Eternal Domination in Grid-like Graphs

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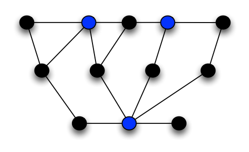
Northwestern Polytechnical University, Xi'an, Sept. 9th, 2019

Domination in Graphs

Dominating set in $G = (V, E)$

$D \subseteq V$ such that $N[D] = V$ — $N[S]$: closed neighborhood of S
i.e., for all $v \in V$, $v \in D$ or $\exists w \in D$ with $vw \in E$.
$\gamma(G)$: minimum size of a dominating set in G.



Computation of $\gamma(G)$: NP-complete [Karp 72], W[2]-hard, no $c \log n$-approximation (for some $c < 1$) [Alon *et al.* 06]

Graph classes: $\gamma(P_n) = \lceil \frac{n}{3} \rceil$, $\gamma(C_n) = \lceil \frac{n}{3} \rceil$, $\gamma(P_n \boxtimes P_m) = \lceil \frac{n}{3} \rceil \lceil \frac{m}{3} \rceil$

$\gamma(P_n \square P_m) = \lfloor \frac{(n+2)(m+2)}{5} \rfloor - 4$ $(16 \leq n \leq m)$ [Gonçalves *al.* 11]

Vizing conjecture: $\gamma(G \square H) \geq \gamma(G)\gamma(H)$

best known: $\gamma(G \square H) \geq \frac{1}{2}\gamma(G)\gamma(H)$ [Clark,Suen 00]

Eternal Domination (one guard move)

[Burger *et al.* 2004]

Two Player Game

Defender first places its k guards at vertices of the graph. Then, **turn-by-turn**

1. *Attacker* attacks one vertex v
2. *Defender* moves **one** guard from $w \in N[v]$ to v.

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$\gamma^{\infty}(G)$: minimum k ensuring guards to win in G.

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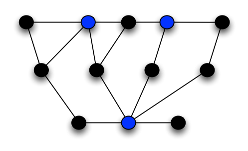

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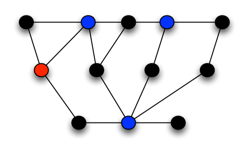

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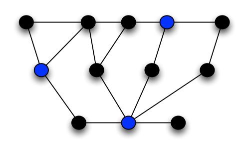

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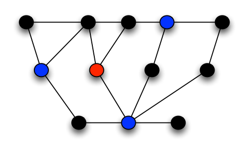

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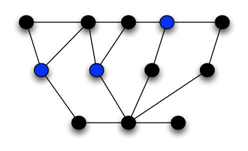

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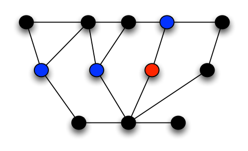

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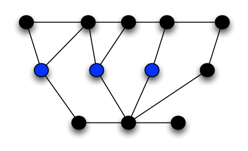

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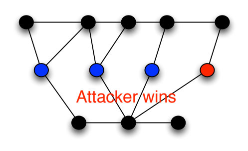

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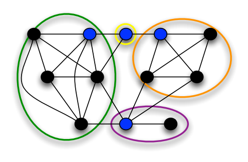

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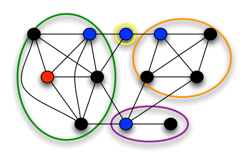

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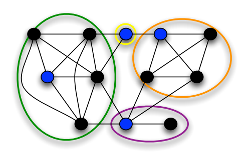

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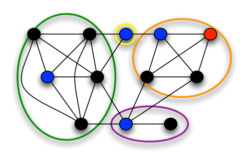

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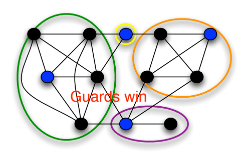

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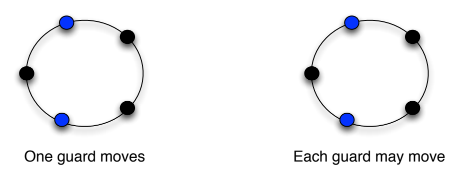

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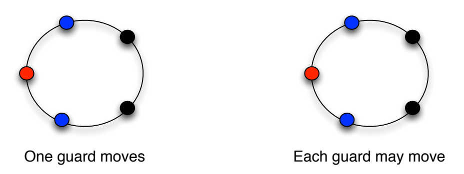

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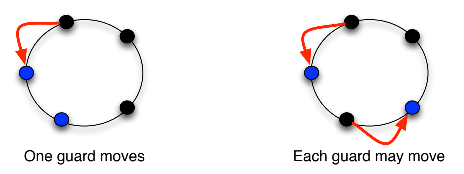

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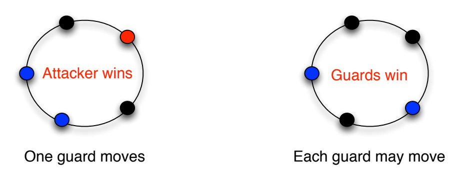

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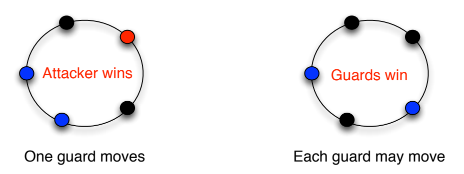

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$\gamma_{all}^{\infty}(G) \leq \alpha(G)$: nice induction proof [Goddard *et al.* 2005]

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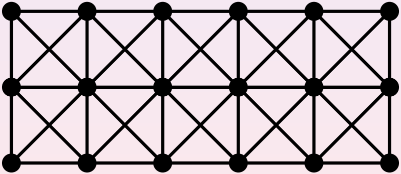

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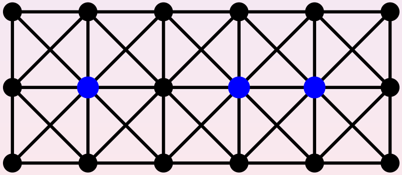

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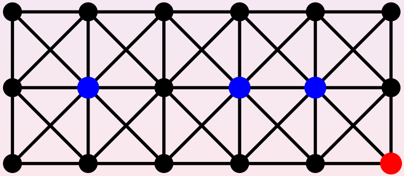

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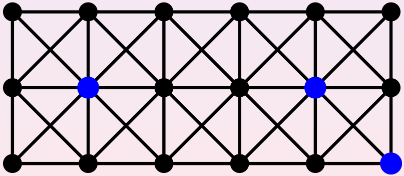

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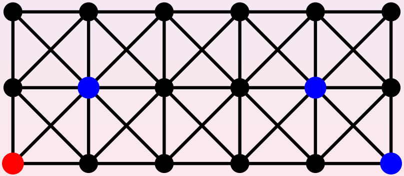

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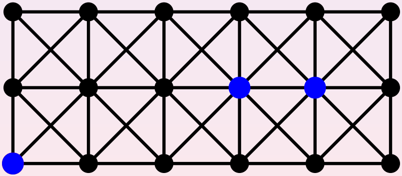

Eternal Domination (all guards move)

[Goddard *et al.* 2005]

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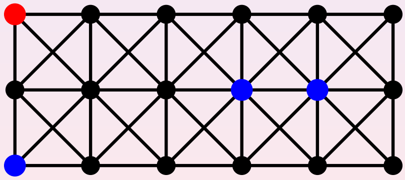

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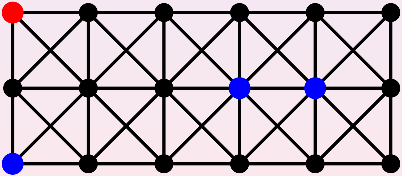

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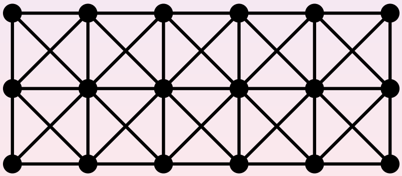

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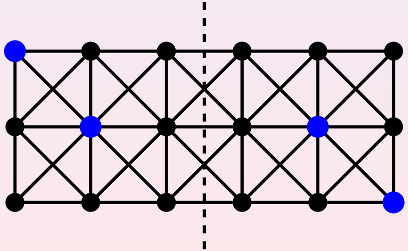

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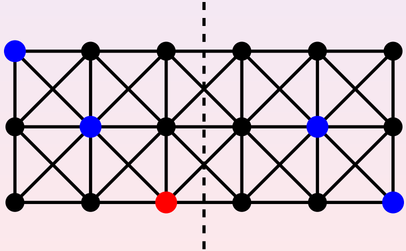

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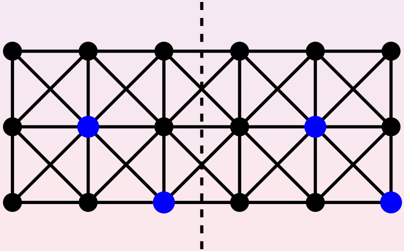

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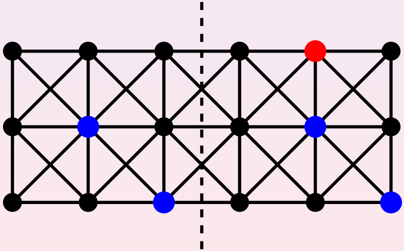

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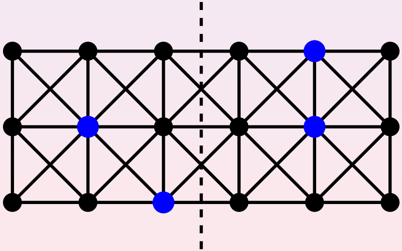

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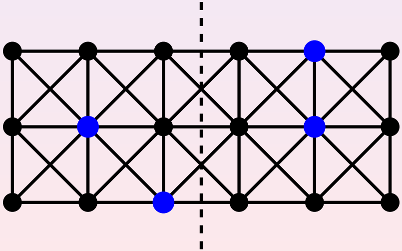
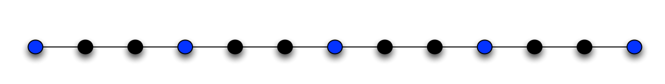


Guards win! (with $2\gamma(P_n \boxtimes P_m)$ guards)

State of the art

- Paths, cycles: $\gamma^{\infty}_{all}(P_n) = \lceil \frac{n}{2} \rceil > \gamma(P_n) = \lceil \frac{n}{3} \rceil$, $\gamma^{\infty}_{all}(C_n) = \lceil \frac{n}{3} \rceil = \gamma(C_n)$



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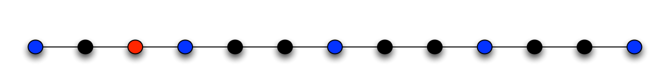

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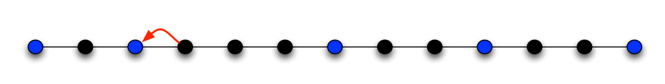

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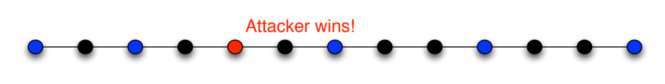

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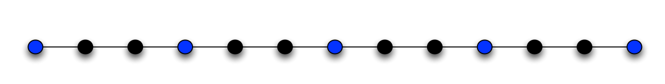

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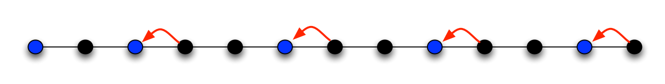

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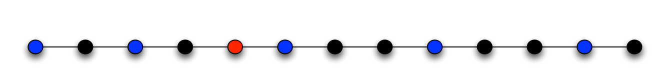

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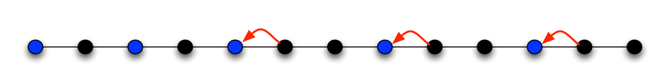

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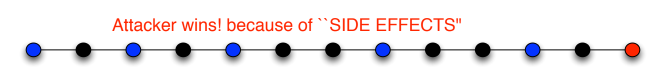

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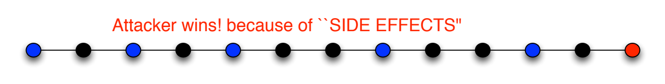


- Deciding whether $\gamma^{\infty}_{all}(G) \leq k$ is NP-hard [Bard *et al.* 2017]
- Linear-time algorithm for trees [Klostermeyer, MacGillivray, 2009]
- $\gamma^{\infty}_{all}(G) = \alpha(G)$ for all proper interval graphs G [Braga *et al.* 2015]
- Linear-time algorithm for all interval graphs [Rinemberg, Soulignac, 2018]

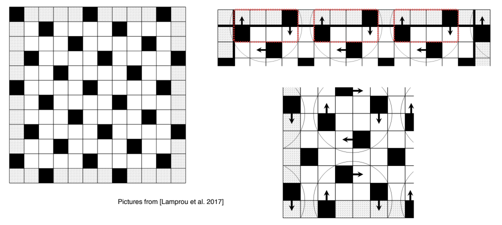
Cartesian Grids

- $\gamma_{all}^{\infty}(P_2 \square P_n) = \lceil \frac{2n}{3} \rceil$ [Goldwasser *et al.* 2013]
- $\lceil \frac{4n}{5} \rceil + 1 \leq \gamma_{all}^{\infty}(P_3 \square P_n) \leq \lceil \frac{4n}{5} \rceil + 5$ [Messinger, 2017]
- $\gamma_{all}^{\infty}(P_4 \square P_n)$ is known [Beaton *et al.* 2014]
- Bounds for $\gamma_{all}^{\infty}(P_5 \square P_n)$ [van Bommel *et al.* 2016]

Theorem [Lamprou *et al.* 2017]

$\gamma_{all}^{\infty}(P_n \square P_m) = \gamma(P_n \square P_m) + O(n+m)$.

$(P_n \square P_m = \lceil \frac{mn}{5} \rceil)$



Pictures from [Lamprou et al. 2017]

Eternal Domination in Strong Grids

[Mc Inerney, N., Pérennes, 2018]

Note that $\gamma(P_n \boxtimes P_m) = \lceil \frac{m}{3} \rceil \lceil \frac{n}{3} \rceil$ and $\alpha(G) = \lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil$ and so

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Theorem [Mc Inerney, N., Pérennes, 2018]

For all $m \geq n$,

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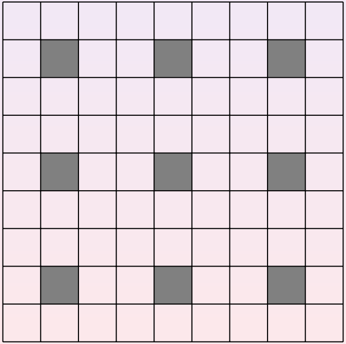
Recursive approach: not based on local patterns but on global movements.

Eternal domination in the torus $C_n \boxtimes C_m$

$$\gamma^{\infty}_{all}(C_n \boxtimes C_m) = \gamma(C_n \boxtimes C_m) = \gamma(P_n \boxtimes P_m) = \left\lceil \frac{m}{3} \right\rceil \left\lceil \frac{n}{3} \right\rceil$$



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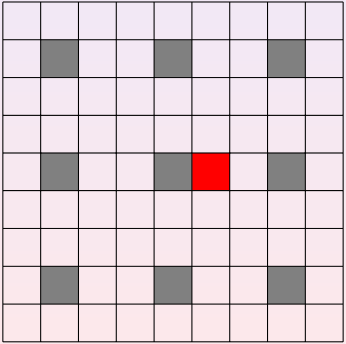

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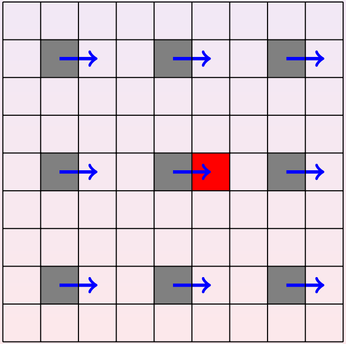

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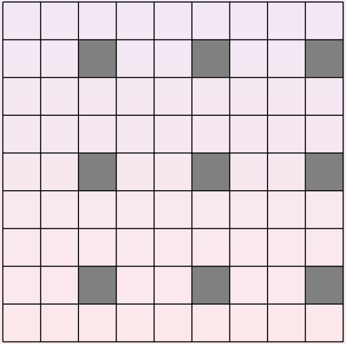

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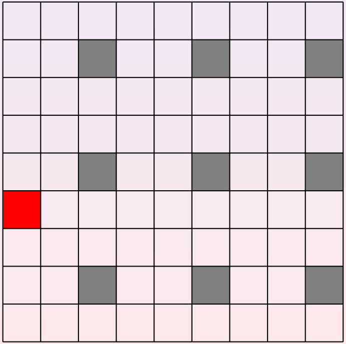

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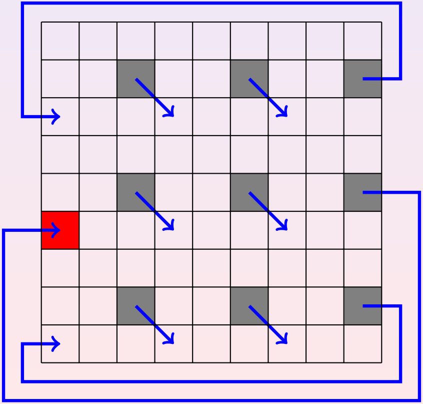

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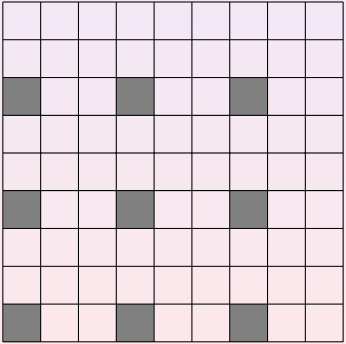

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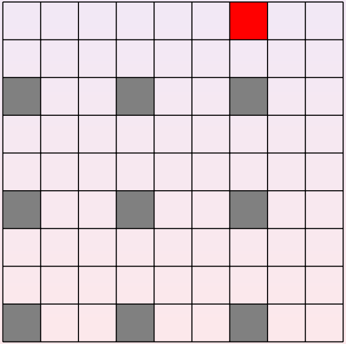

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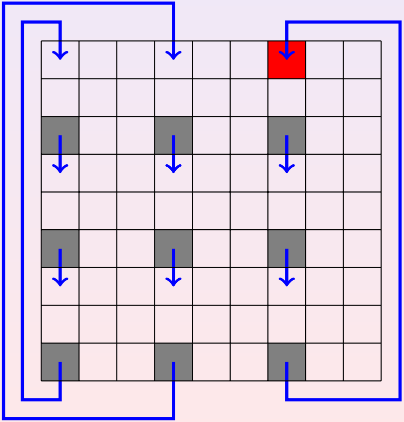

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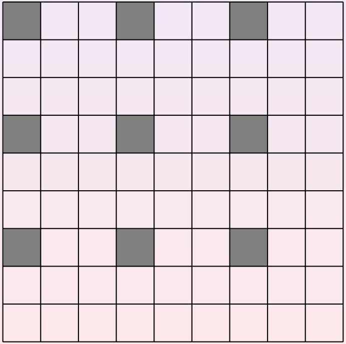
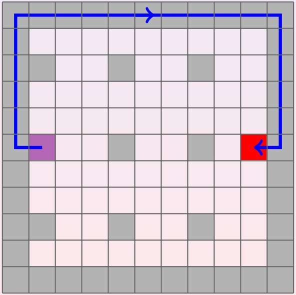


Easy in the torus because we can wrap around $\rightarrow$ impossible in the grid!

Back to the Grid: Key Lemma

Teleportation (case of one guard)

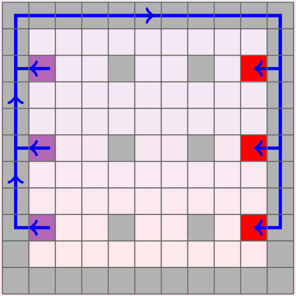
If there is one guard on each border vertex, then one guard may "teleport" using the borders of the grid.



Back to the Grid: Key Lemma

Teleportation

If there are α guards on each border vertex, then $\beta \leq \alpha$ guards may teleport using the borders of the grid.



Upper Bound Overview $\gamma_{all}^{\infty}(P_n \boxtimes P_m) = \gamma(P_n \boxtimes P_m) + O(m\sqrt{n})$

Configuration

Multi-set $C = \{v_i \mid 1 \leq i \leq k\}$ giving the positions of the k guards.

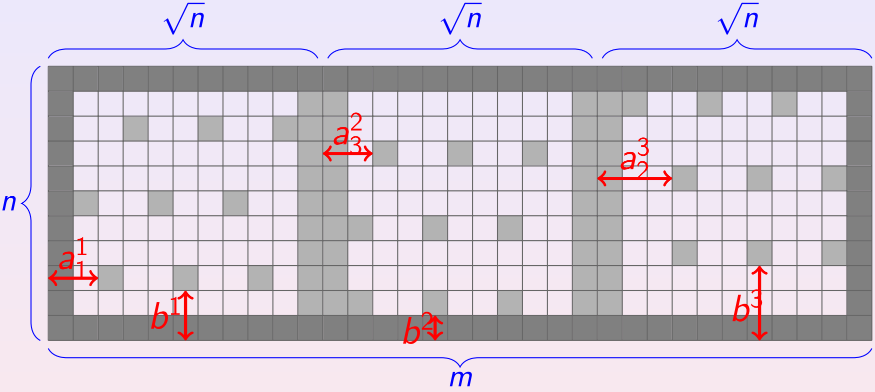
Configurations of the winning strategy: *SetWinConf*

Set of configurations that dominate the grid.

Attacks split into 3 types: Horizontal, Vertical, and Diagonal.

We show: for any attack at a vertex $v \in V(P_n \boxtimes P_m)$, the guards can move from a configuration $C \in SetWinConf$ to a configuration $C' \in SetWinConf$ where $v \in C'$.

Configuration $C \in SetWinConf$

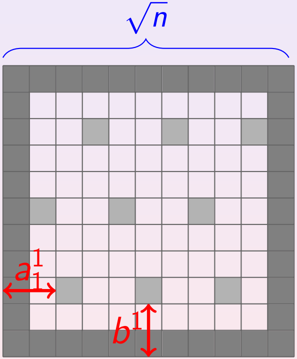


1 guard on vertices in light gray.

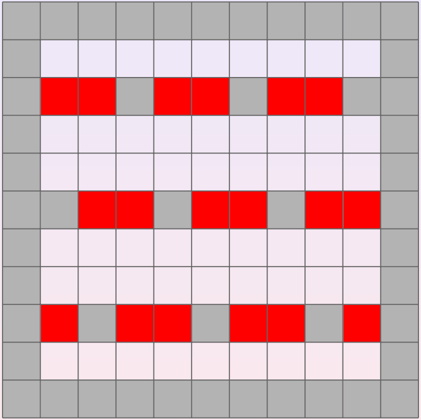
$O(\sqrt{n})$ guards on vertices in dark gray.

$\gamma(P_n \boxtimes P_m) + O(m\sqrt{n})$ guards total.

Block of Configuration $C \in SetWinConf$

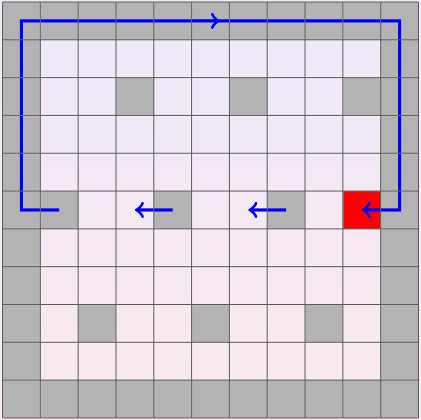


Horizontal Attacks



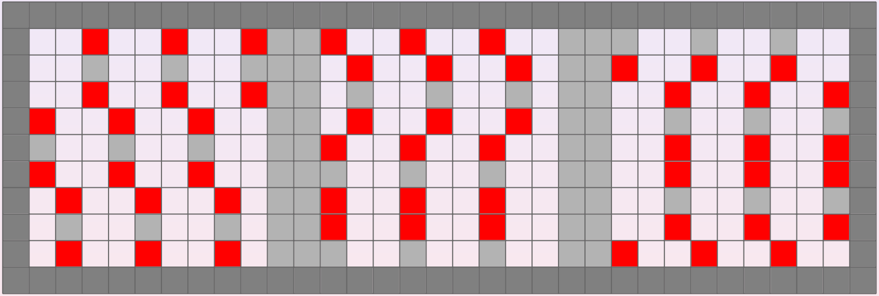
Horizontal attacks may only occur at vertices in red.

Horizontal Attacks - attack at red vertex



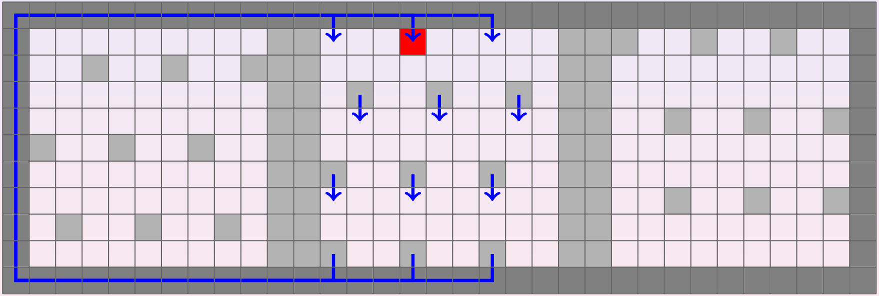
Only guards in same row and block move (except borders maybe).

Vertical Attacks



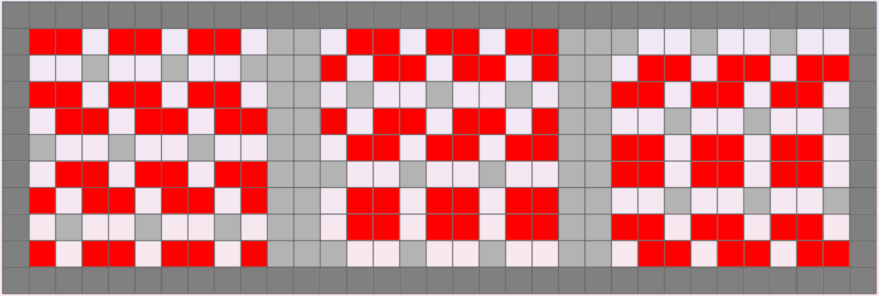
Vertical attacks may only occur at vertices in red.

Vertical Attacks - attack at red vertex



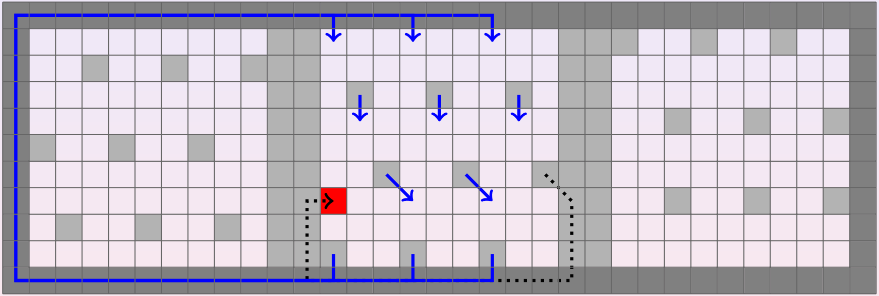
Only guards in same block move (except borders maybe).

Diagonal attacks



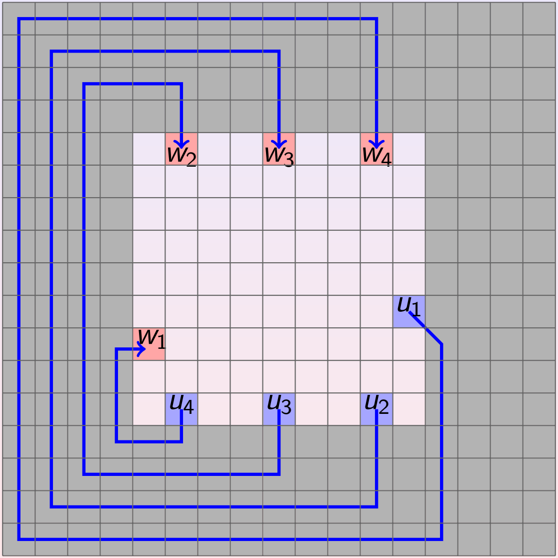
Diagonal attacks may only occur at vertices in red.

Diagonal Attacks - attack at red vertex



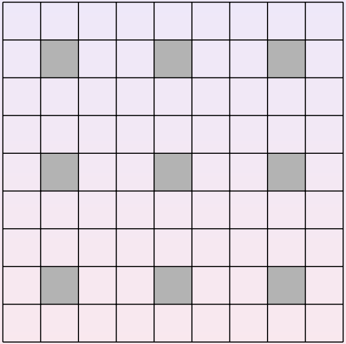
Guards in closest row (and block) move like in Horizontal and Vertical case at once and the rest in the same block move like in Vertical case.

At most 1 guard at each vertex



Lower Bound Idea of Proof $\gamma_{all}^{\infty}(P_n \boxtimes P_m) = \gamma(P_n \boxtimes P_m) + \Omega(m+n)$

At least 2 guards needed in each 4×5 subgrid on the border.



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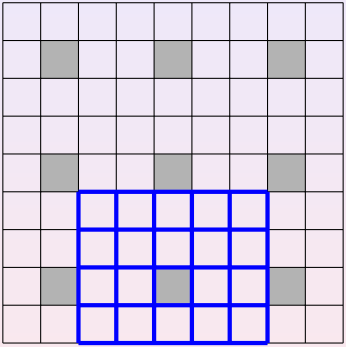

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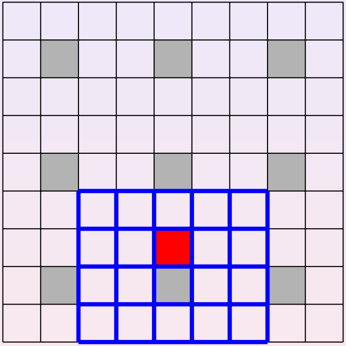

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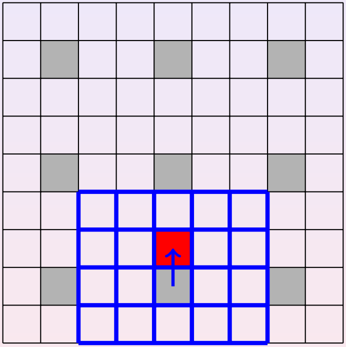

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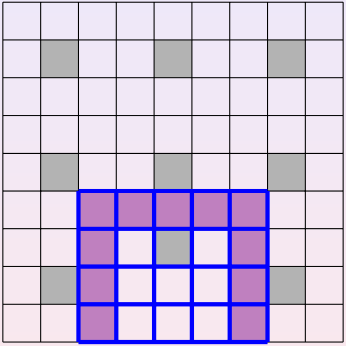

Lower Bound Idea of Proof $\gamma_{all}^{\infty}(P_n \boxtimes P_m) = \gamma(P_n \boxtimes P_m) + \Omega(m+n)$

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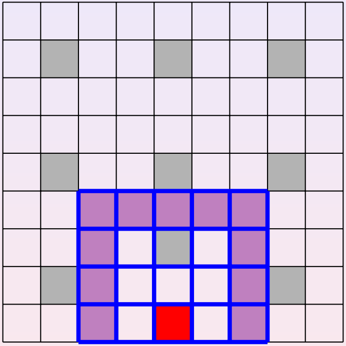
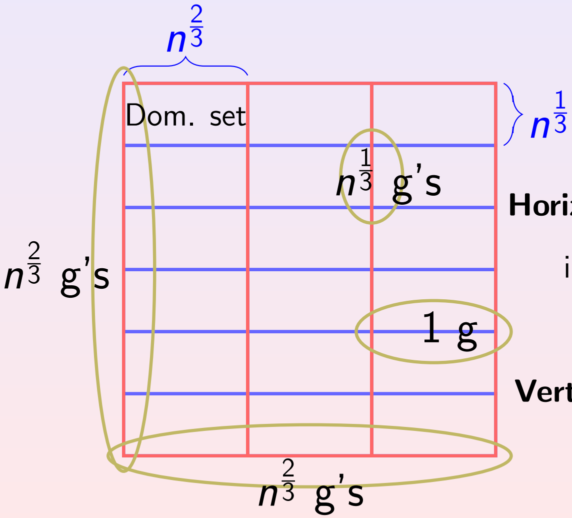


Vertices dominated by > 1 guard, and/or some guards dominate ≤ 6 vertices
Double counting argument leads to result.

Further Work

- Tighten bounds for strong grids.
- All-guards move model is *NP*-hard but unknown if in *NP*.
  - Is it *PSPACE*-complete? *EXPTIME*-complete?
- For all Cayley graphs G obtainable from abelian groups, $\gamma_{all}^{\infty}(G) = \gamma(G)$ [Goddard *et al.* 2005].
  - Prove $\gamma_{all}^{\infty}(H) = \gamma(H) + o(\gamma(H))$ for truncated Cayley graphs H obtained from abelian groups by generalizing our technique.

Gen. tech. - Cartesian Grid $\gamma_{all}^{\infty}(P_n \Box P_n) = \gamma(P_n \Box P_n) + O(n^{\frac{5}{3}})$



Horiz. attack: move only in small rectangle.

Vert. attack: move only in red rectangle.

Thanks!