Brief Introduction to Parameterized Algorithms

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How to face NP-hard problems? (but not only)

- Exact exponential algorithms $O(c^n)$, $c > 1$
- Approximation algorithms, heuristics not always possible/desired.
- Particular instances e.g., bipartite graphs, planar graphs...
- ...
- Parameterized Complexity: "Refined" analysis of the complexity depending not only on the size of the instance but on some "parameter" describing the "structure" of the instance.

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- Exact exponential algorithms $O(c^n)$, $c > 1$
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- Particular instances e.g., bipartite graphs, planar graphs...
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- Parameterized Complexity: "Refined" analysis of the complexity depending not only on the size of the instance but on some "parameter" describing the "structure" of the instance.

Two very nice books:

M. Cygan, F.V. Fomin, L. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, S. Saurabh: Parameterized Algorithms. Springer 2015.

F.V. Fomin, D. Lokshtanov, S. Saurabh, M. Zehavi: Kernelization. Theory of Parameterized Preprocessing. Cambridge University Press 2019.

Parameterized Complexity in a nutshell

Some well known NP-hard problems

Vertex Cover:

Inputs: a graph $G = (V, E)$, and $k \in \mathbb{N}$;

Output: does there exist $Q \subseteq V$, $|Q| \leq k$, $\forall e \in E$, $Q \cap e \neq \emptyset$?

best known $O(1.2114^n)$ [Bourgeois *et al.*, 12], 2-approximation (maximal matching)

Clique:

Inputs: a graph $G = (V, E)$, and $k \in \mathbb{N}$;

Output: does there exist $Q \subseteq V$, $|Q| \geq k$, $\forall u, v \in Q$, $\{u, v\} \in E$?

best known $O(1.1888^n)$ [Robson *et al.*, 01], $O(n(loglogn)^2 / log^3 n)$-approximation [Feige 04]

Proper Coloring:

Inputs: a graph $G = (V, E)$, and $k \in \mathbb{N}$;

Output: $\chi(G) \leq k$?, i.e., is there a proper coloring $c : V \rightarrow \{1, \cdots, k\}$?

Some well known NP-hard problems

What if the size k of the solution is a fixed parameter?

k-VERTEX COVER:

Input: a graph $G = (V, E)$;

Output: does there exist $Q \subseteq V$, $|Q| \leq k$, $\forall e \in E$, $Q \cap e \neq \emptyset$?

k-CLIQUE:

Input: a graph $G = (V, E)$;

Output: does there exist $Q \subseteq V$, $|Q| \geq k$, $\forall u, v \in Q$, $\{u, v\} \in E$?

Polynomial-time solvable!: try $n^{O(k)}$ possibilities.

k-PROPER COLORING:

Input: a graph $G = (V, E)$;

Output: $\chi(G) \leq k$?, i.e., is there a proper coloring $c : V \to \{1, \cdots, k\}$?

Still NP-hard for any fixed $k \geq 3$! (constant-time in planar graphs if $k \geq 4$)

The class XP

Parameter

A parameter is given by a polynomial-time computable function, which maps instances of our problem to natural numbers.

e.g., size of the solution, genus, treewidth, number of vertices to be removed to obtain a bipartite graph, *etc.*

Class XP

A problem is in XP parameterized by k if there exists an algorithm which solves the problem in time $O(n^{f(k)})$ for some function f.

If the parameter k is the size of the solution:

- k-Vertex-Cover and k-Clique are in XP.
- k-Coloring $\notin$ XP for any $k \geq 3$.

Let's go further: example of k-Vertex-Cover

Vertex Cover of $G = (V, E)$: set $Q \subseteq V$ s.t., $\forall e \in E$, $e \cap Q \neq \emptyset$.

Trivial lemmas: let Q be a Vertex Cover of G, then

- for all $\{u, v\} \in E$, $u \in Q$ or $v \in Q$ or both.
- if $|Q| \leq k$, then $|E| \leq k|V|$.

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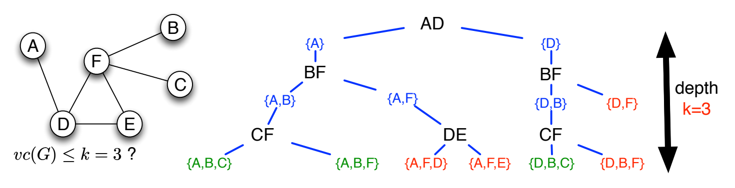
Branch & Bound Algorithm (BB) for deciding if $vc(G) \leq k$

If $|E| > 0$ and $k = 0$, *Return* ∞. **Else if** $|E| = 0$, *Return* 0.
Else if $|E| = 1$, *Return* 1
Else Let $\{u, v\} \in E$, *Return* $\min\{BB(G \setminus u, k-1), BB(G \setminus v, k-1)\} + 1$



Limited recursion depth + bounded # of edges $\Rightarrow$ Complexity : $O(k2^k \cdot |V|)$
linear in $|V|$, $|V|$ and k are "separated"

Fixed Parameter Tractable (FPT) algorithms

FPT Problem

A problem is fixed parameter tractable (FPT) parameterized by a parameter k if there exists an algorithm which solves the problem in time $f(k) \cdot n^{O(1)}$.

VERTEX-COVER is FPT parameterized by the size of the solution.

Which problems are FPT?

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The W-Hierarchy

$FPT \subseteq W[1] \subseteq W[2] \subseteq \cdots$.

$W[]$ defined using *weft* of Boolean circuits

It is strongly believed that $FPT \neq W[1]$.

- k-CLIQUE is $W[1]$-complete
- k-DOMINATING SET, k-SET COVER, k-HITTING SET are $W[2]$-complete.

but... k-CLIQUE FPT in planar graphs

Let's go even further: example of k-VERTEX-COVER

Simple lemmas: let Q be a Vertex Cover of $G = (V, E)$ of size $\leq k$, then

- if $v \in V$ has degree $> k$, then $v \in Q$;
- if no vertex with degree $> k$, then $|E| \leq k^2$.

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Kernelization Algorithm (Ker) for deciding if $vc(G) \leq k$

Remove isolated vertices
If $|E| = 0$, *Return TRUE*. **Else if** $k = 0$, *Return FALSE*
Else if no vertex of degree $> k$ and $|E| > k^2$, *Return FALSE*
Else if v is a vertex of degree $> k$. Return $Ker(G \setminus v, k-1)$.
Else Return $BB(G, k)$

Complexity: $O(2^k k^3 + |V|^2)$

polynomial-time "data-reduction" to an instance of order k^2.

Kernelization vs. FPT

Kernelization

A kernelization for a parameterized problem Π is an algorithm that takes an instance (x, k) and maps it in time polynomial in $|x|$ and k to an instance (x', k') s.t.

- $(x, k) \in \Pi \Leftrightarrow (x', k') \in \Pi$,
- $k' + |x'| \leq g(k)$ where g is a function called the size of the kernel.

"Preprocessing" algorithm that reduces to an instance of size $\leq$ a function only of k.

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Kernelization $\Leftrightarrow$ FPT

$\Rightarrow$ $n^{O(1)}$ "data-reduction" $+$ brute force on instance of size dependent only on k

$\Leftarrow$. Assume there exists an algorithm solving the problem in time $f(k)n^{O(1)}$.

- if $n \leq f(k)$, nothing to be done;
- else, $f(k)n^c \leq n^{c+1}$ and the algorithm is polynomial in n. Apply it and return a trivial *Yes* or *No* instance.

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Can we always get a kernel of polynomial size?

- OK for k-Vertex-Cover, k-Feedback Vertex-Set, k-Planar Dominating Set...
- NOK for k-Path... (method of compositionability)

Linear Kernel for k-Vertex-Cover

Fractional relaxation (LP) for Vertex Cover:

$$\begin{array}{llcll} \text{Min.} & \sum_{v \in V} x_v & & & \\ \text{s.t.:} & x_v + x_u & \geq & 1 & \forall \{u,v\} \in E \\ & x_v & \geq & 0 & \forall v \in V \end{array}$$

Theorem: Optimal solution: $(x_v)_{v \in V}$

Let $V_1 = \{v \in V \mid x_v > 1/2\}$ and $V_{1/2} = \{v \in V \mid x_v = 1/2\}$.
Then, $\exists$ optimal (Integral) Vertex-Cover Q such that $V_1 \subseteq Q \subseteq V_1 \cup V_{1/2}$

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Linear Kernel (LK) for deciding if $vc(G) \leq k$

If $|E| = 0$, *Return TRUE*
Remove isolated vertices
Let $(x_v)_{v \in V}$ be an optimal solution obtained by LP
If optimal fractional solution $> k$, *Return FALSE*

Else let $V_1 = \{v \in V \mid x_v > 1/2\}$.
  If $V_1 \neq \emptyset$ then *Return* $LK(G \setminus V_1, k - |V_1|)$.
  Else Return $BB(G, k)$

When $V_1 = \emptyset$, $x_v = 1/2$ for all $v \in V$, and so $|V| \leq 2k$ kernel of linear size.

A bit on scheduling

A bit of Scheduling from parameterized point of view

"Surprisingly", scheduling problems have received "few" attention in the context of parameterized complexity until the last 5 years.

"Reason": "Many numerical input data, which alone render many problems NP-hard" [Mnich,Wiese 15]

Some known results on Makespan minimization... according to different parameters

- on identical machines: FPT parameterized by the size of the solution [Alon *et al.* 98]
- without preemption: FPT parameterized by the maximum processing time of a job [Mnich,Wiese 15]
- unrelated machines: FPT parameterized by the # of distinct processing times and the # of machines [Mnich,Wiese 15]
- unrelated machines: FPT parameterized by the treewidth of the primal graph [Jansen,Maack,Solis-Oba 20]
- ...

Parameterized Complexity vs. Approximation

Efficient Polynomial Time Appoximation Scheme (EPTAS)

For all $\epsilon > 1$, ϵ-approximation algorithm running in time $f(\frac{1}{\epsilon})n^{O(1)}$.

Theorem: EPTAS $\Rightarrow$ FPT parameterized by the size of the solution [Bazgan 95]

Proof for minimization problem:

Apply the EPTAS for $\epsilon = \frac{1}{k+1}$.

If the obtained solution has value $\leq k$, then *TRUE*.

Else, $OPT \geq Value(Sol)/(1 + \frac{1}{k+1}) \geq (k+1)/(1 + \frac{1}{k+1}) > k$, then *FALSE*.

A simple example: width of dependency DAG

Inputs:	set T of n tasks, unit length, with individual deadlines $d(t)$ and precedence constraints (partial order P).
Output:	Is there a schedule of T with at most k late task?

Theorem: above problem is $W[1]$-hard parameterized by k [Fellows,McCartin 03]

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width of P: $w(P) = \min. \#$ of chains forming a partition of P

Theorem: above problem is FPT in $w(P)$ and k [Fellows,McCartin 03]

- Compute (in time $O(n^{2,5})$) a partition of P into $w(P)$ chains [Brightwell 94];
- If a maximal element e has deadline $\geq n$, put it in last position and recurse on $T \setminus \{e\}$;
- Else branch on the at most $w(P)$ maximal elements (any maximal element put in last position will be late).

Limited recursion depth $\Rightarrow$ Complexity $O(w(P)^{k+1}n + n^{2,5})$.

Powerfull Meta-Theorems for FPT algorithms

Several powerfull Meta-Theorems for FPT (in graphs)

Minor Graph Theorem [Robertson, Seymour 04]

Every minor-closed property is recognizable in time $O(n^3)$ time.

(Finite number of minimal obstructions and $O(f(|V(H)|)n^3)$ algorithm for deciding if an n-node graph admits H as minor.)

(but very, very,... huge constants and non-constructive algorithm)

Courcelle's Theorem [Courcelle 90]

Every graph property definable in the monadic second-order logic of graphs can be decided in linear time on graphs of bounded treewidth.

Bi-dimensionnality theory [Demaine,Hajiaghayi 08]

FPT and EPTAS algorithms in bounded genus graphs for many problems.

(Based on the Grid-Minor Theorem and on Courcelle's Theorem.)

Meta-Kernelization [Bodlaender *et al.* 16]

Linear Kernels for huge family of graph's problems.

Conclusion

FPT in P: Not only NP-hard problems are concerned, e.g.:

- Radius and Diameter can be solved in $2^{O(k \log k)} n^{1+O(1)}$-time, where k is treewidth. [Abboud,Vassilevska Williams,Wang 16]
  (sub-cubic algorithms are unexpected in general graphs)
- Maximum Matching can be solved in $O(k^4 n + m)$-time where k is either the modular-width or the P_4-sparseness. [Coudert,Ducoffe,Popa 19]

Scheduling, many open problems:

M. Mnich, R. Bevern: Parameterized complexity of machine scheduling: 15 open problems. Comput. Oper. Res. 100: 254-261 (2018)

Bring Parameterized algorithms to practice

E.g., last three Meta-theorems of previous slide are based on the computation of "good" tree-decompositions.

Computing treewidth: NP-hard, not-approximable, FPT (but not practical), $O(2^{O(k)} n)$ algorithm that decides if a graph has treewidth at most $5k + 4$.

Practical exact or approximation algorithms?

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Merci !