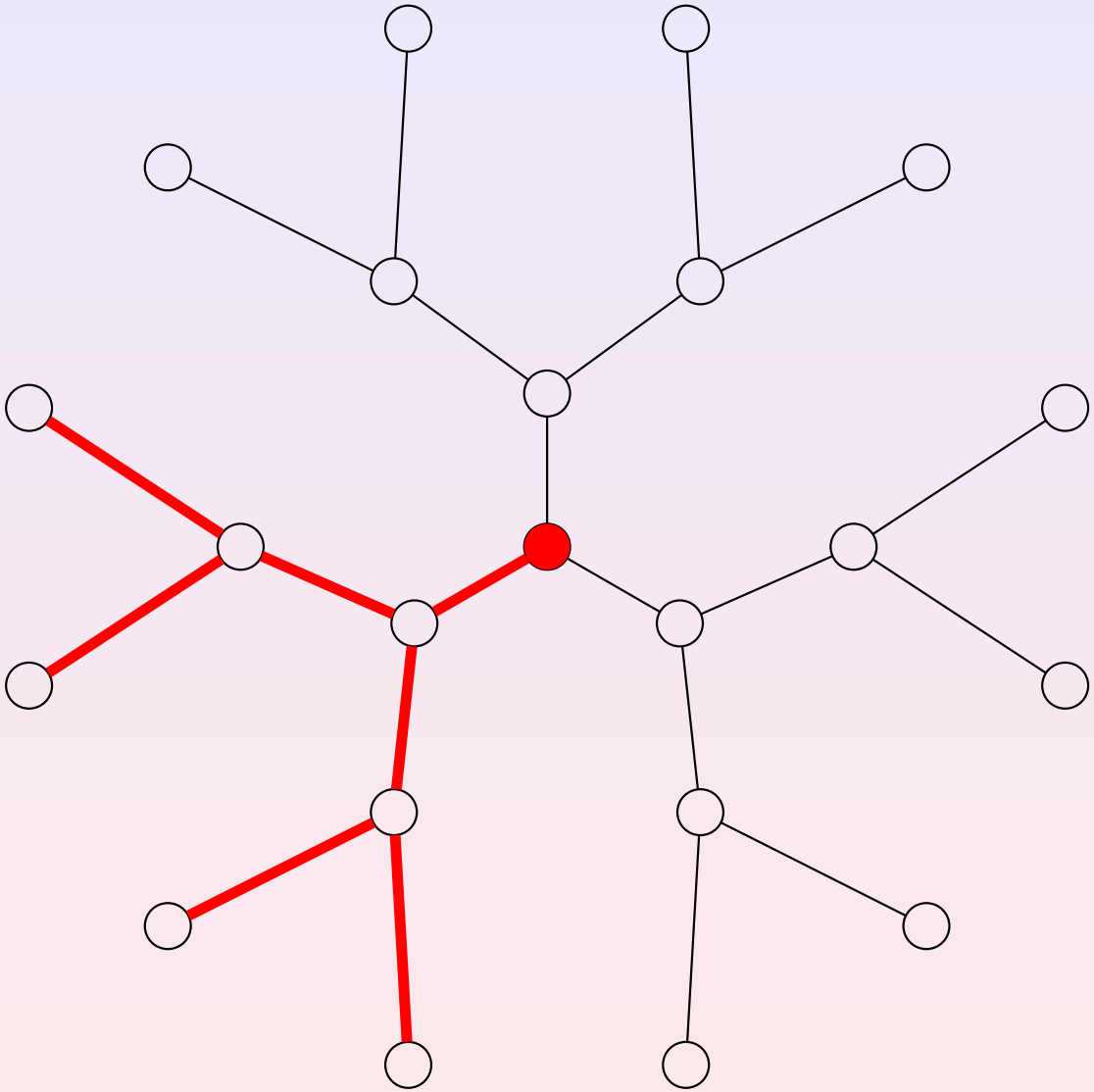

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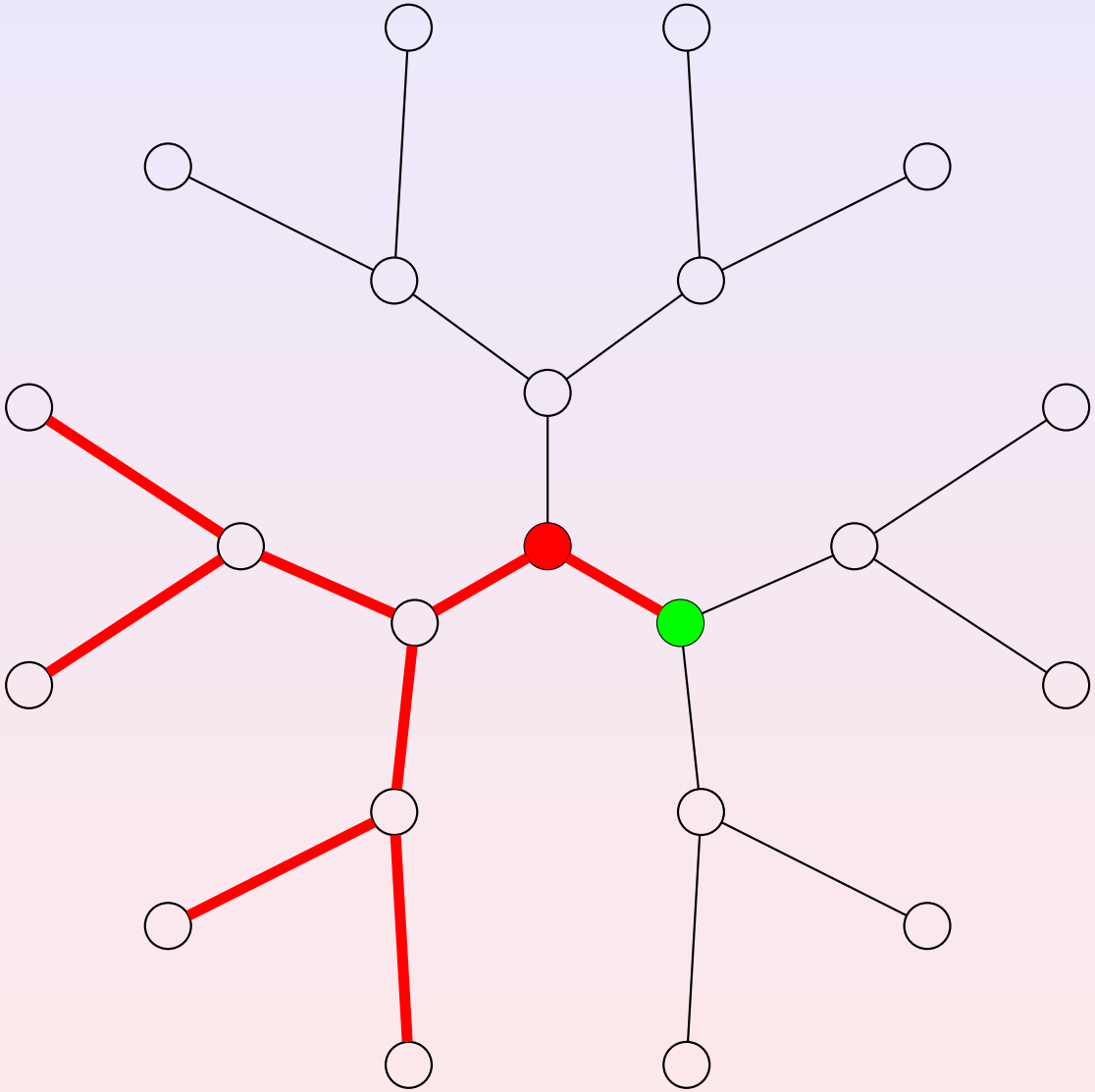

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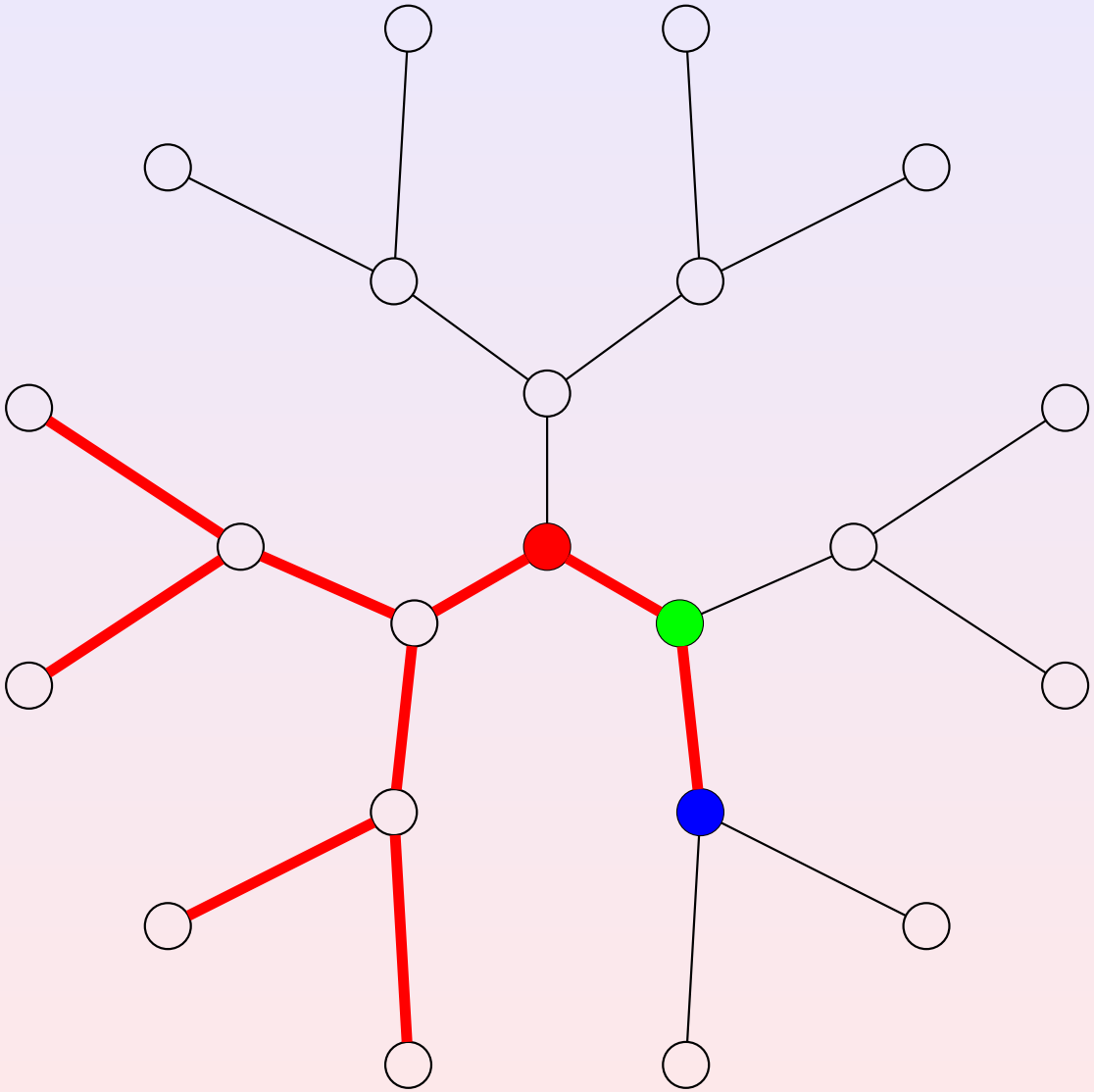

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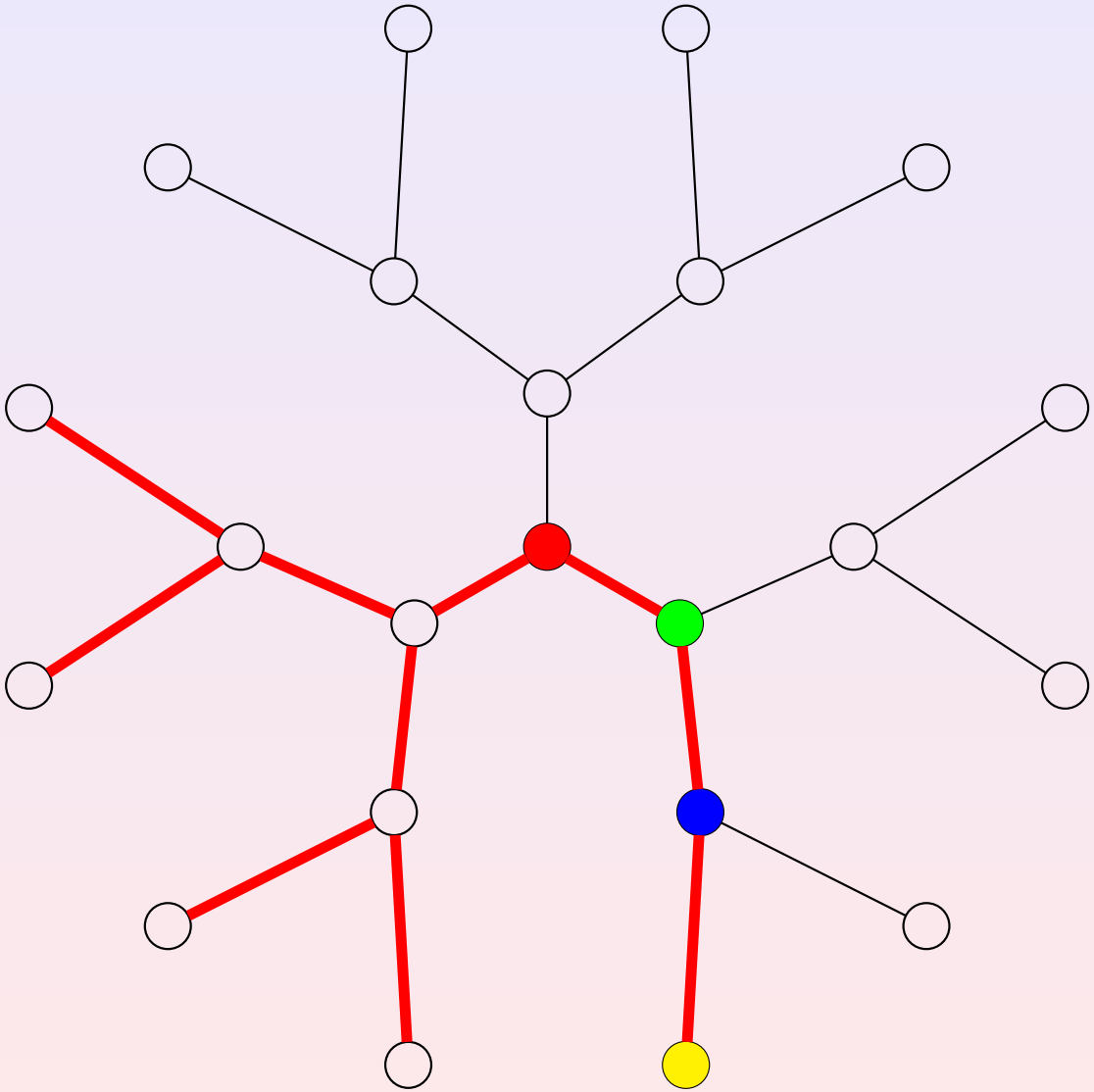

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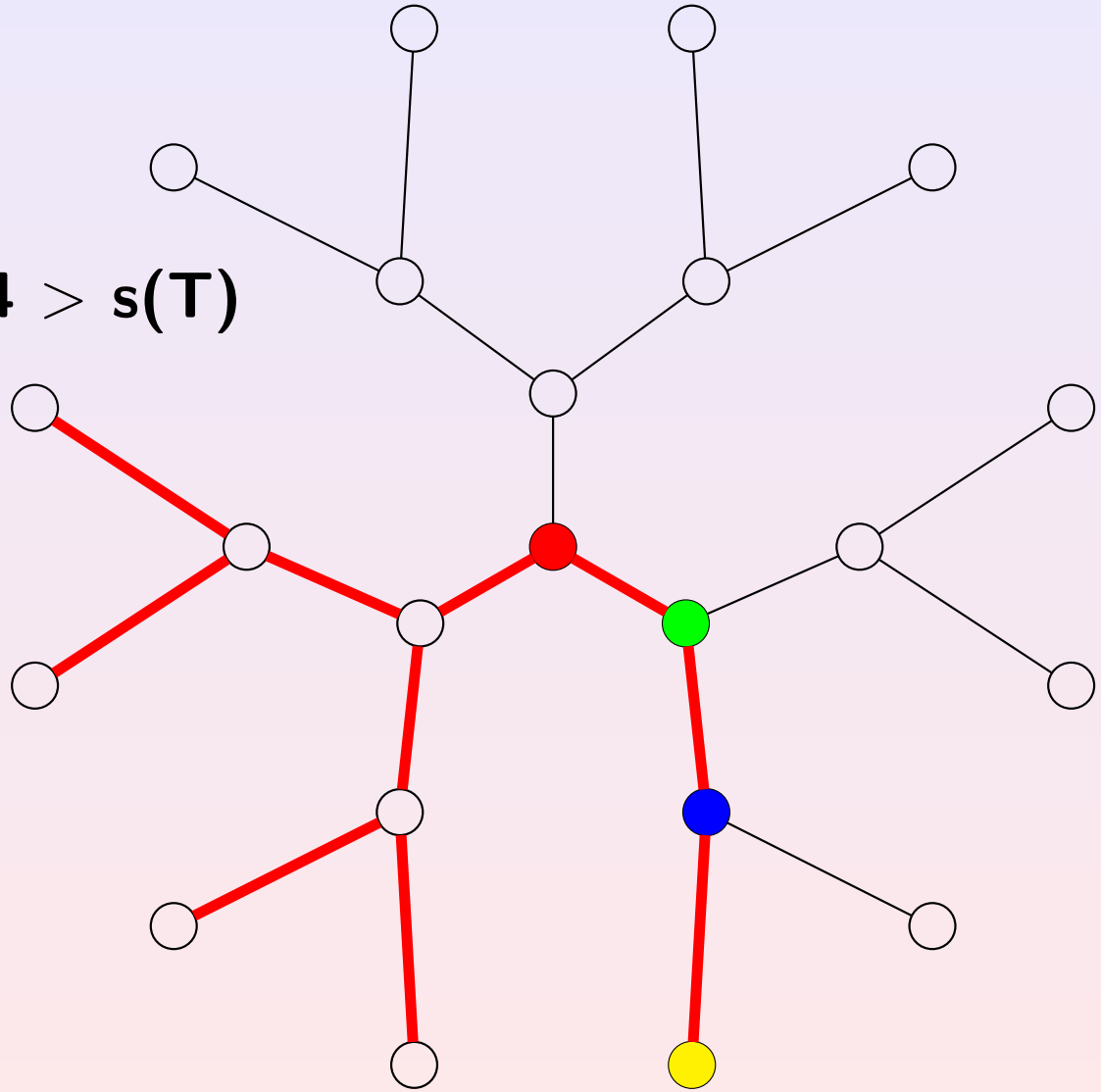
$\mathbf{cs(T) \geq 4 > s(T)}$



Price of connectedness

Barrière, Fraigniaud, Santoro and Thilikos. [WG, 2003]

For any tree T, $s(T) \leq cs(T) \leq 2\ s(T) - 2$.
Moreover, these bounds are tight.

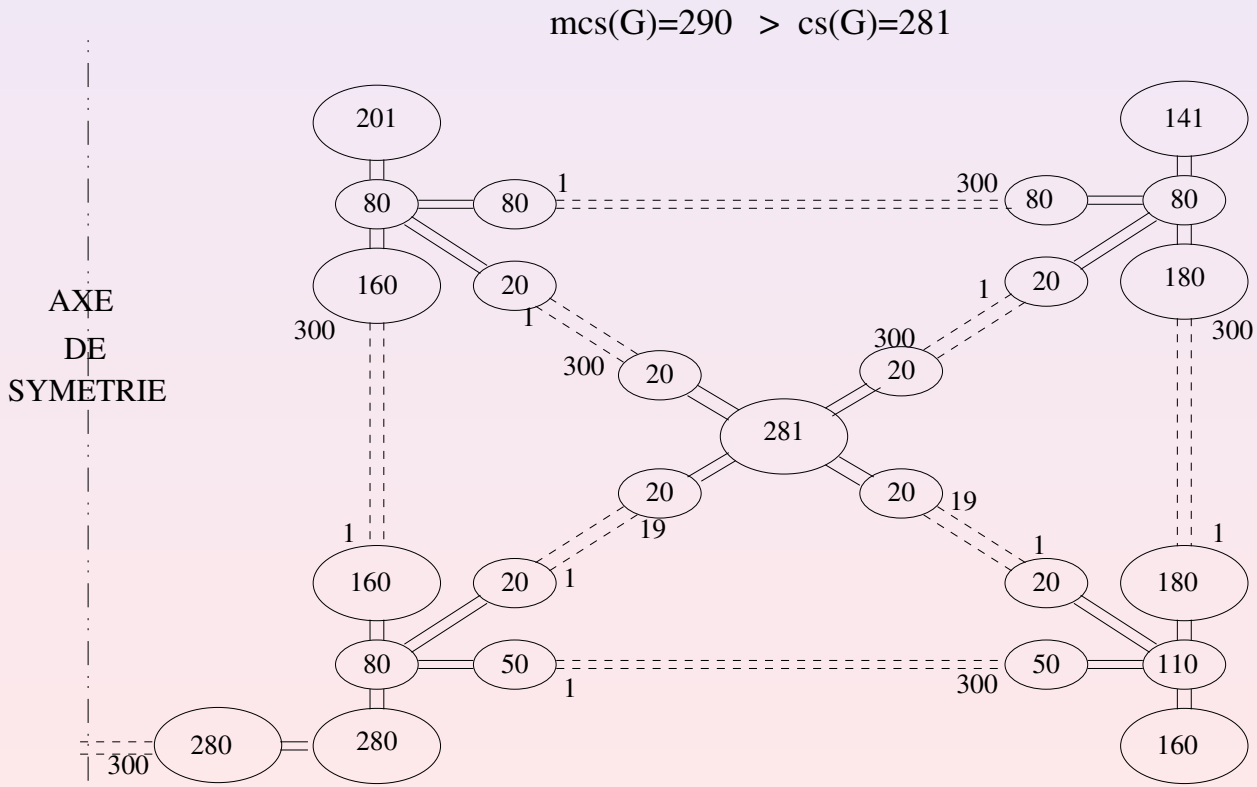
Fraigniaud and Nisse [LATIN, 2006]

For any n-node graph G, $s(G) \leq cs(G) \leq (1 + \log n)\ s(G)$.

Price of connectedness

In the connected case, Recontamination helps

Yang, Dyer, Alspach. Sweeping Graphs and Digraphs. [ISAAC04]



Related Work

Ratio monotone connected search over search

	in trees	in arbitrary n-node graphs
invisible fugitive	2(tight) [BFF^+03]	$O(\log n)$ [$FN06$]
visible fugitive	1 [trivial]	

An overview of monotony in graph searching

	search in arbitrary graphs	connected search in trees	connected search in arbitrary graphs
invisible fugitive	monotone [$BS91, L93$]	monotone [BFF^+02]	not monotone [$YDA04$]
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Visible Connected Graph Searching

Theorem 1

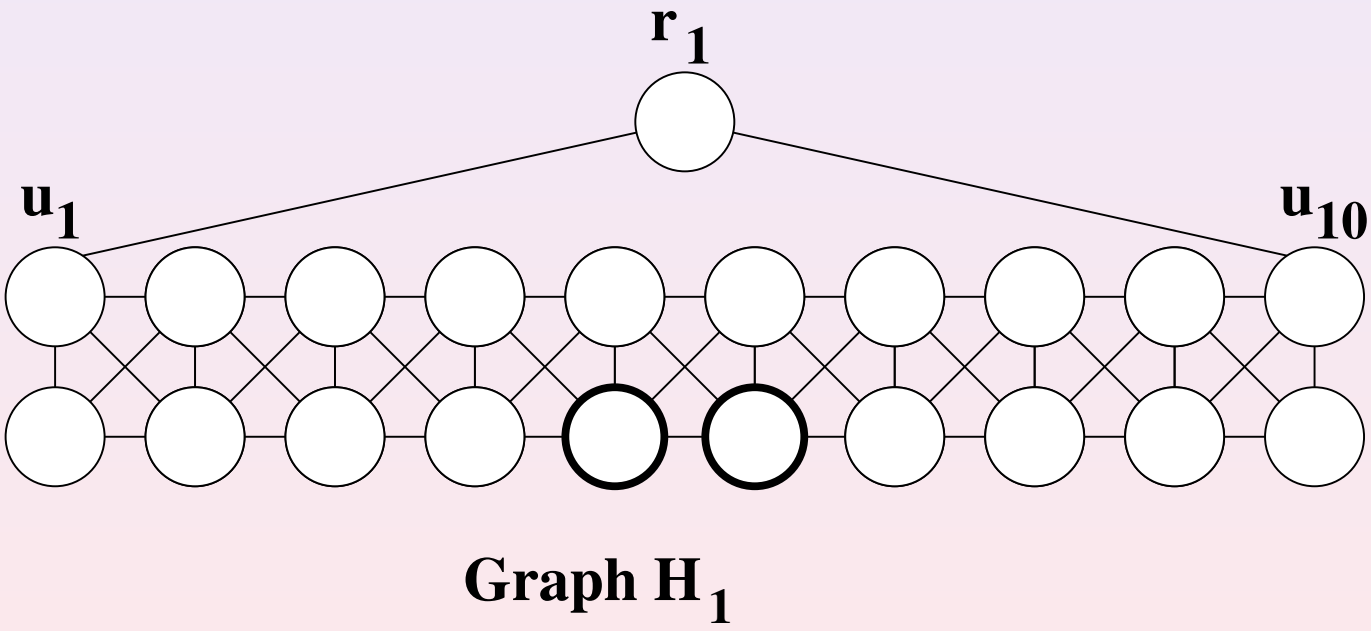
For any n_0, there is $n \geq n_0$ and an n-node graph G such that any monotone connected visible search for G uses at least $\Omega(\mathbf{vs}(G) \cdot \log n)$ searchers.

More precisely

We propose a sequence of n_i-node graph $(\mathcal{G}_i)_{i \in N}$, such that, the sequence $(n_i)_{i \in N}$ is strictly increasing, and, for any $i > 1$, $\mathbf{vs}(\mathcal{G}_i) \leq 5$ and $\mathbf{mcvs}(\mathcal{G}_i) \geq \Omega(\log n_i)$.

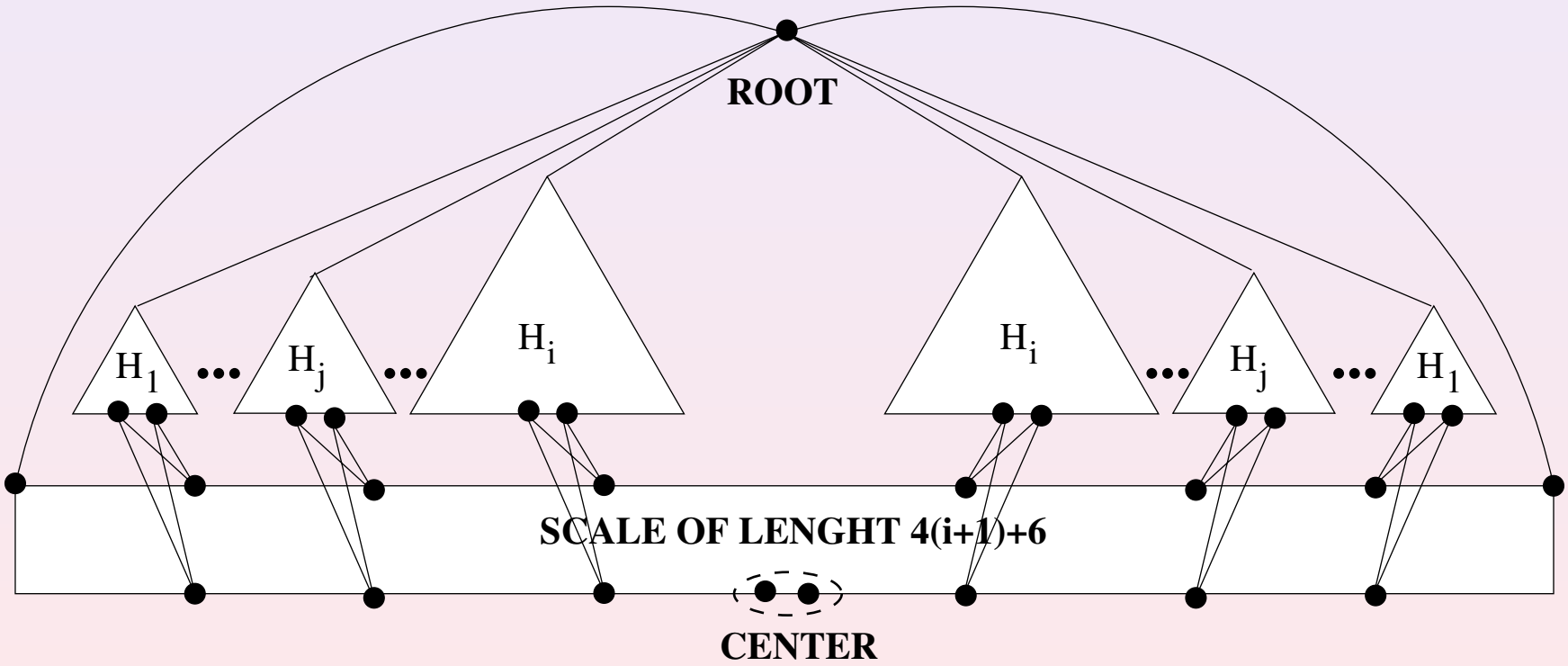
Squetch of the proof by induction on i

The sequence of graphs $(H_i)_{i \in N}$ is defined recursively.
H_1 consists of a scale of length 10 plus a root r_1.



Squetch of the proof by induction on i

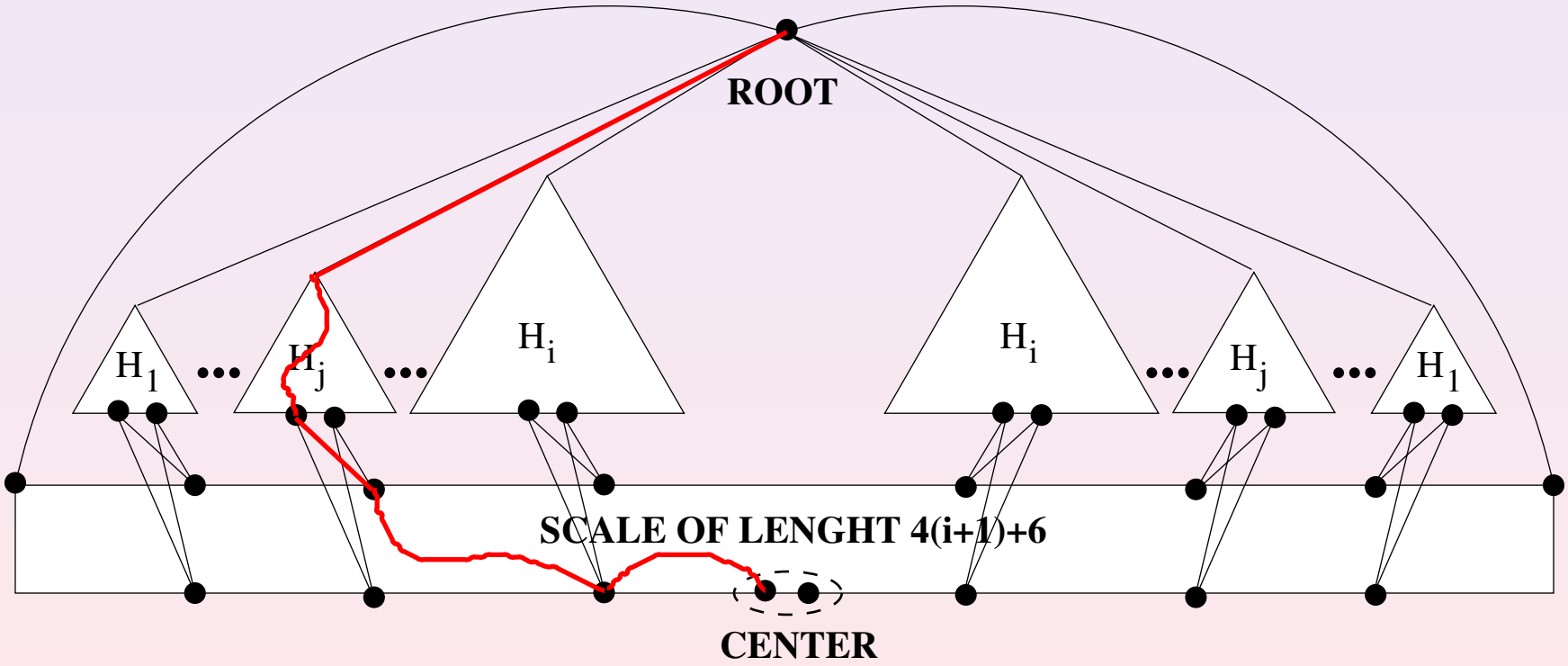
For any $i > 1$, the graph H_{i+1} is built
using 2 copies of $H_1, \cdots, H_i$ and a scale of length $4(i+1)+6$.



Graph H_{i+1}

Squetch of the proof by induction on i

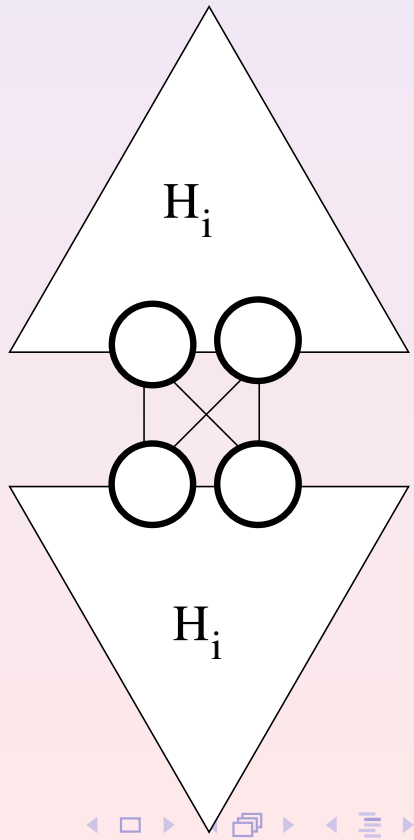
Lemma: Catching the fugitive in the root, starting from the center requires at least $2i$ searchers.



Graph H_{i+1}

Squetch of the proof by induction on i

For any $i \geq 1$, $\mathcal{G}_i$ consists of 2 copies of H_i linked by a 4-clique.

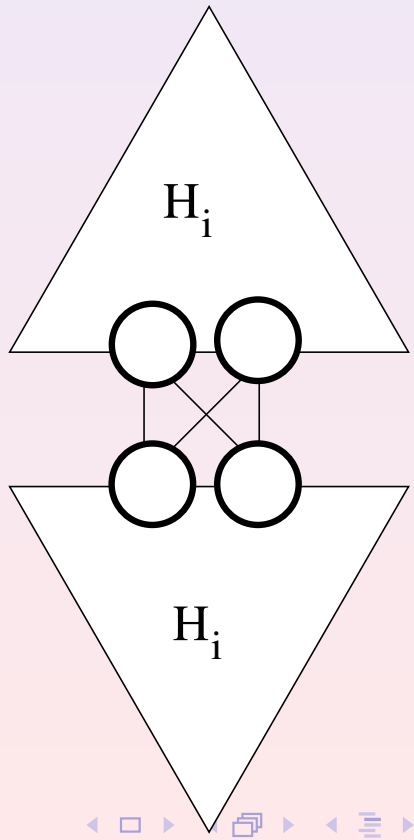


Squetch of the proof by induction on i

For any $i \geq 1$, $\mathcal{G}_i$ consists of 2 copies of H_i linked by a 4-clique.

Visible search of G_i

Using tree-decomposition, we prove by induction that $vs(G_i) \leq 5$.



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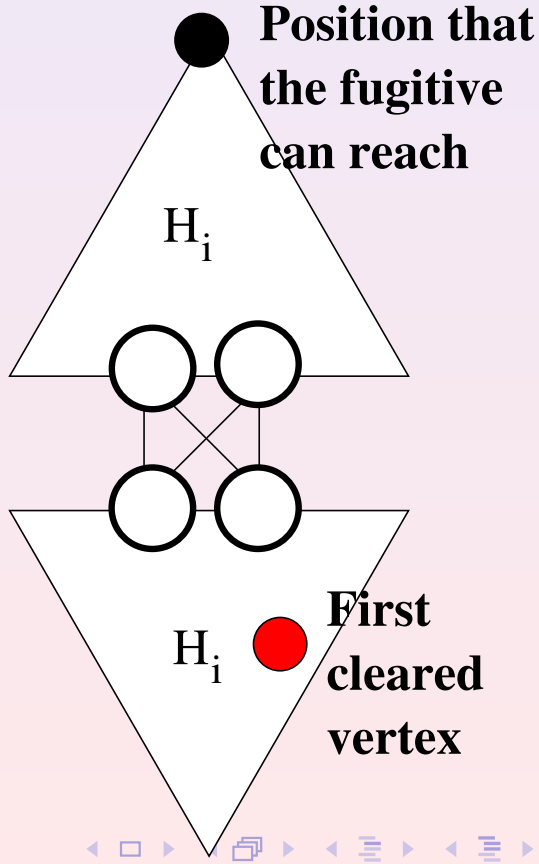
Whatever the first cleared vertex is, the fugitive can flee in the other copy of H_i.



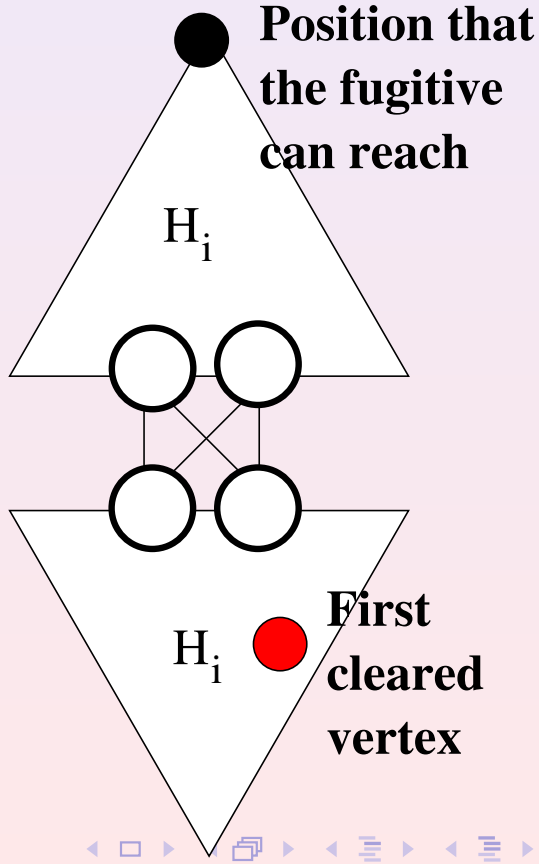
Squetch of the proof by induction on i

For any $i \geq 1$, $\mathcal{G}_i$ consists of 2 copies of H_i linked by a 4-clique.

Visible search of G_i

Using tree-decomposition, we prove by induction that $vs(G_i) \leq 5$.

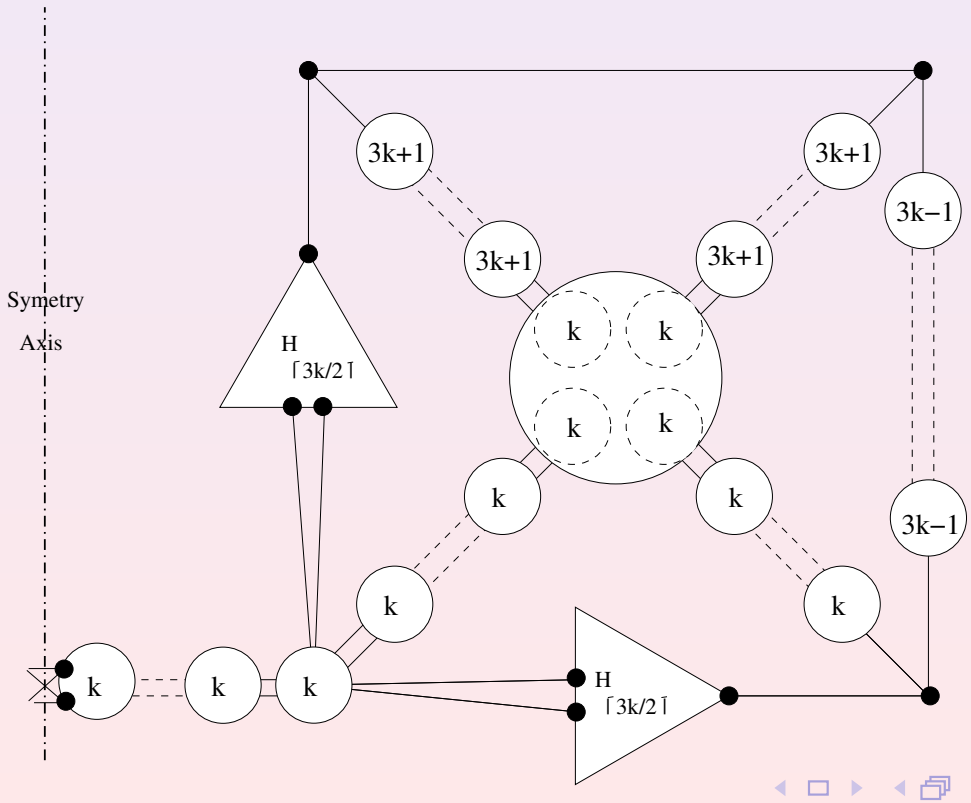
Whatever the first cleared vertex is, the fugitive can flee in the other copie of H_i.
Thus, $2i$ searchers are required, i.e. about $\mathbf{vs}(G_i) \log n$ searchers.



Monotony in Visible Connected Graph Searching

Theorem 2

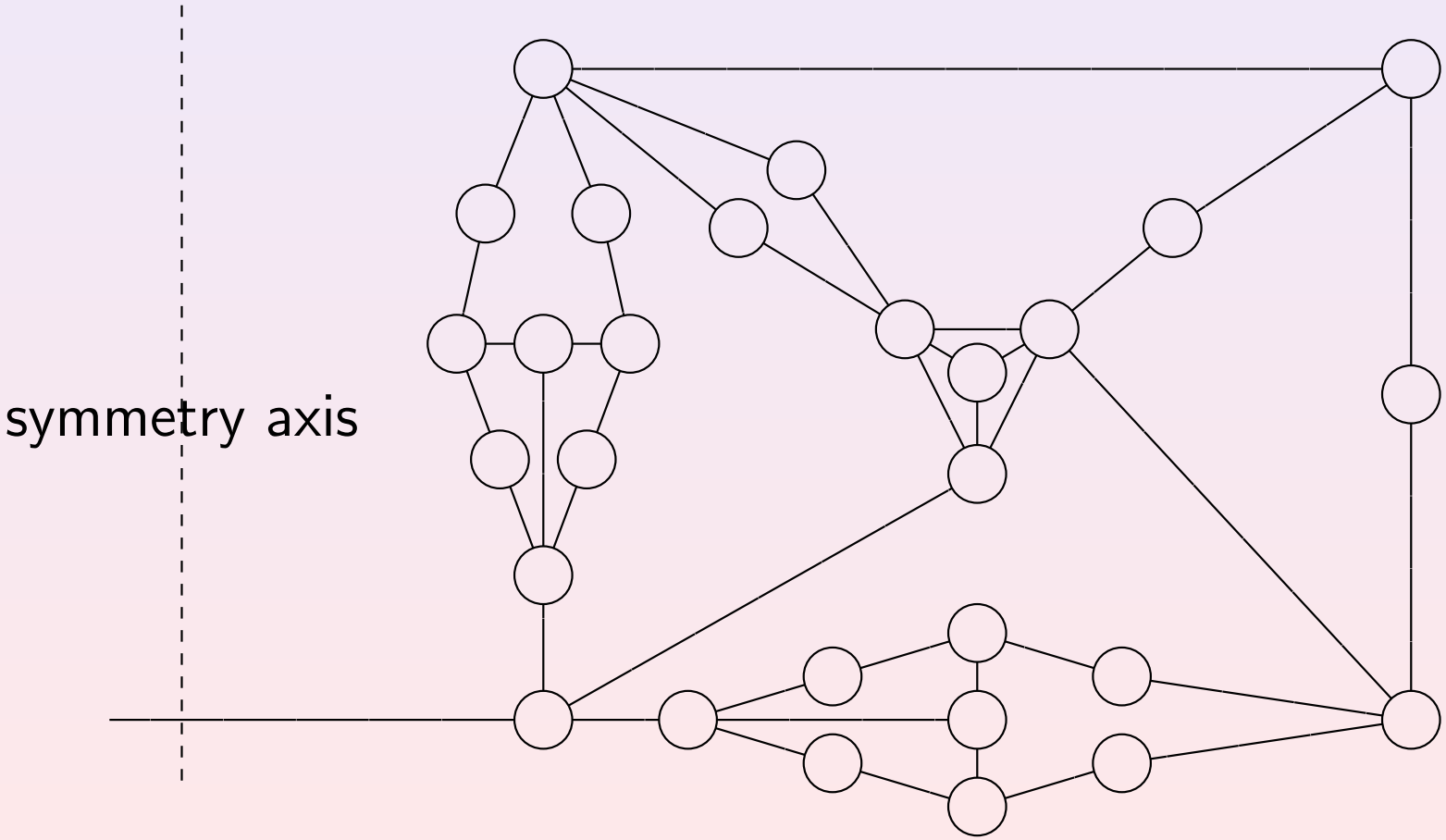
For any $k \geq 4$, there exists a graph G such that $\mathbf{cvs}(G) = 4k + 1$ and any monotone connected visible search strategy uses at least $4k + 2$ searchers.



Monotony in Visible Connected Graph Searching

Recontamination helps

Let G be the graph below: $mcvs(G) > cvs(G) = 4$.



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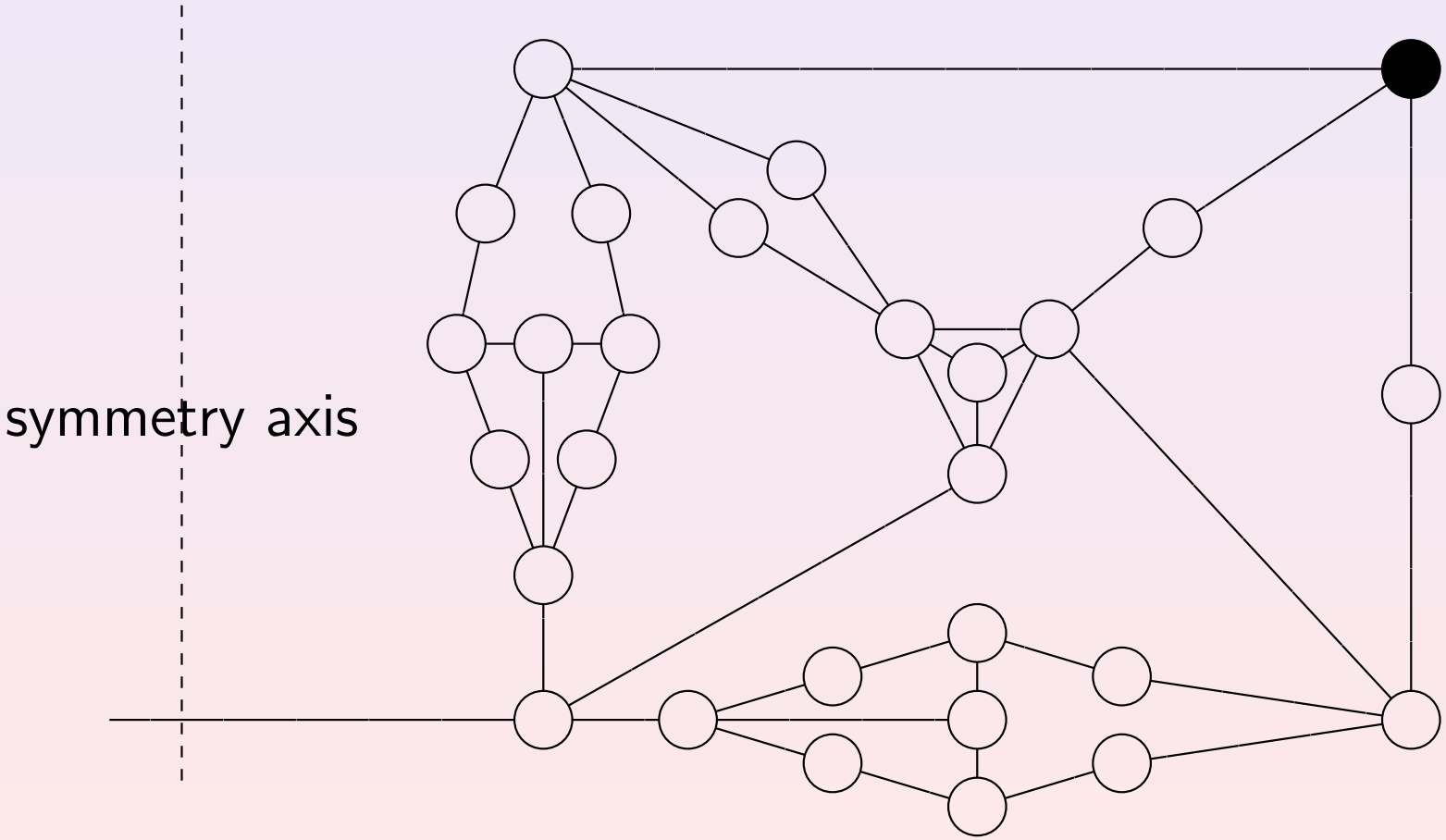

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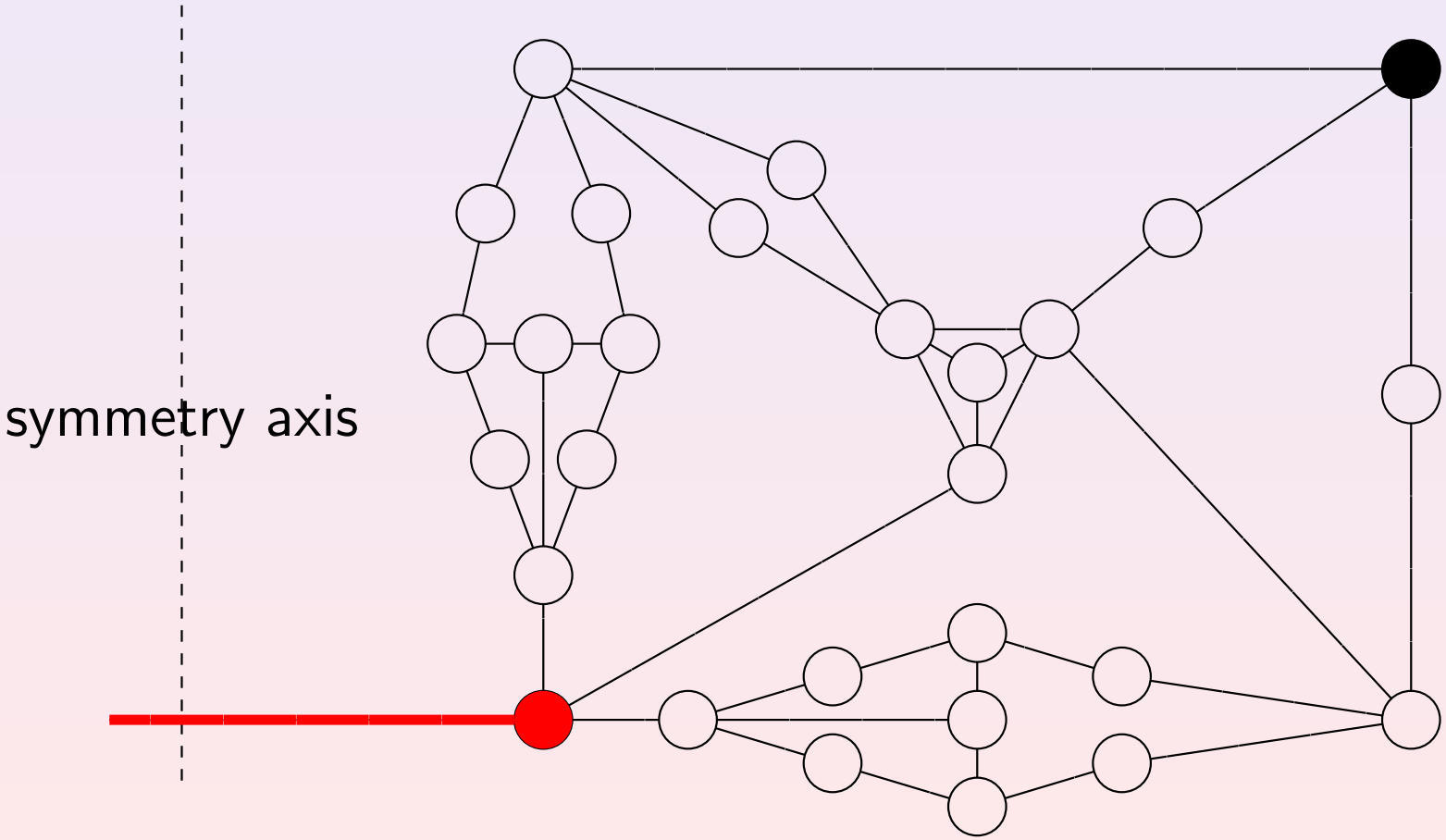

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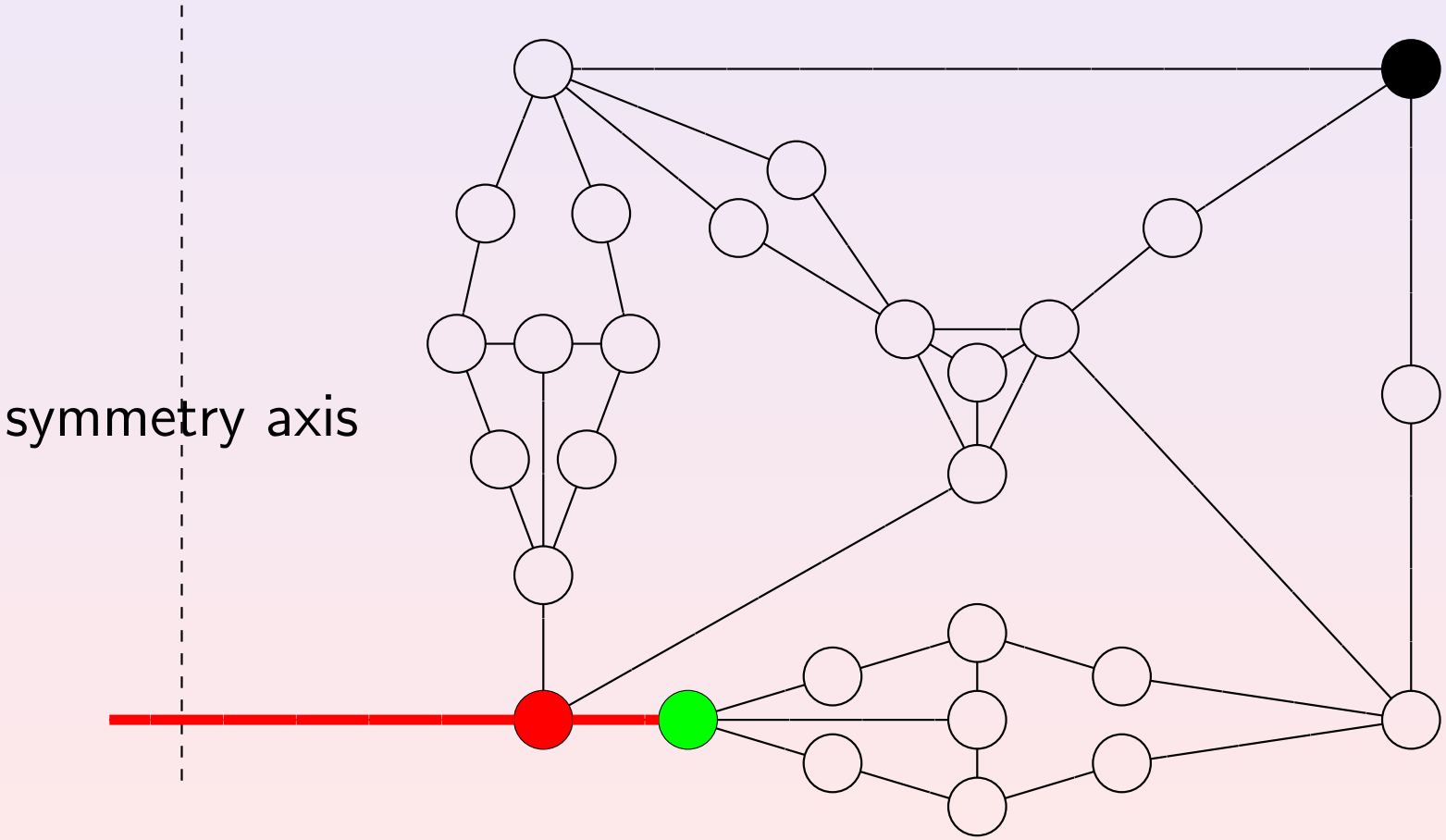

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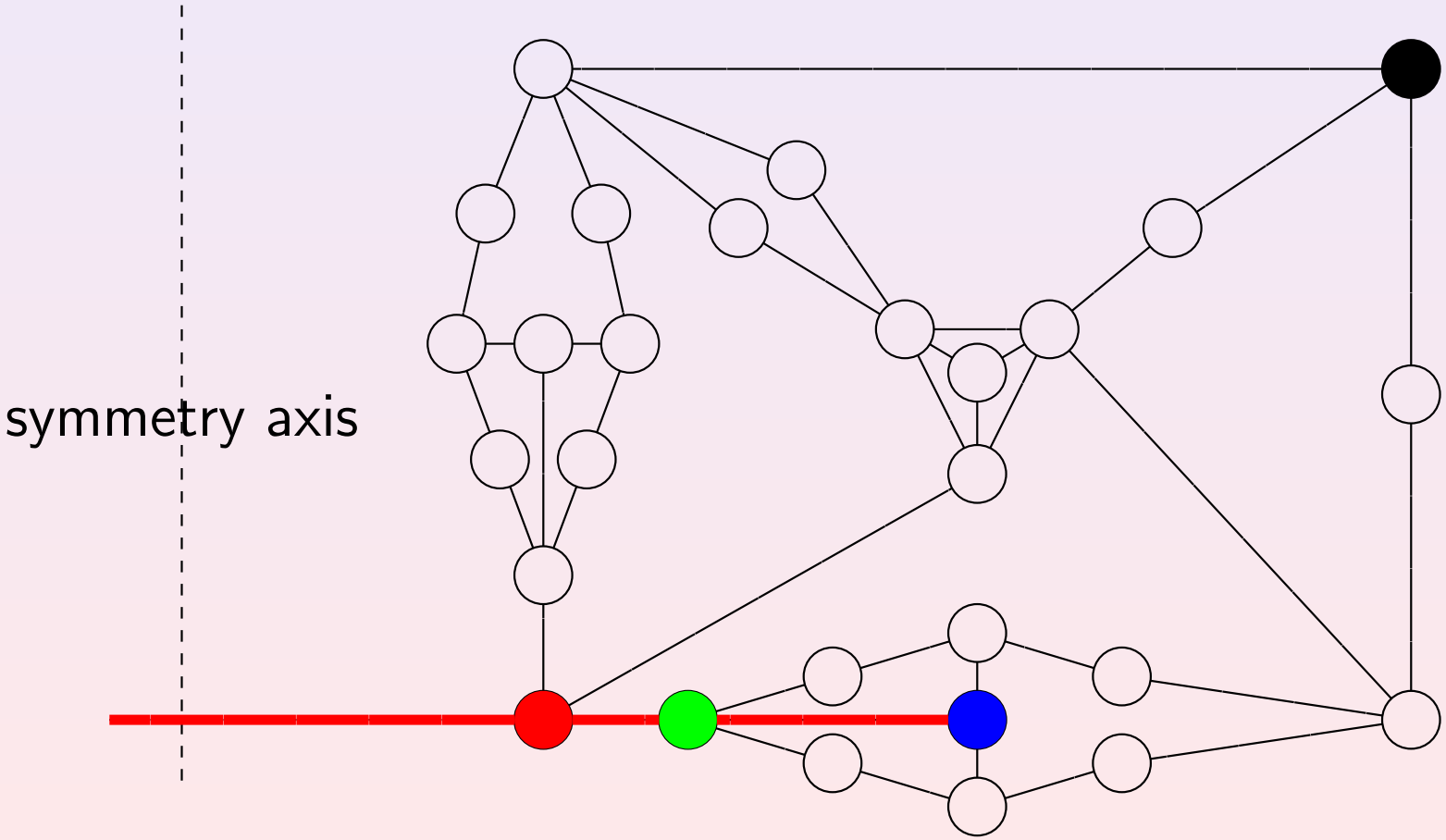

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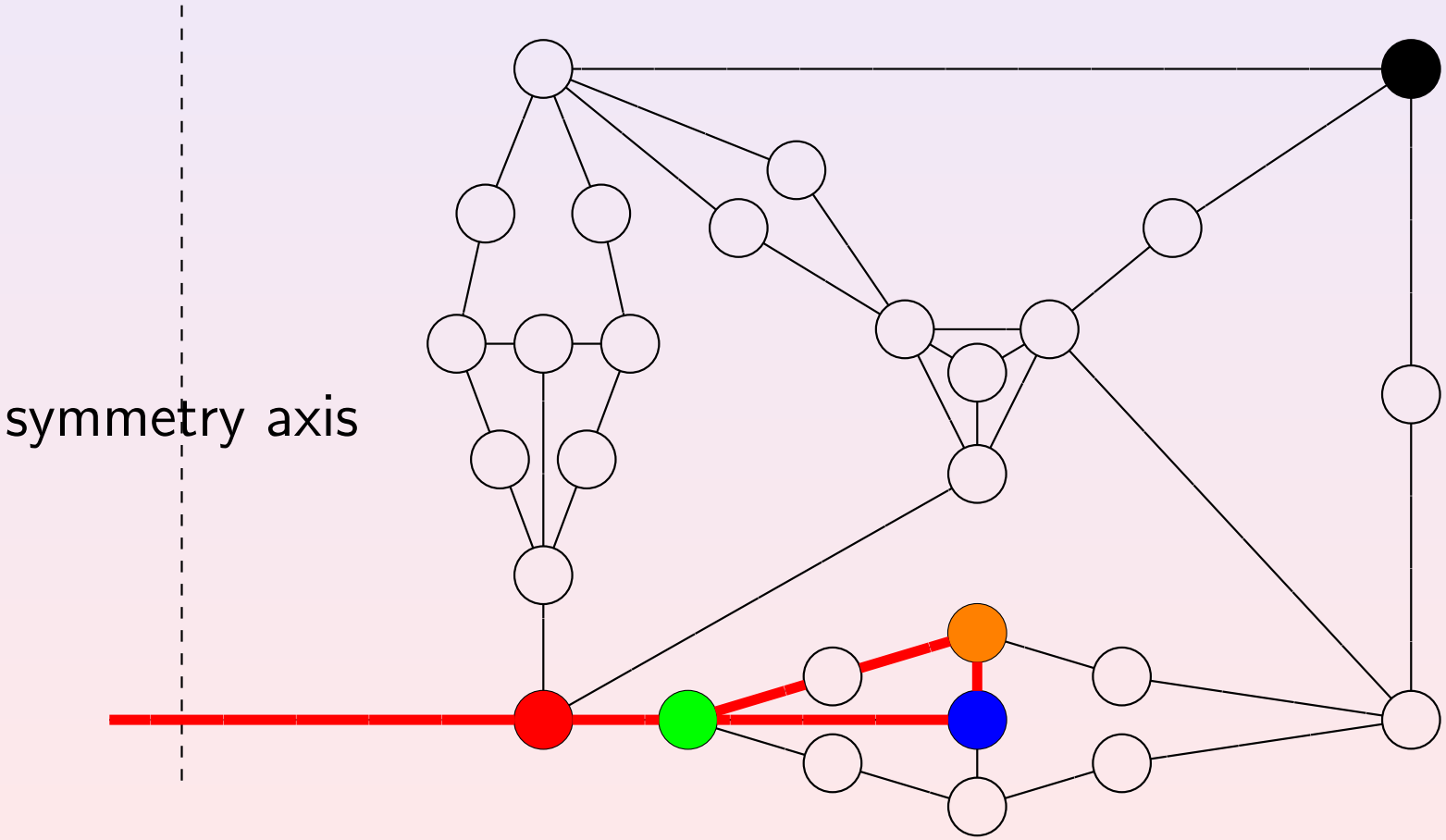

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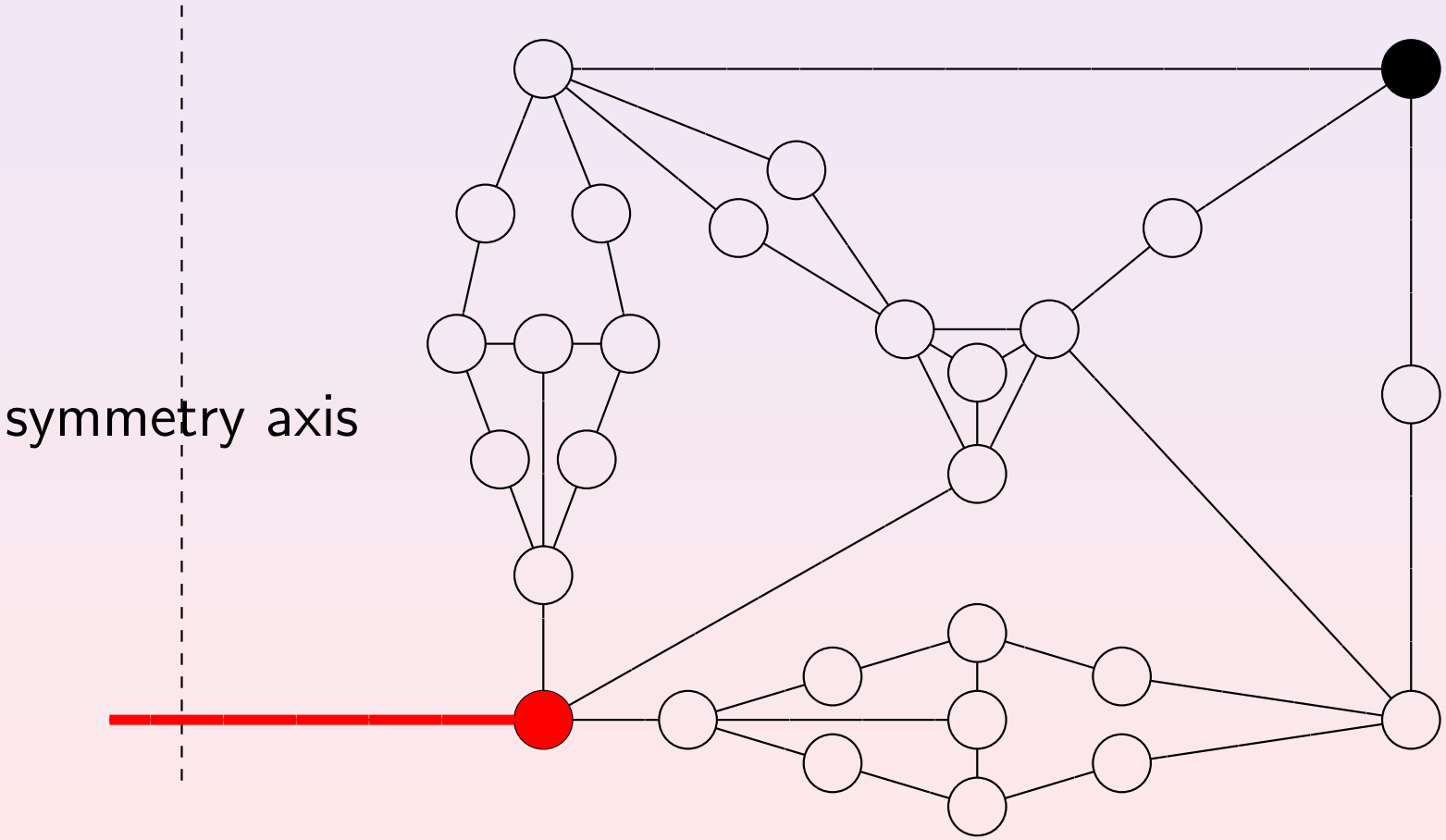

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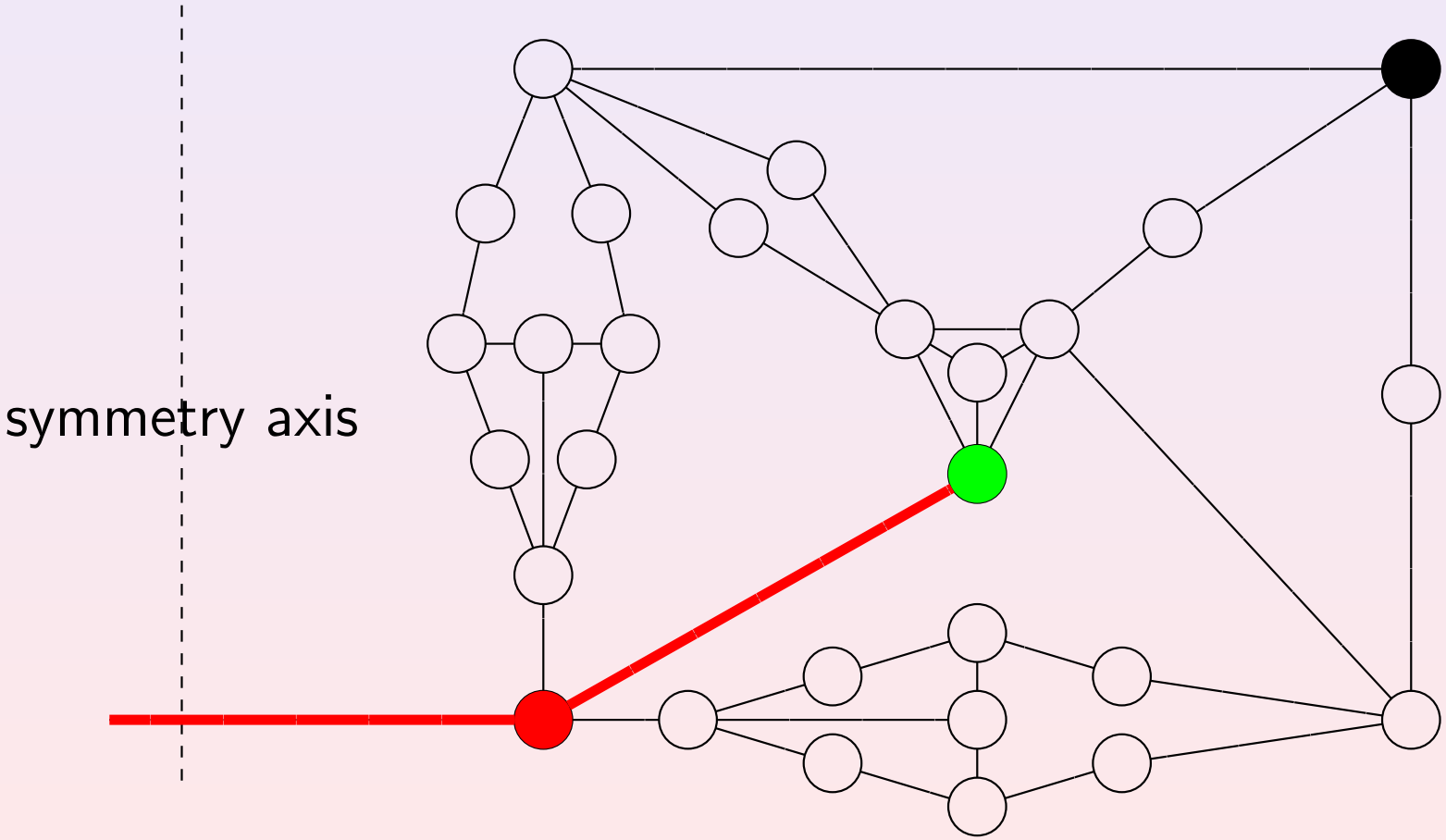

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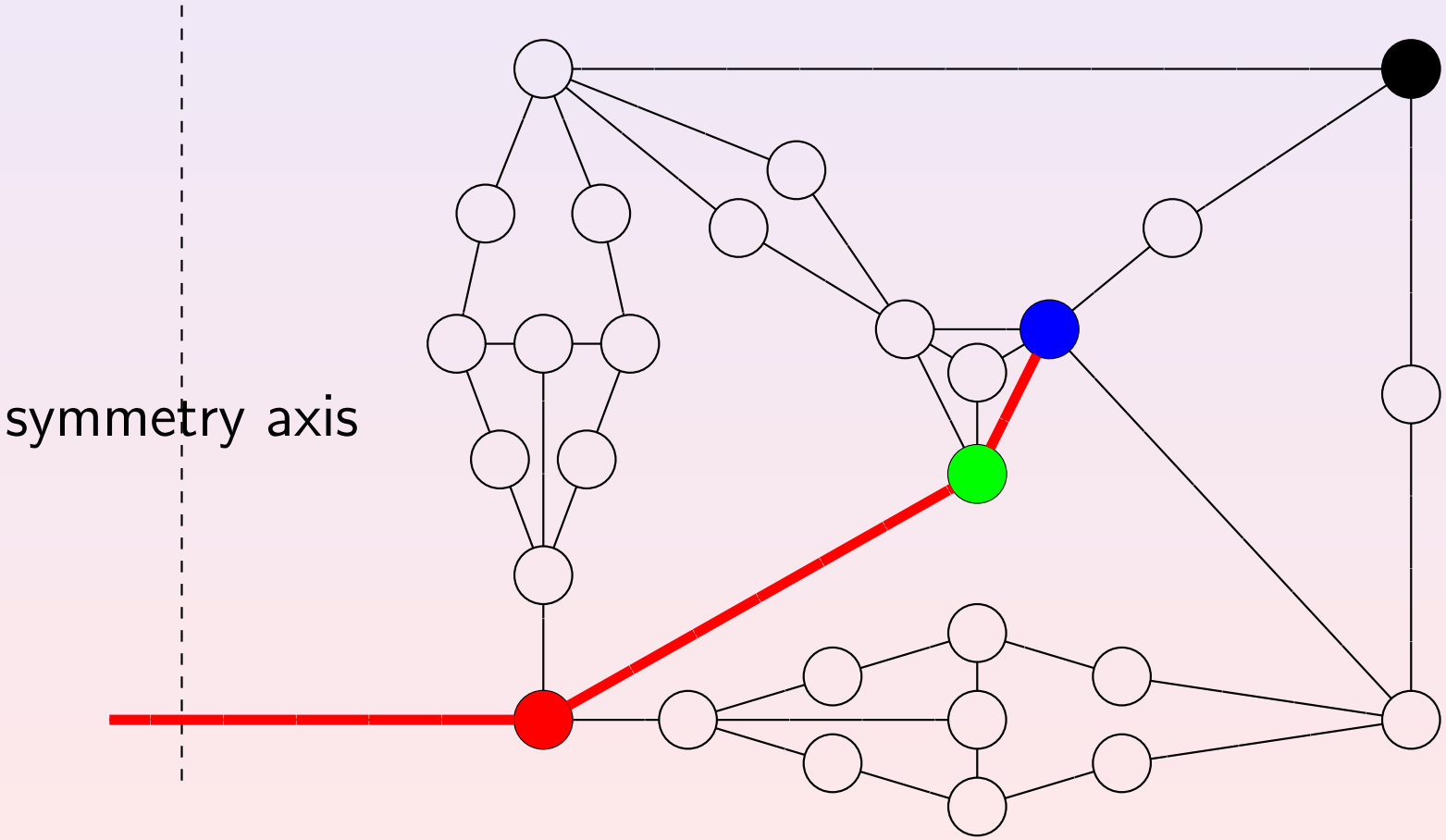

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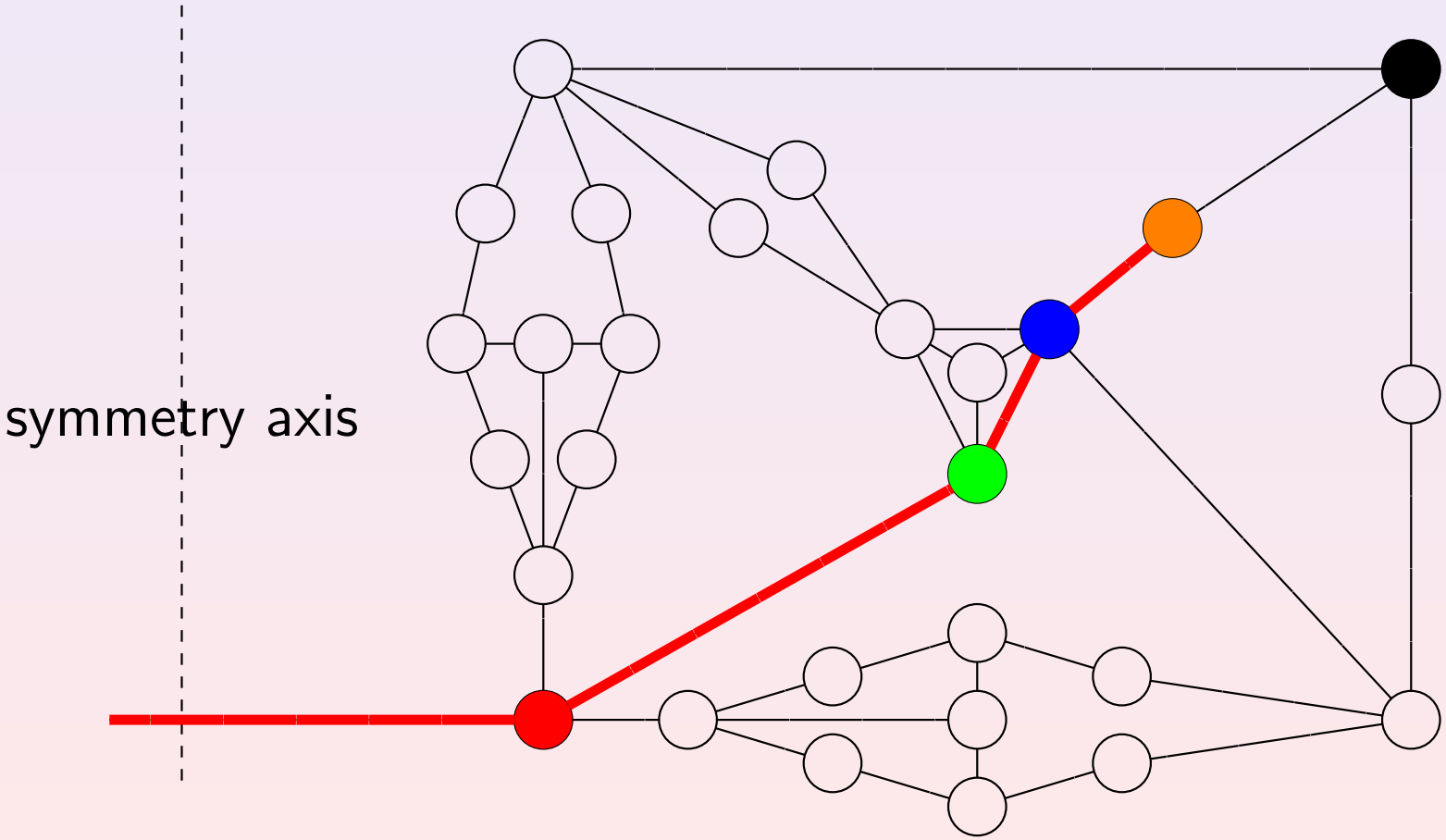

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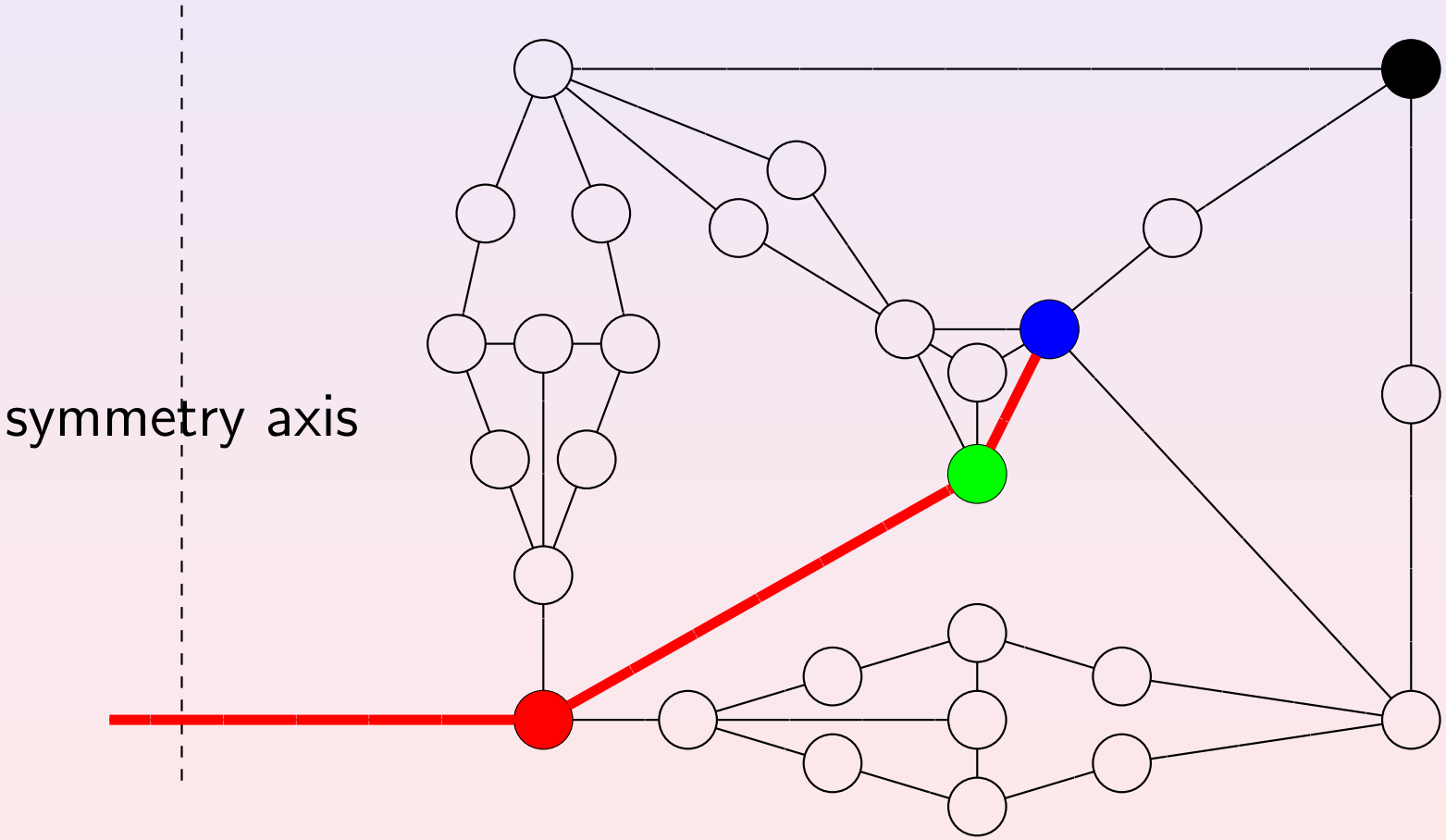

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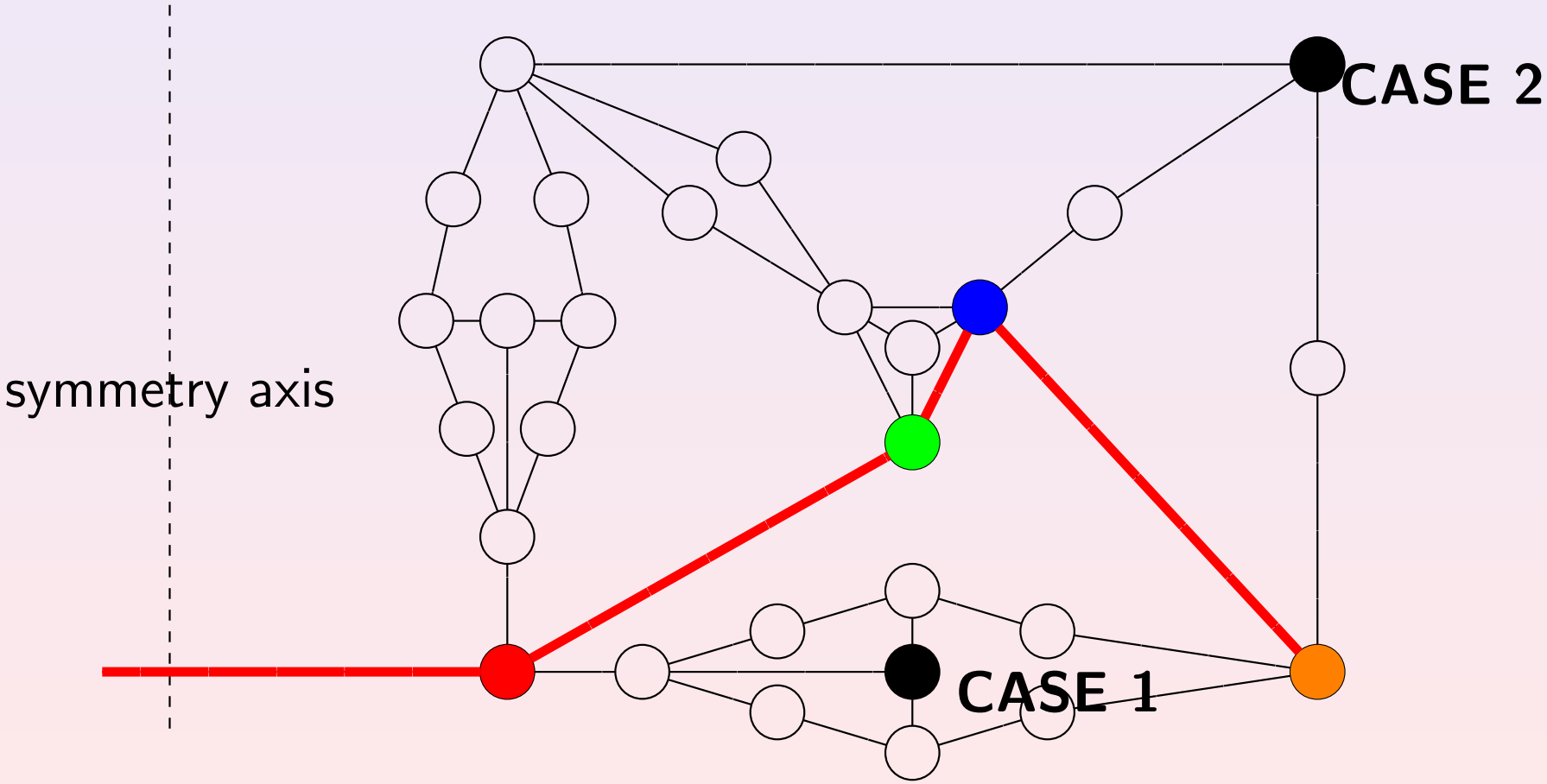

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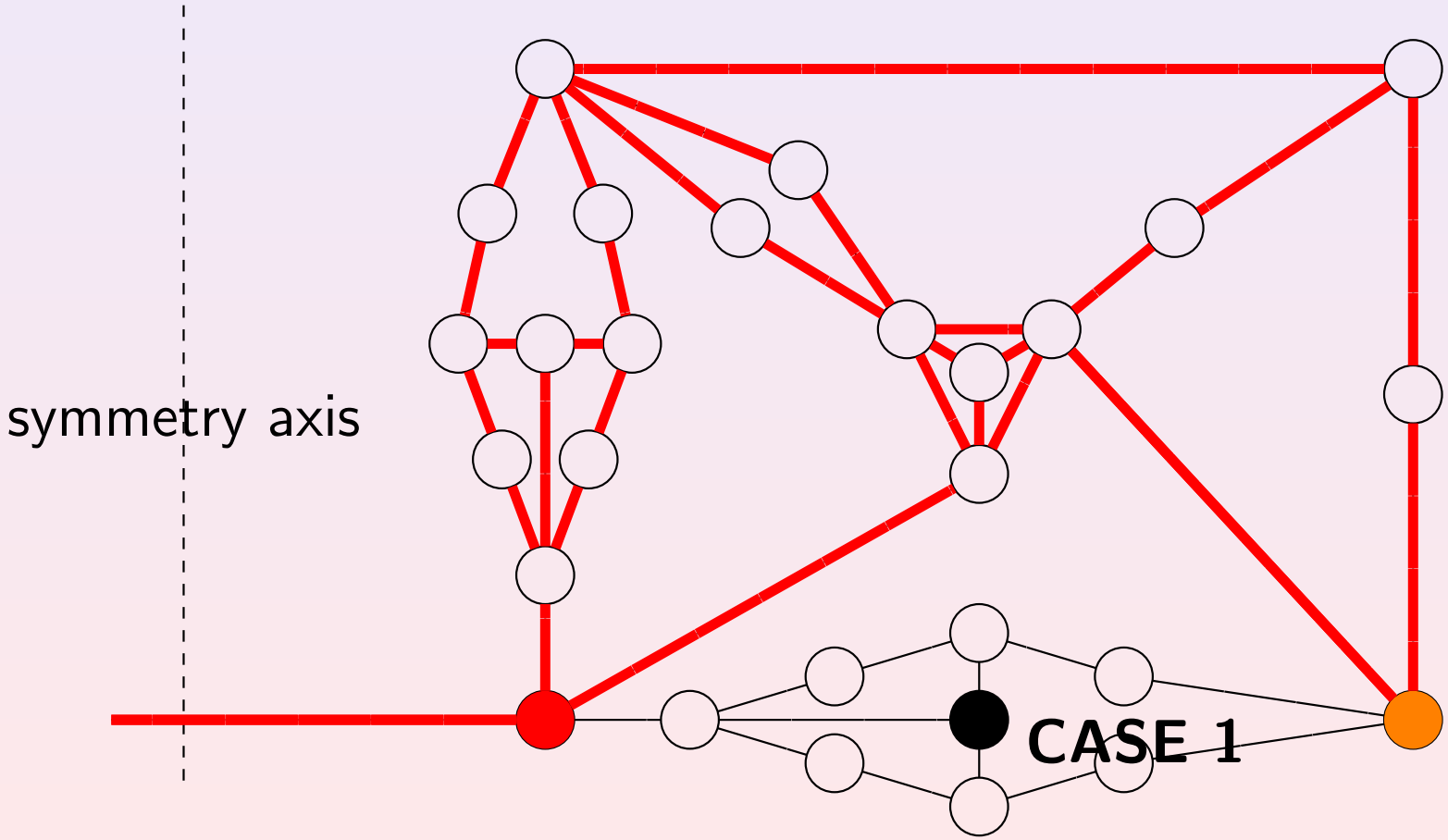

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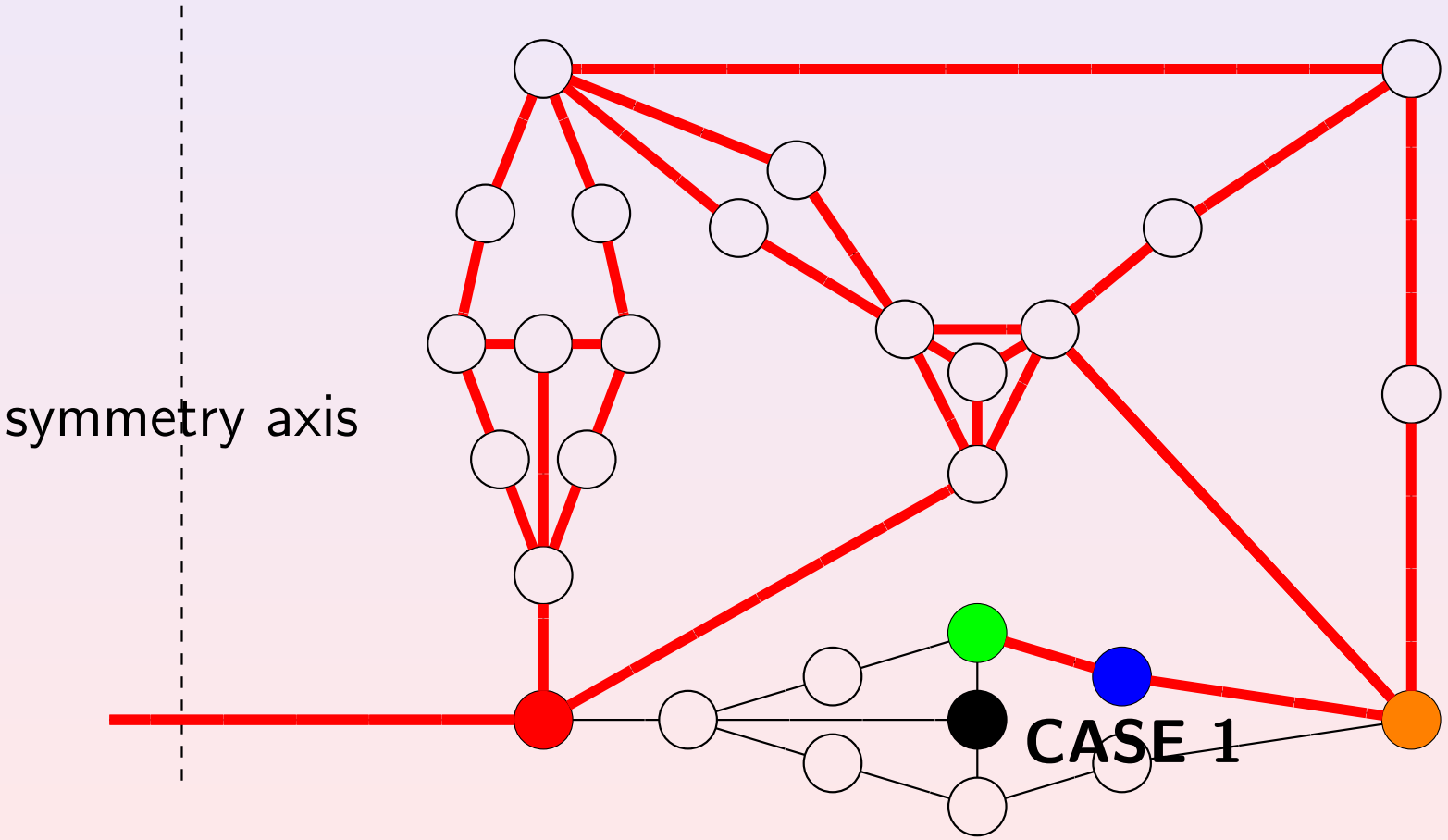

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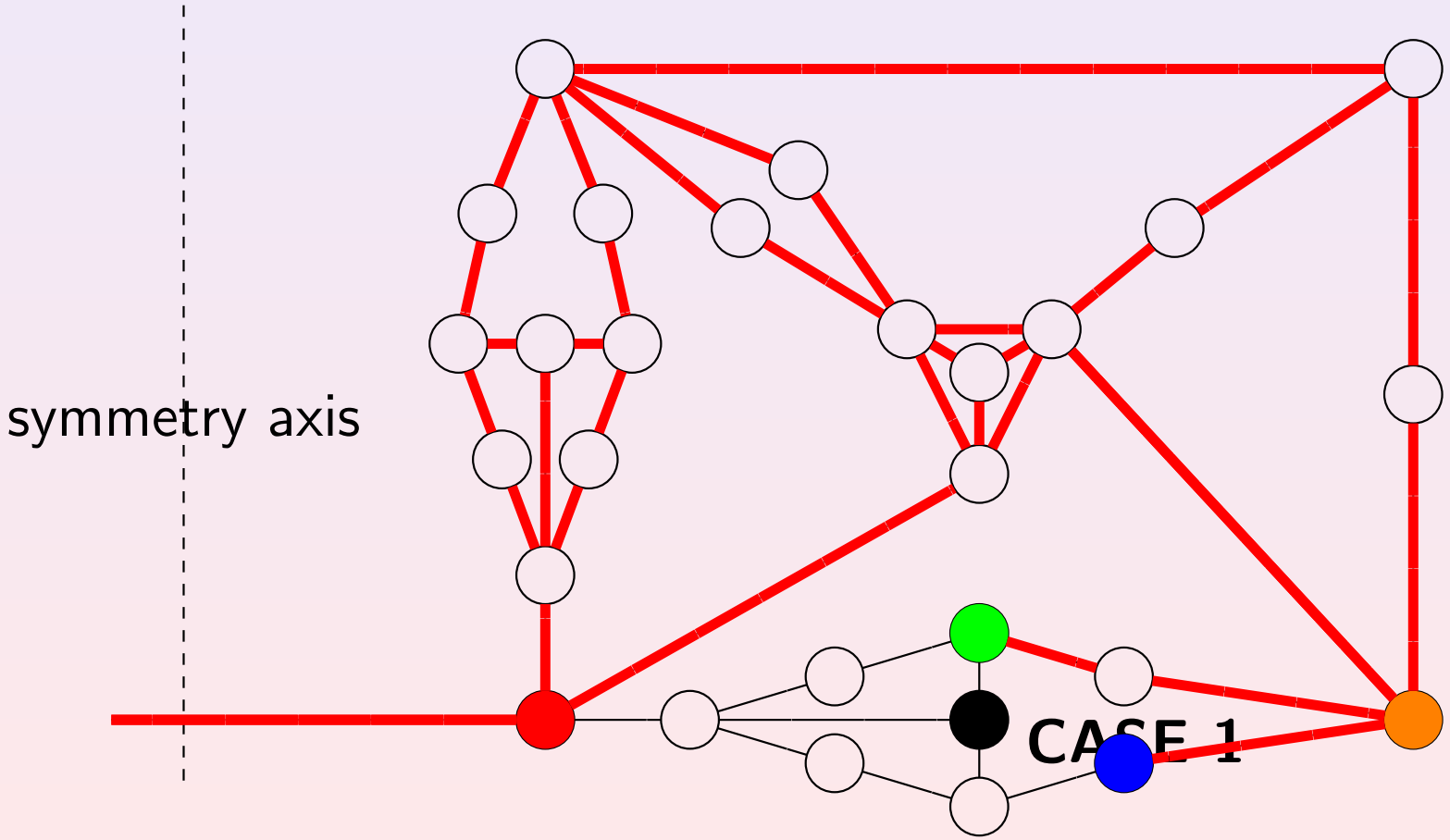

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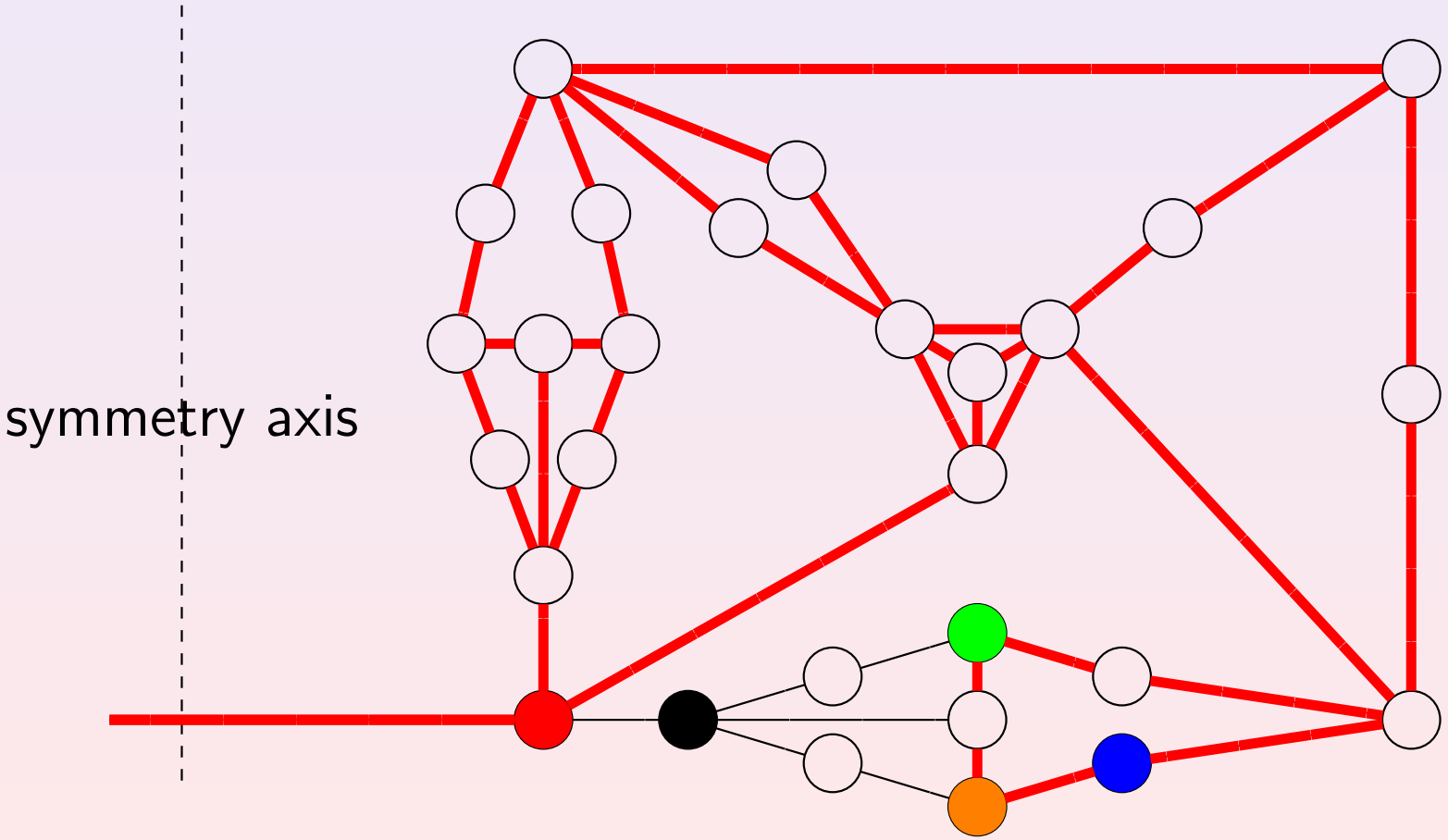

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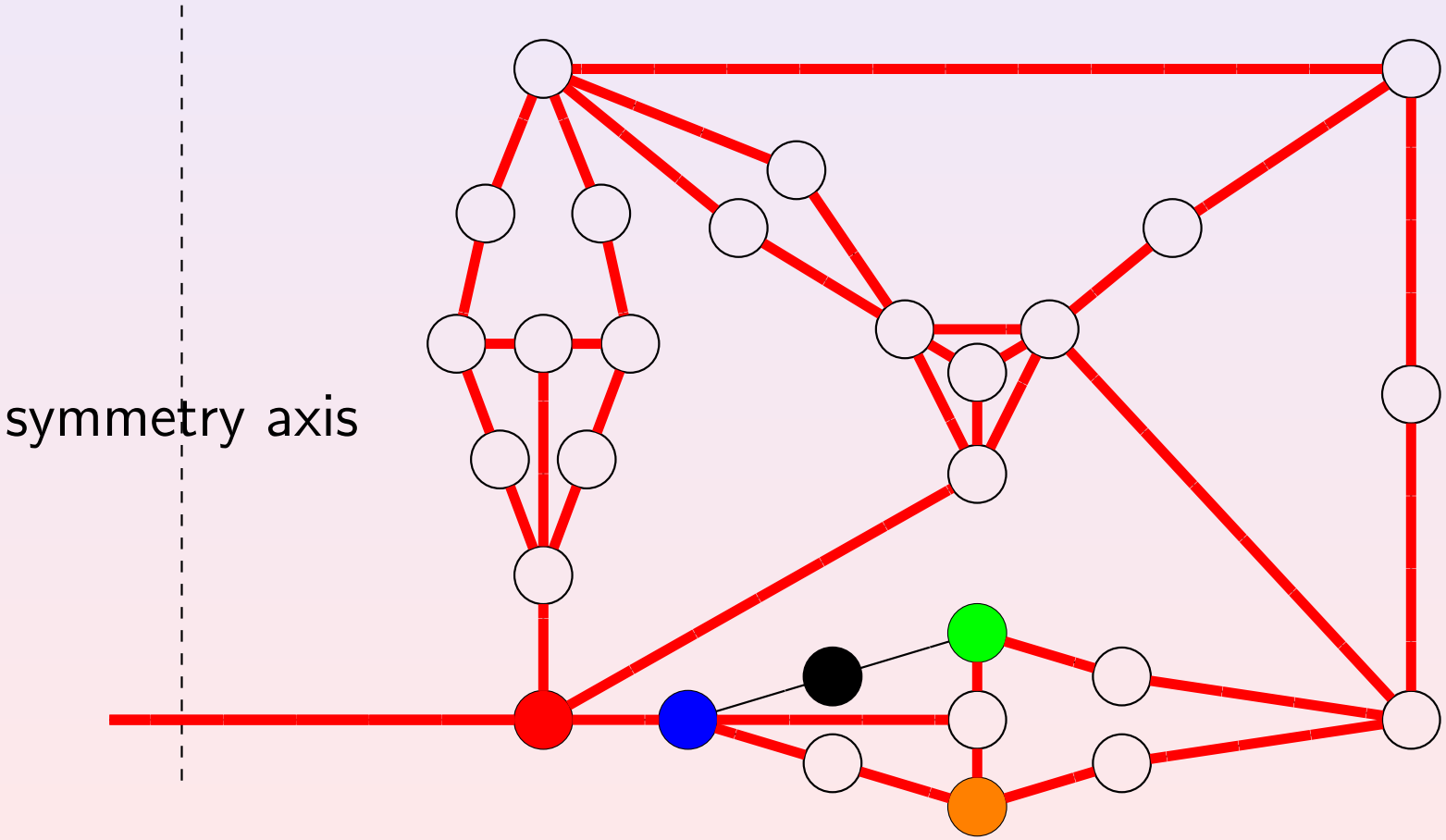

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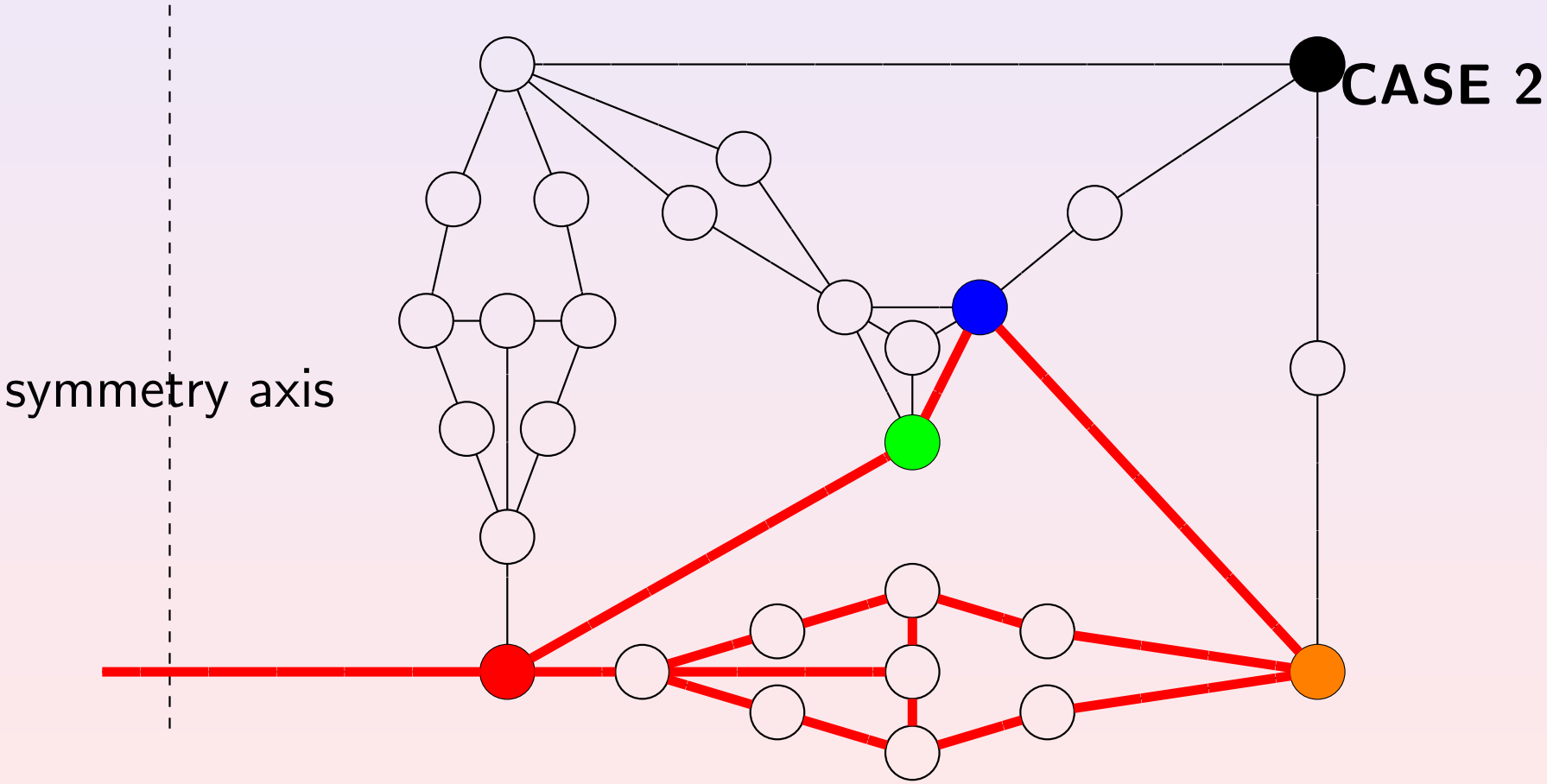

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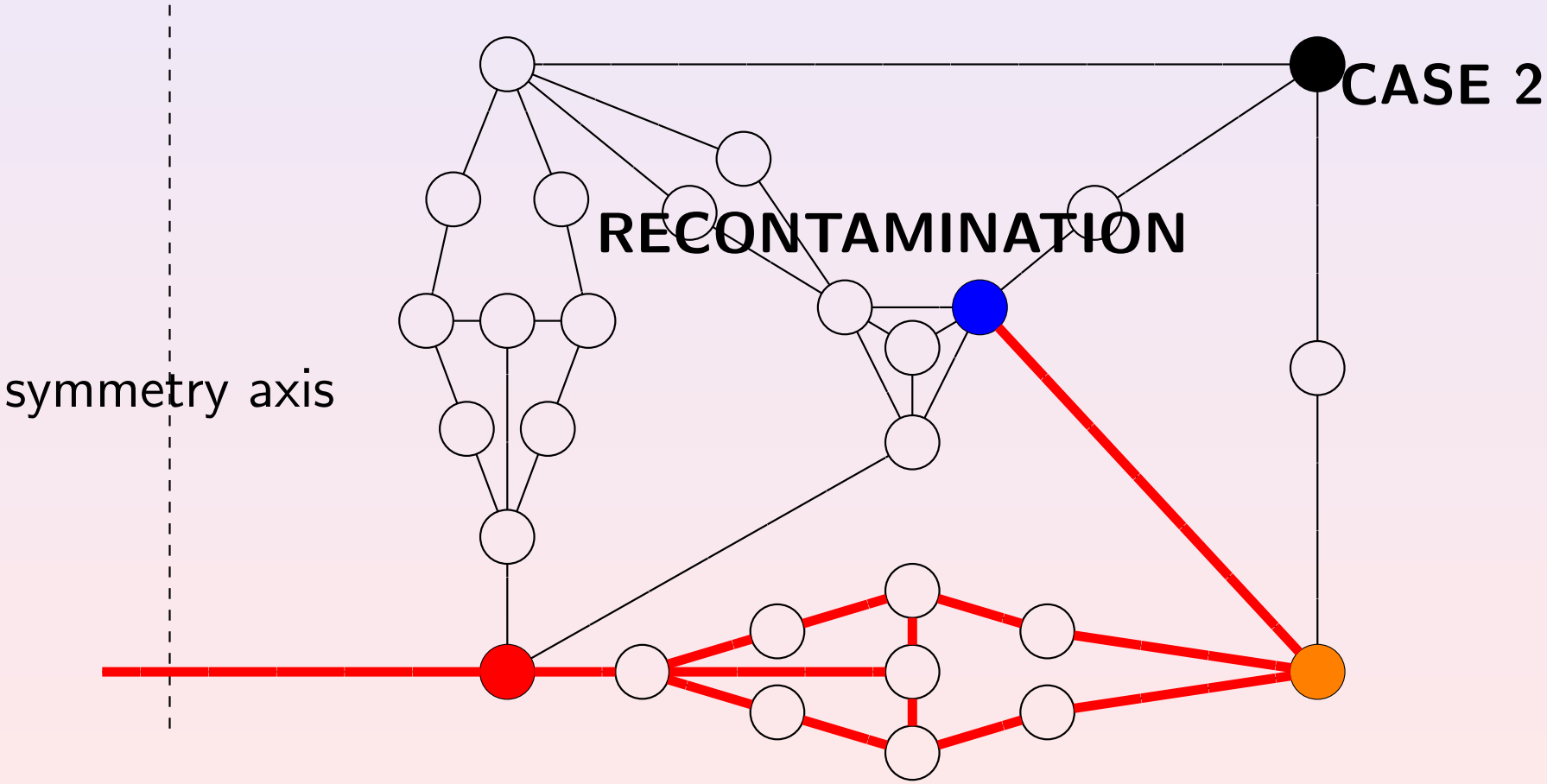

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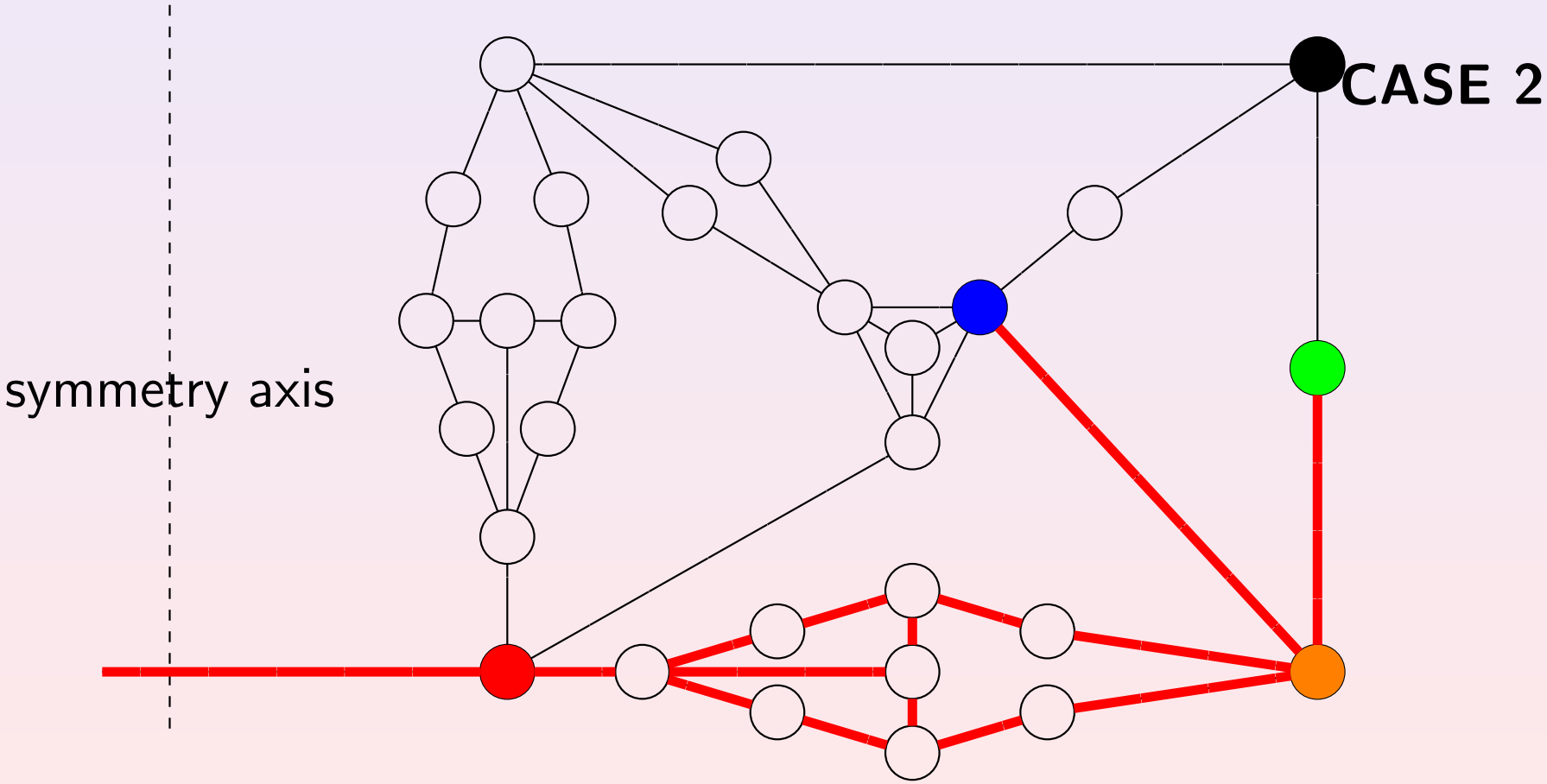

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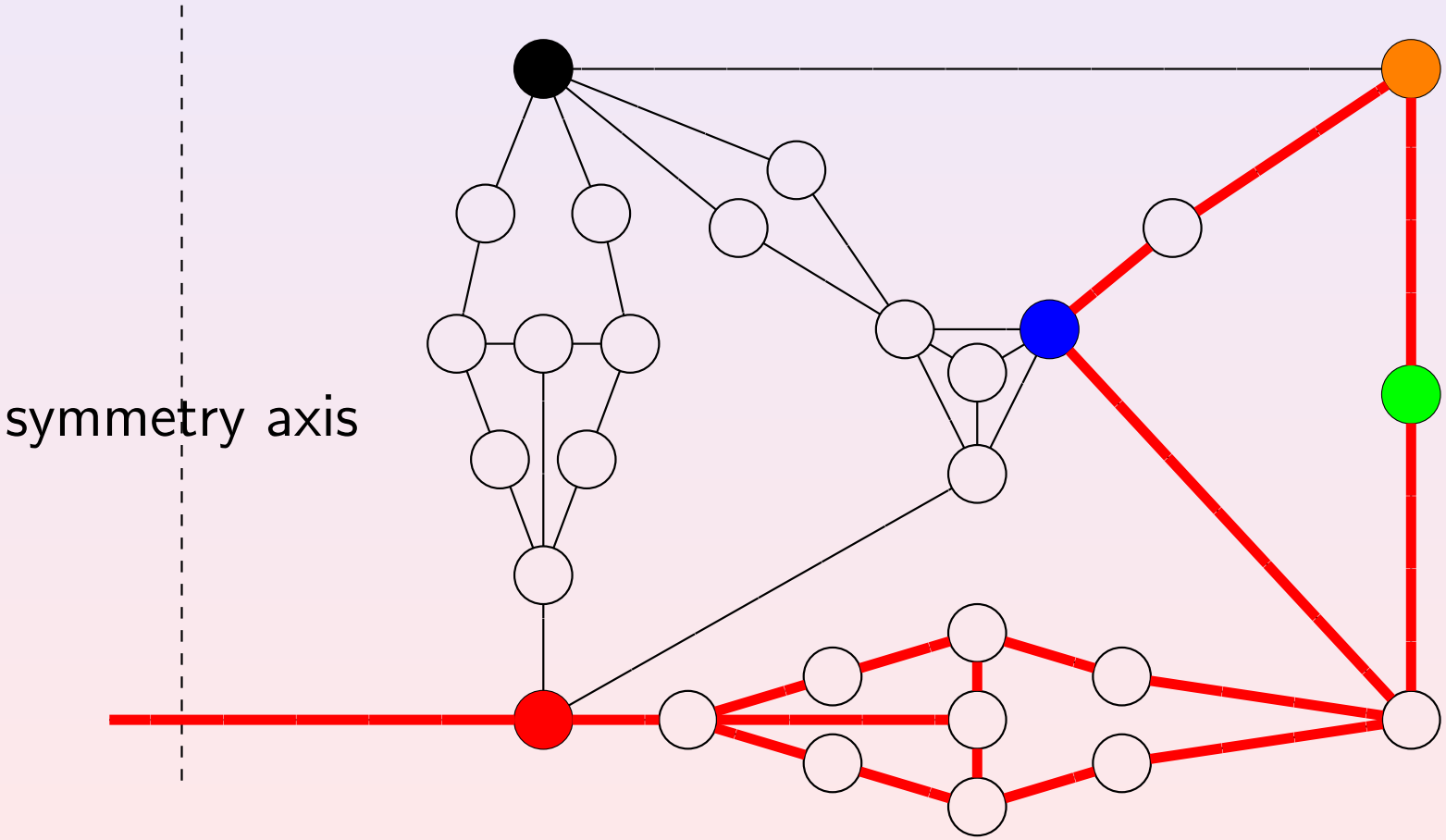

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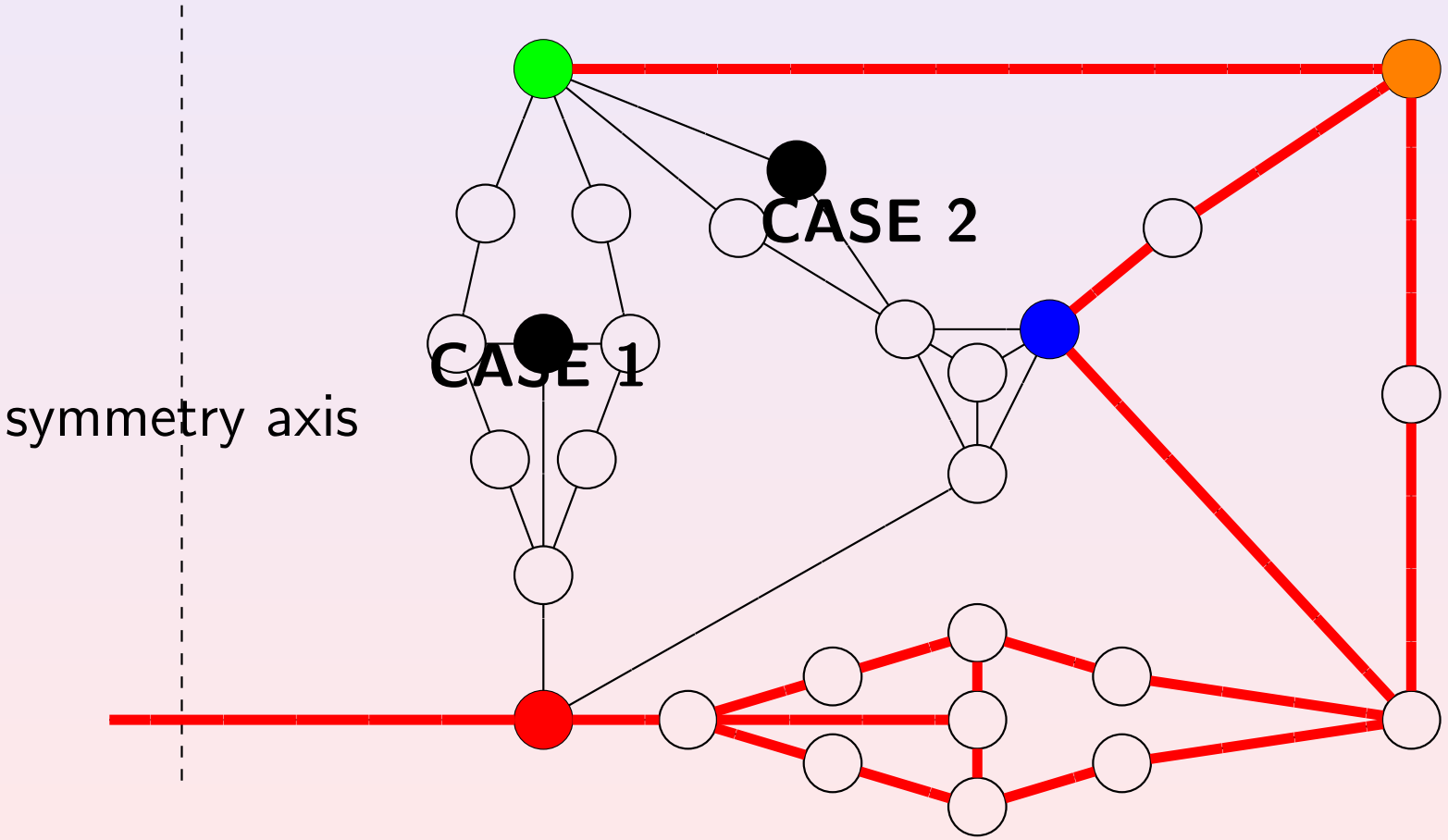

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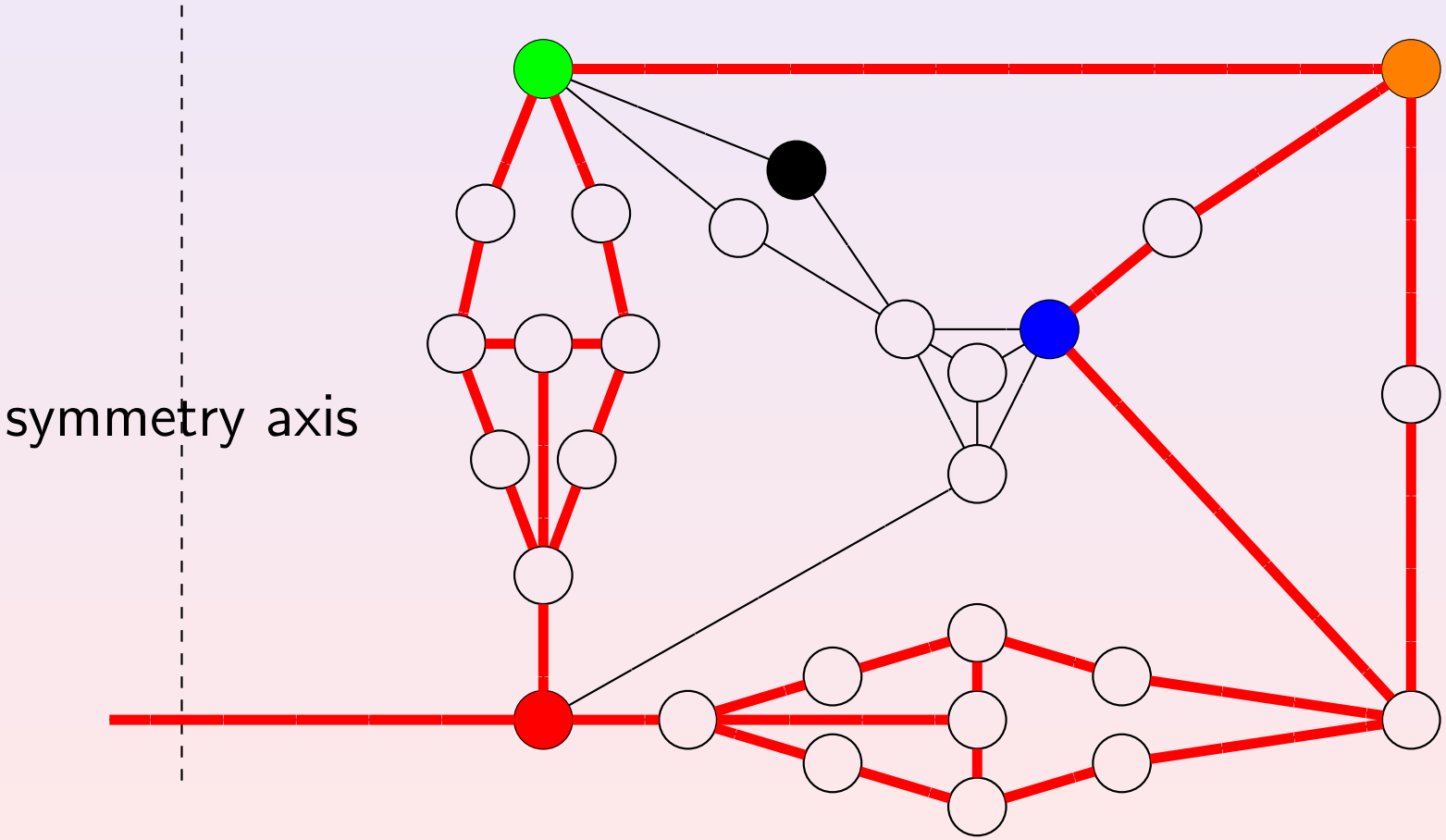

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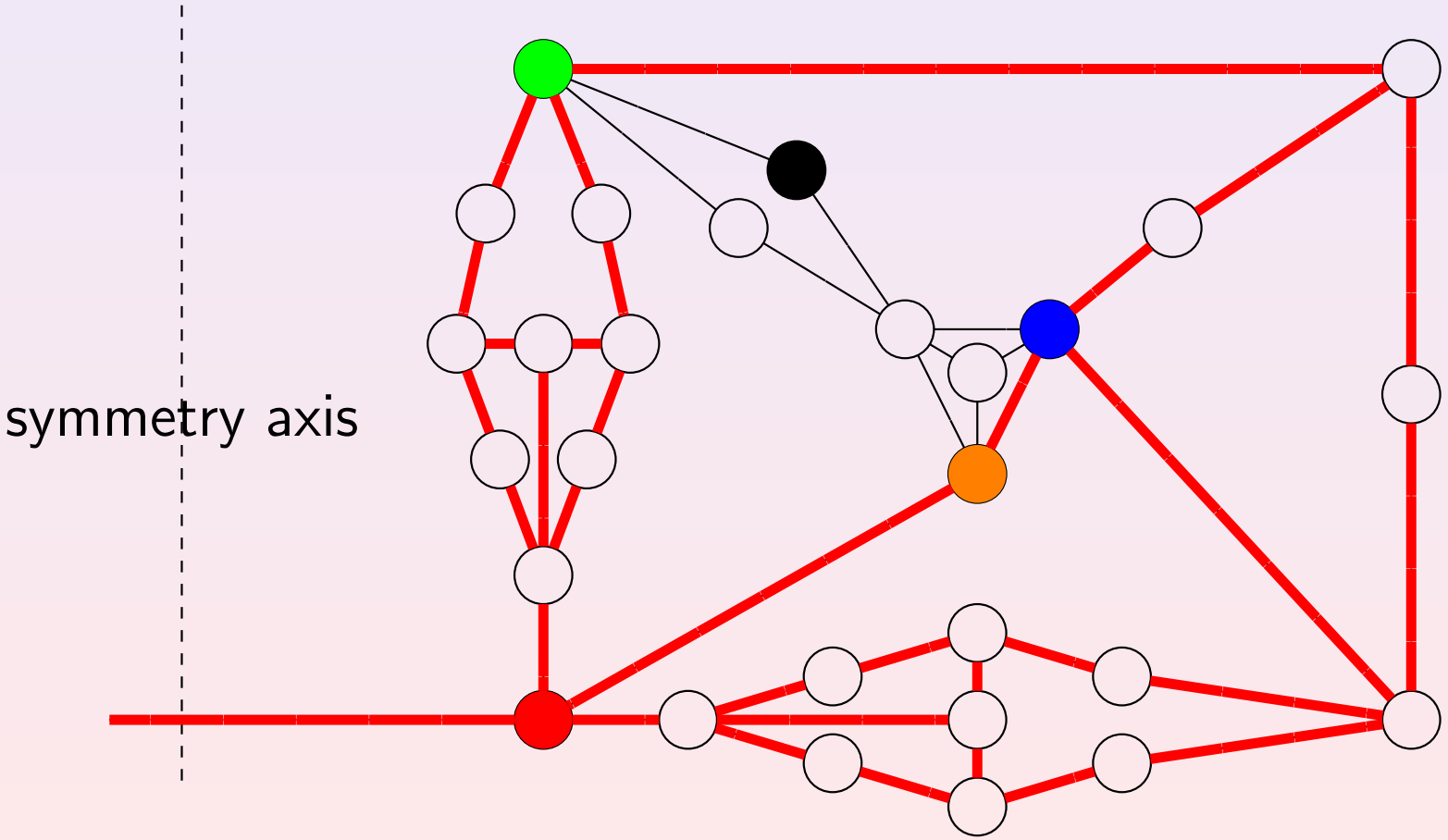

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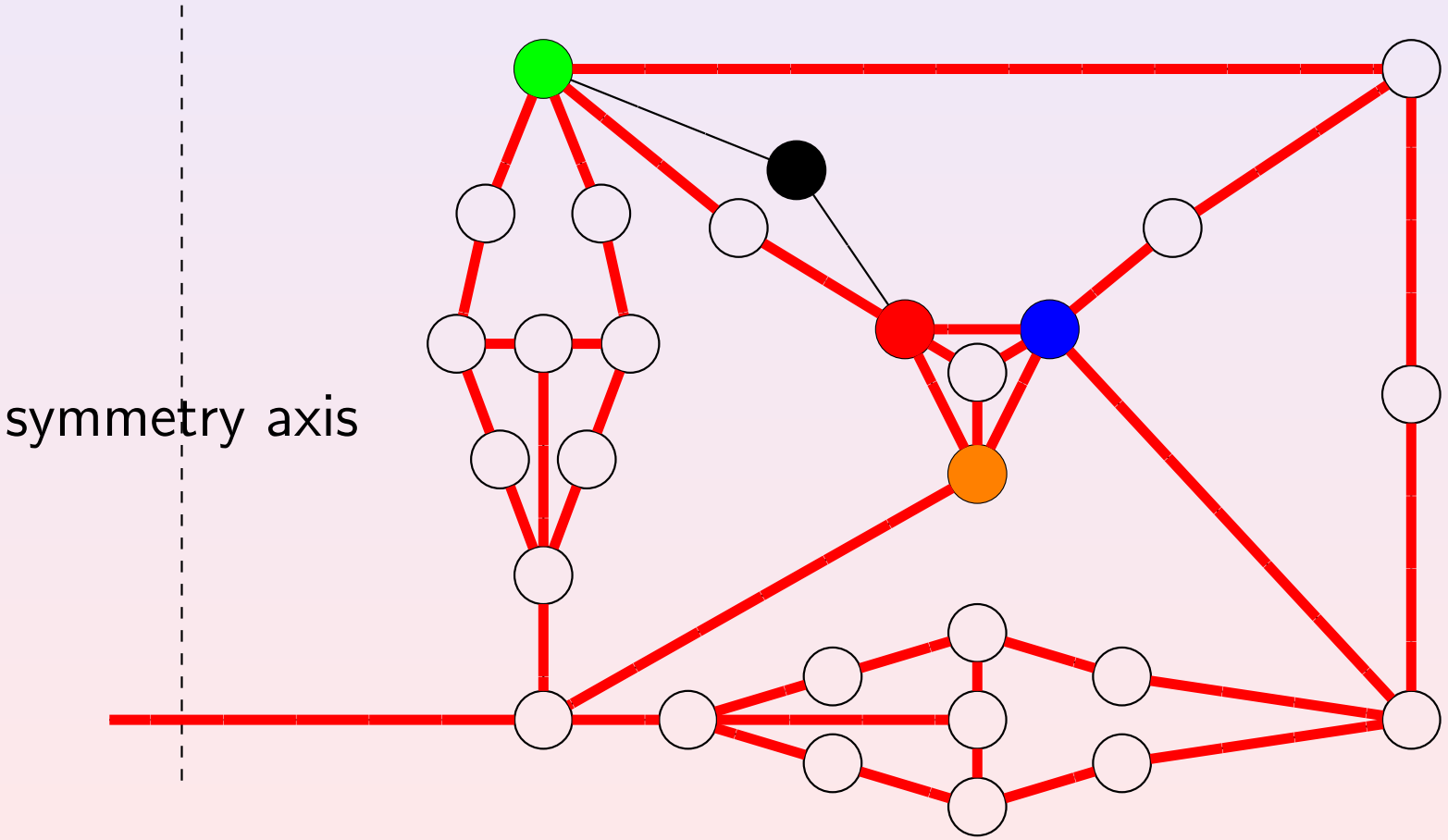

Conclusion and Further Work

Cost of connectivity

In visible Graph Searching

- $\mathbf{cvs}(G)/\mathbf{vs}(G) = \Theta(\log n)$
- recontamination helps

Open Problems

- $\mathbf{cs}(G)/\mathbf{s}(G) = O(\log n)$, what is the tight bound?
- What is the minimum k such that there is a graph G with $cs(G) = k$ and recontamination helps?
  The smallest known values is $k = 281$...
- Are there graphs G, such that $mcvs(G) > cvs(G) = 3$?
  I.e., does recontamination helps to catch a visible fugitive in a connected way using at most 3 searchers?