Tree-decompositions with bags of small diameter

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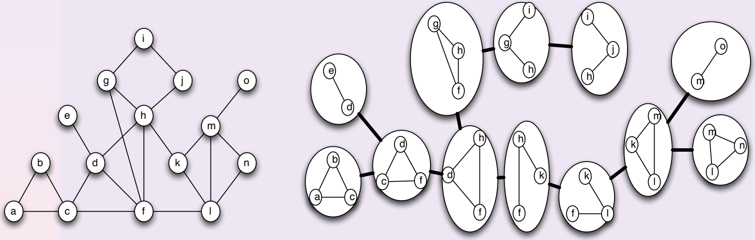
COATI Christmas seminar, December 2022

joint work with Thomas Dissaux, Guillaume Ducoffe and Simon Nivelle

Tree-Decompositions

[Robertson and Seymour 83]

Tree-decomposition: Representation of a graph as a Tree with connectivity properties



Tree T + family $\mathcal{X} = (X_t)_{t \in V(T)}$ of "bags" (sets of vertices of G)
Important: intersection of two adjacent bags = separator of G

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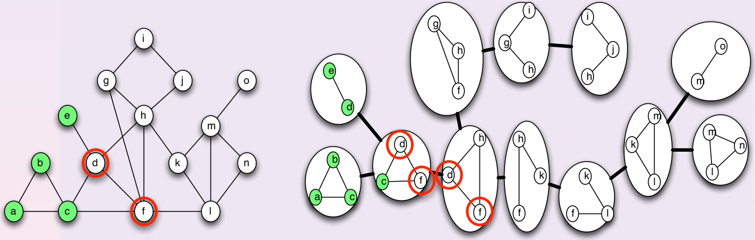

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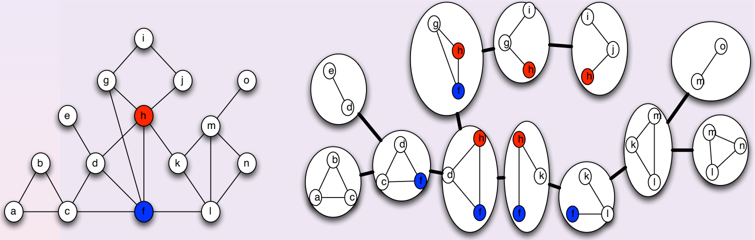

Tree T + family $\mathcal{X} = (X_t)_{t \in V(T)}$ of "bags" (sets of vertices of G)
Important: intersection of two adjacent bags = separator of G

- $\bigcup_{t \in V(T)} X_t = V(G)$;
- for any $uv \in E(G)$, there exists a bag X_t containing u and v;
- for any $v \in V(G)$, $\{t \in V(T) \mid v \in X_t\}$ induces a subtree.

Tree-Decompositions [Robertson and Seymour 83]

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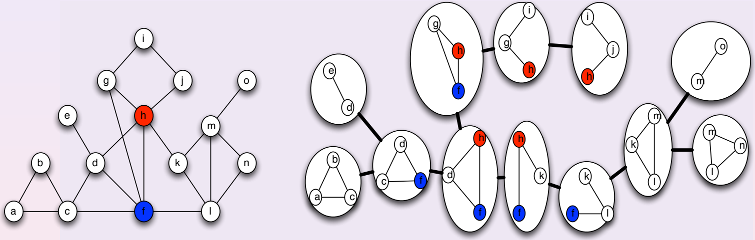

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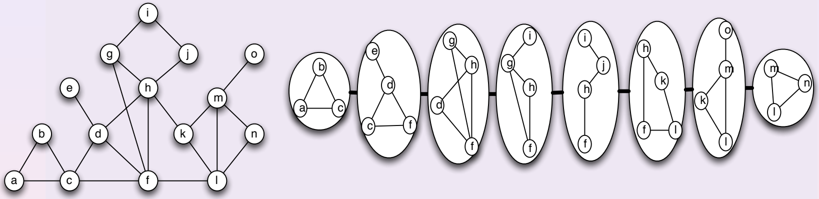
Important: intersection of two adjacent bags = separator of G

Width of $(T, \mathcal{X})$: size of largest bag (minus 1)
Treewidth of a graph G, $tw(G)$: min width over all tree-decompositions.

Path-Decompositions

[Robertson and Seymour 83]

Path-decomposition: Representation of a graph as a Path with connectivity properties



Sequence $(X_1, \cdots, X_q)$ of "bags" (sets of vertices of G) s.t.

- $\bigcup_{1 \leq i \leq q} X_i = V(G)$;
- for any $uv \in E(G)$, there exists a bag X_i containing u and v;
- for any $1 \leq i \leq j \leq k \leq q$, $X_i \cap X_k \subseteq X_j$.

Width of $(T, \mathcal{X})$: size of largest bag (minus 1)
Pathwidth of a graph G, $pw(G)$: min width over all path-decompositions.

Many important Algorithmic Applications of *tw*

- cornerstone of Graph Minors Theorem [Robertson and Seymour 1983-2004]

  $\Rightarrow$ any graph property ($\Pi(G) \leq k$) that is closed under minor is FPT in k

- problems expressible in MSOL solvable in polynomial time in graphs of bounded treewidth (dynamic programming) [Courcelle, 90]

  any such problem is FPT in *tw*

- design of sub-exponential algorithms in some graph classes (e.g., planar, bounded genus, H-minor-free...) (bi-dimensionality) [Demaine *et al.* 04]
- design of FPT algorithms (meta-kernelization/protrusions) [Fomin *et al.* 09]

Main Problem: Computing tree-decomposition

Deciding if $tw(G) \leq k$? $\Rightarrow$ Very hard!

Exact algorithms

- NP-hard if k part of the input [Arnborg,Corneil,Prokurowski 87]
- FPT: algorithm in time $O(2^{k^3} n)$ [Bodlaender,Kloks 96]
- "practical" algorithms only for graph with treewidth ≤ 4 e.g., [Sanders 96]
- Branch & Bound algorithms (for small graphs) [Bodlaender *et al.* 12] [Coudert,Mazauric,N. 14]

Approximation algorithms

- 2-approximation in time $O(2^k n)$ [Korhonen 21]
- $\sqrt{\log OPT}$-approximation in polynomial-time (SDP) [Feige *et al.* 05]
- assuming Small Set Expansion Conjecture, no poly-time constant-ratio approximation [Wu,Austrin,Pitassi,Liu 14]
- $3/2$-approximation in planar graphs in time $O(n^3)$ [Seymour,Thomas 93]

Heuristics

- Mainly based on local complementations of edges (minimum fill-in: perfect elimination ordering of vertices) [Bodlaender *et al.*]

Approach: focus on other measure(s)

Two main problems:

What to do when the treewidth is large? how to compute "good" decompositions?

Instead of constraining the size of bags $\Rightarrow$ constraint bags' metric/structural properties

Some examples

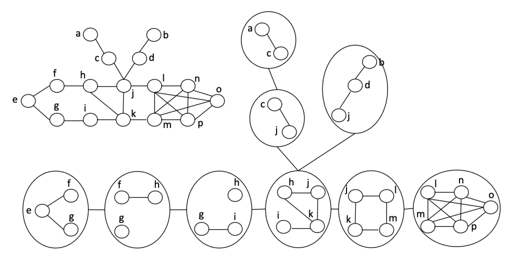
- bags' diameter (*treelength*) [Dourisboure,Gavoille 07, Lokstanov 10, Coudert,Ducoffe,N. 16]
  PTAS for TSP when bounded treelength, metric dimension FPT in treelength+max. degree...
- bags with short dominating path [Kosowski,Li,N.,Suchan 15]
  compact routing in distributed computing
- bags' chromatic number (*tree-chromatic number*) [Seymour 16]
- bags' radius (*treebreadth*) [Dragan,Köhler 14, Ducoffe,Legay,N. 20]
- bags' independence number (*tree-independence number*) [Dallard,Milanic,Storgel 21]
  Maximum Weight Independent Packing problem FPT in tree-indep. number...

In this talk, we focus on bags' diameter

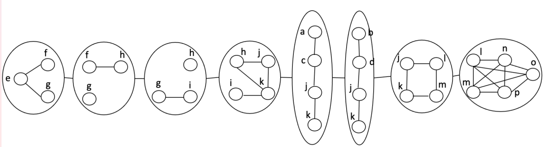
Treelength and pathlength

Length of $(T, \mathcal{X})$: $\ell(T, \mathcal{X}) = \max_{t \in V(T)} \max_{u,v \in X_t} dist_G(u, v)$. [Dourisboure,Gavoille 07]

Treelength of G, $t\ell(G)$: min. length among all tree-decompositions.



Pathlength of G, $p\ell(G)$: min. length among all path-decompositions. [Dragan,Köhler 14]



Tree/path-length *vs.* Tree/path-width

Incomparable in general:

- **Cliques:**
- **Cycles:**

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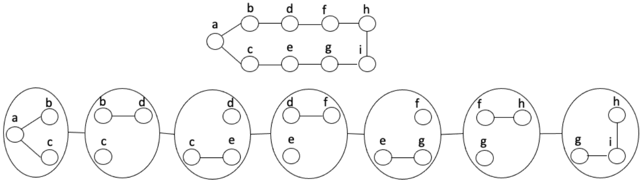

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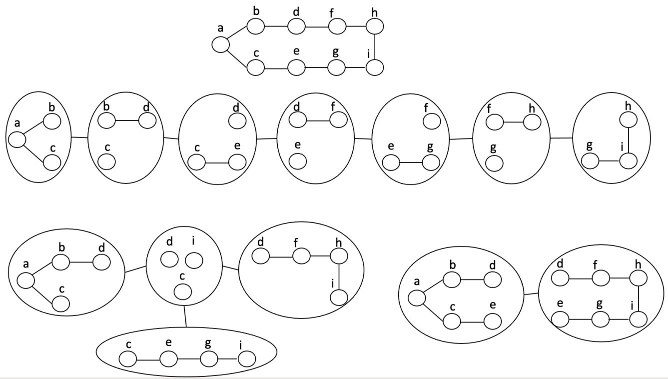

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A subgraph H of a graph G is **isometric** if the distances are "preserved".
That is, if $dist_H(u, v) = dist_G(u, v)$ for every $u, v \in V(H)$.

Tree/path-length closed under taking isometric subgraph [Dourisboure,Gavoile 07]

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Let $is(G)$ be the length of a largest isometric cycle in G.

Corollary: For any graph G, $\lceil \frac{is(G)}{3} \rceil \leq t\ell(G)$ and $\lfloor \frac{is(G)}{2} \rfloor \leq p\ell(G)$.

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Cliques and large isometric cycles are the "single" extreme cases. [Coudert,Ducoffe,N. 16]

$tl(G) = \Theta(tw(G))$ in any apex-free graph G with bounded largest isometric cycles.

Computation of tree/path-decompositions

	Treewidth $tw(G) \leq k$?	**Pathwidth** $pw(G) \leq k$?	**Treelength** $t\ell(G) \leq k$?	**Pathlength** $p\ell(G) \leq k$?
k part of the input	NP-complete [Arnborg et al. 87]			
exact FPT (parameter k)	in time $2^{O(k^3)}n$ [Bodlaender,Kloks 96]		NP-c for $k=2$ [Lokshtanov 10]	NP-c for $k=2$ [Ducoffe,Legay,N. 20]
approximation algorithms (in general graphs)	$tw \log^{\frac{1}{2}}(tw)$ in time $n^{O(1)}$ [Feige et al. 08]; $2 \cdot k$ in time $2^{O(k)}n$ [Korhonen 21]	$pw \log^{\frac{3}{2}}(pw)$ in time $n^{O(1)}$ [Feige et al. 08]	$3 \cdot t\ell$ in time $O(n)$ [Dourisboure,Gavoille 07]; no $\frac{3}{2}$-approx unless $P=NP$ [Lokshtanov 10]	$2 \cdot p\ell$ in time $O(n)$ [Dragan et al. 17]
planar graphs	Open; $\frac{3}{2}$-approx in time $O(n^3)$ [Seymour,Thomas 93]	NP-complete [Monien, [Sudborough 88]	Open	

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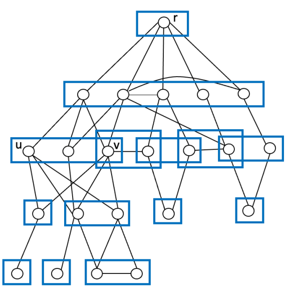
Algorithm based on a particular BFS (LexM)

Roughly: 2 vertices are in a same bag if

- they are in the same BFS-level
- there is a path between them with internal vertices further from the root



3 approximation for treelength

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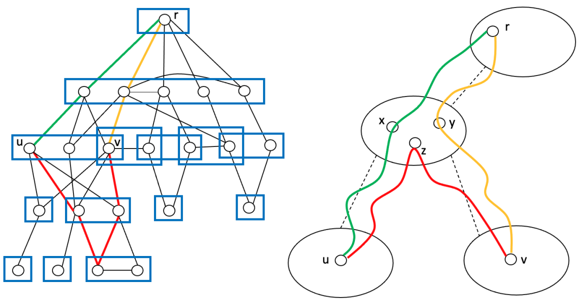

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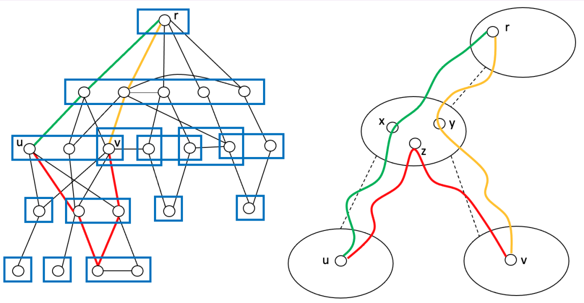
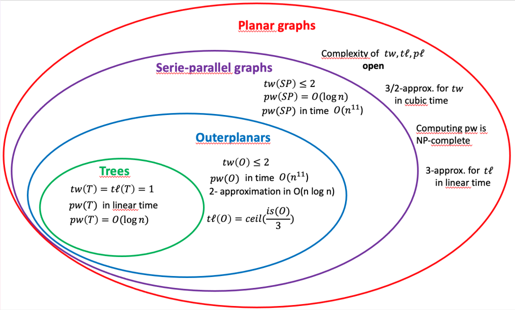
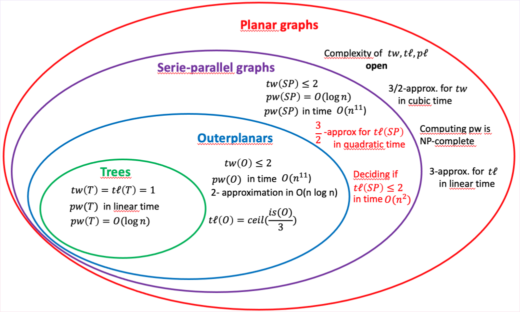


Better approximation in general graphs? in planar graphs? Use this for treewidth?

Planar graphs: known results



Planar graphs: known results and our contributions



Treelength in Serie-Parallel

[Dissaux,Ducoffe,N.,Nivelle, LAGOS 21]

- $\frac{3}{2}$-approx. in $O(n^2)$-time;
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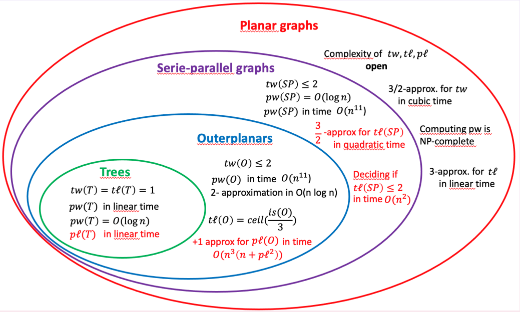

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Pathlength in Outerplanars

[Dissaux,N., LATIN 22]

- $p\ell(T)$ in linear time in trees;
- Cycles: $pl(C_n) = \lfloor \frac{n}{2} \rfloor$;
- $(+1)$-approximation in poly-time.

Treelength in Serie-Parallel graphs

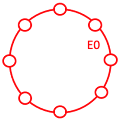
(2-connected) Serie-Parallel graphs

Serie parallel $=$ K_4-minor free graphs

G serie-parallel $\Leftrightarrow$ Nested Ear decomposition [Eppstein 92]

Recursive construction:

- Start with graph G_0 that consists of a cycle E_0;
- At step $i > 0$, obtain G_i by adding an ear E_i (a path) attached, in a nested way, to a previous ear E_j, $j < i$.



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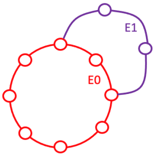

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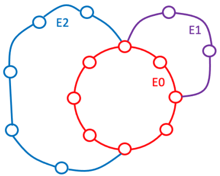

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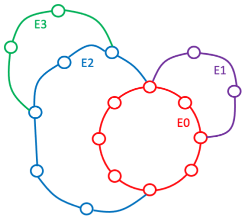

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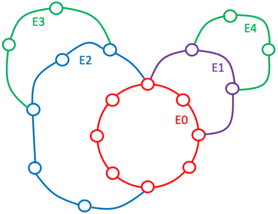

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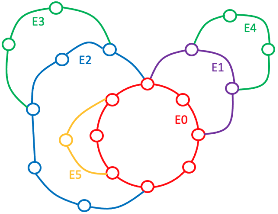

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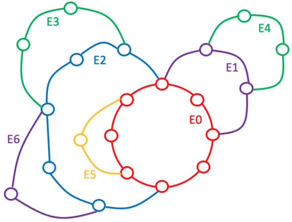

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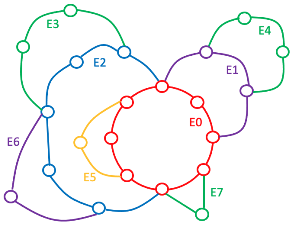

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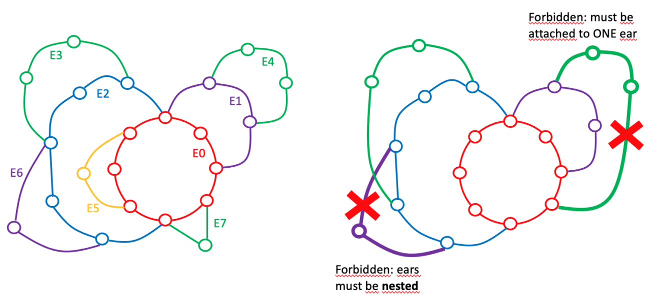
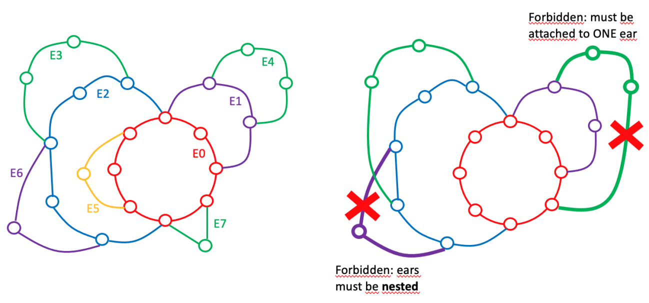


Serie parallel $=$ K_4-minor free graphs

G serie-parallel $\Leftrightarrow$ isometric Nested Ear decomposition **[Dissaux,Ducoffe,N.,Nivelle 21]**

Recursive construction:

- Start with graph G_0 that consists of a **largest isometric** cycle E_0;
- At step $i > 0$, obtain G_i, **isometric subgraph**, by adding an ear E_i (a path) attached, in a nested way, to a previous ear E_j, $j < i$.



Isometric nested Ear decomposition can be computed in time $O(n^2)$.

$\frac{3}{2}$-approximation for $t\ell$ in Serie-Parallel graphs

Very simple algorithm [Dissaux,Ducoffe,N.,Nivelle 21]

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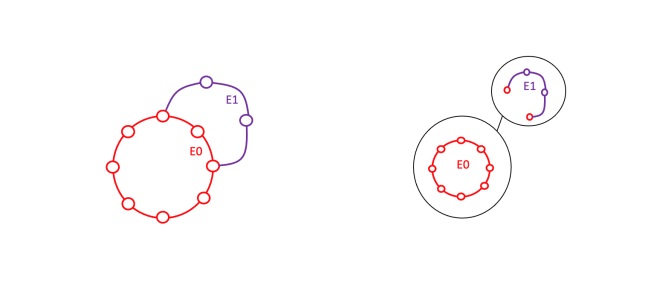

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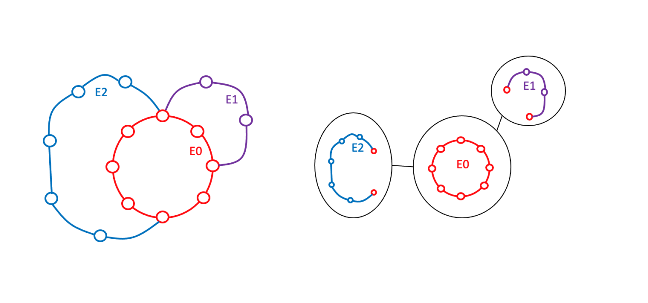

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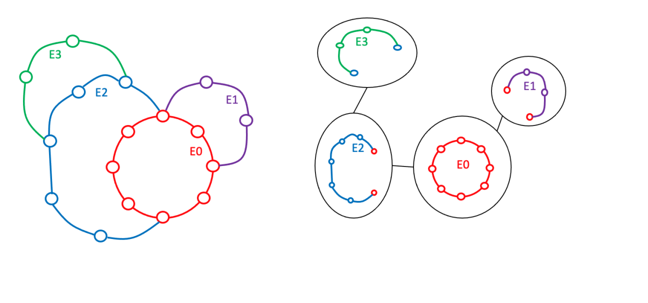

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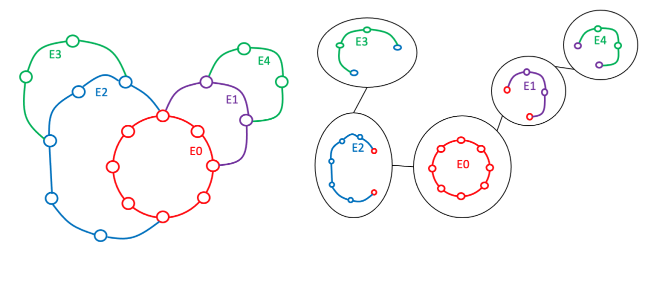

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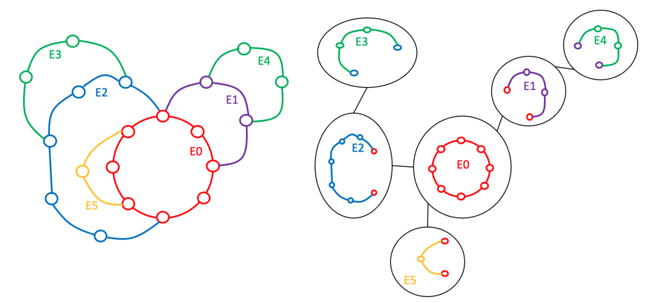

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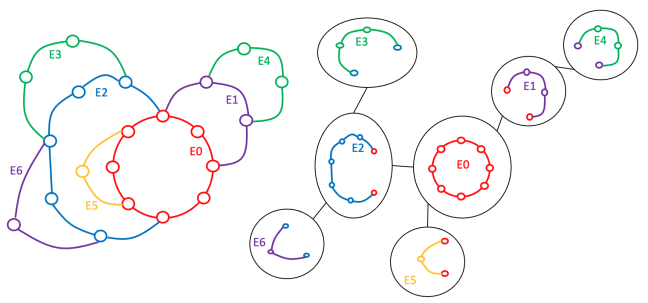

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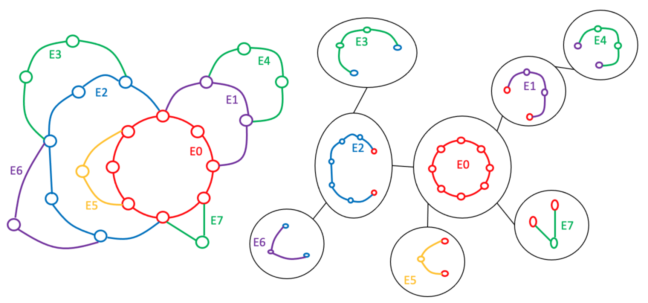

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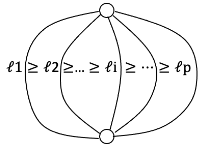
The simplest (?) subclass of Serie-Parallel graphs

Melon graph: paths linking two vertices



Theorem: [Dissaux,Ducoffe,N.,Nivelle LAGOS 21]

Let G be a melon graph with paths of lengths $\ell_1 \geq \cdots \geq \ell_p$

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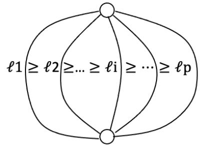

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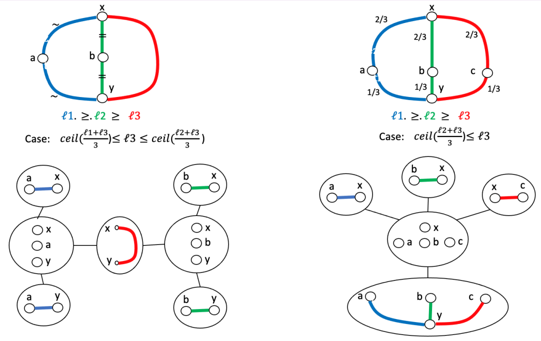

Deciding if $t\ell(G) \leq 2$ in Serie-Parallel graphs

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Characterization by forbidden isometric subgraphs [Dissaux,Ducoffe,N.,Nivelle LAGOS 21]

Let G be a Serie-Parallel graphs. Then, $t\ell(G) \leq 2$ if and only if $is(G) \leq 6$ and G has no Dumbo graph as isometric subgraph.

Polynomial-time algorithm that, given G Serie-parallel:

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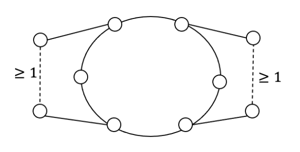
Dumbo graph:



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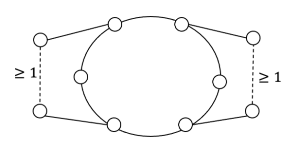


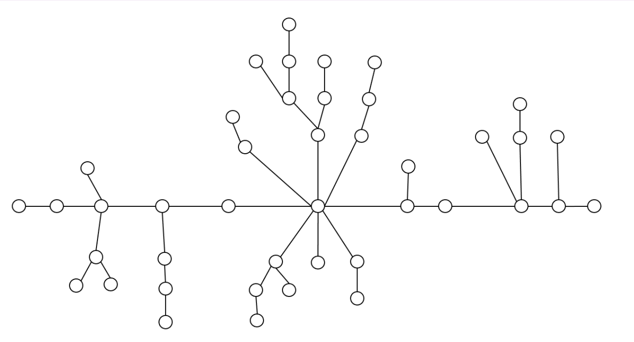
Proof:

- by induction on the number of Ears;
- must ensure that: if a forthcoming ear is attached to two vertices x and y, then there is a bag containing them;
- tedious case analysis depending on the length of the ears.

Pathlength in Outerplanar graphs

Pathlength of trees

Linear time algorithm for any tree T [Dissaux,N. LATIN 22]



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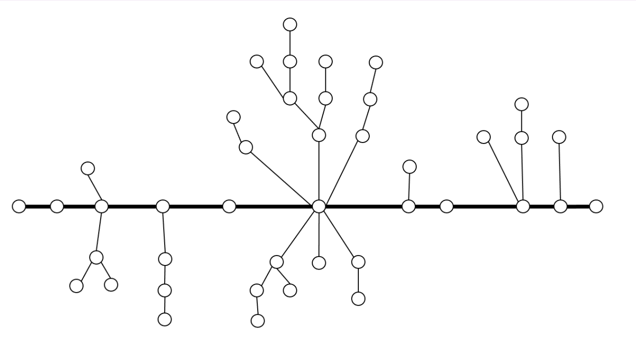

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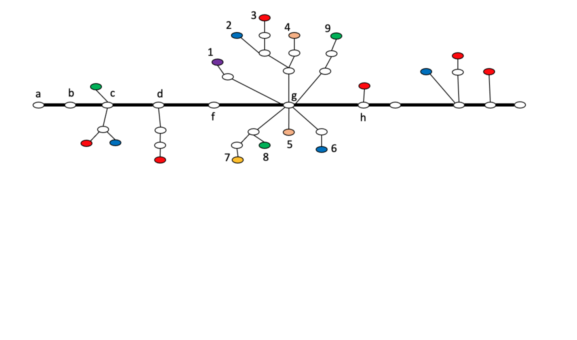

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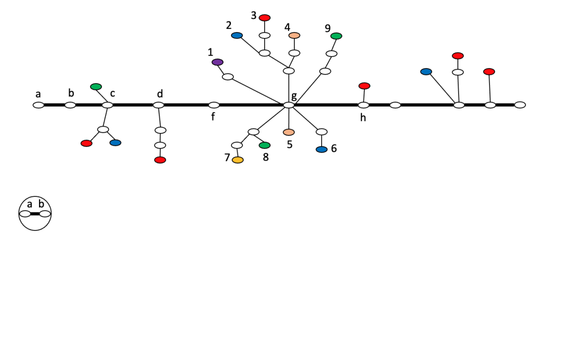

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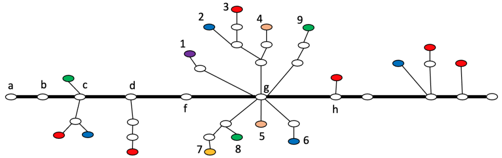

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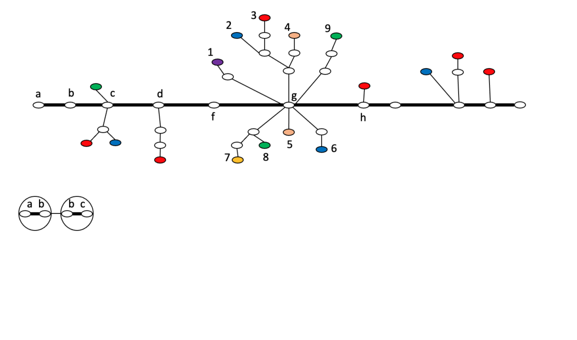

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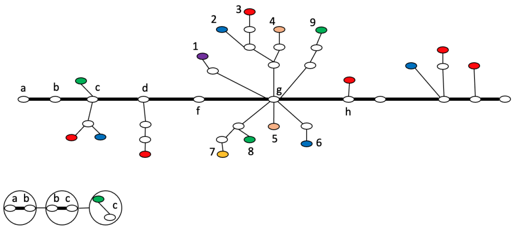

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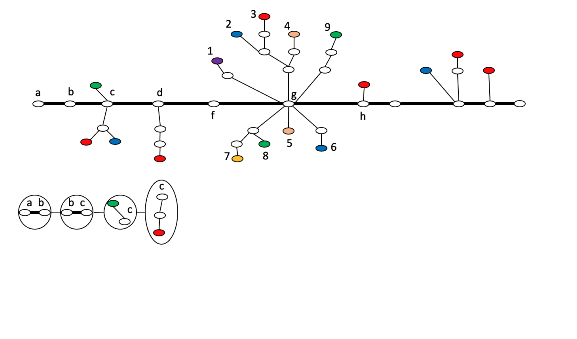

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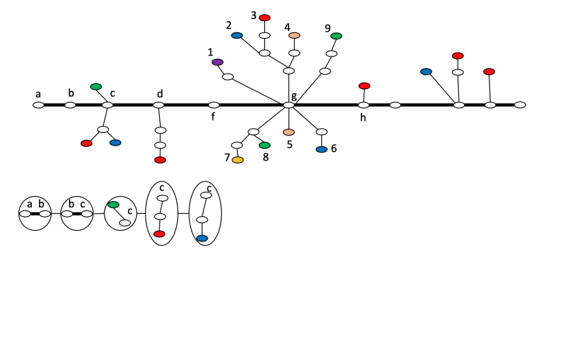

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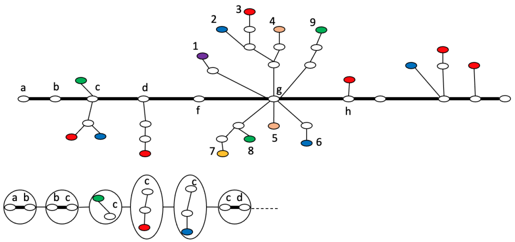

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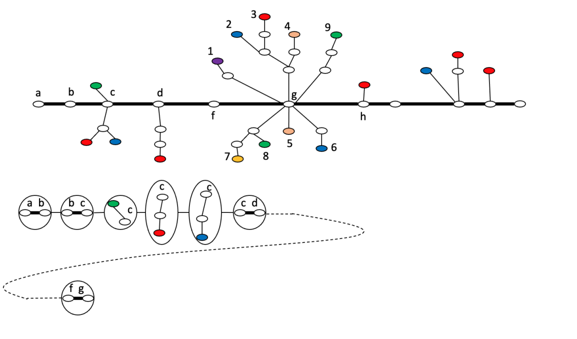

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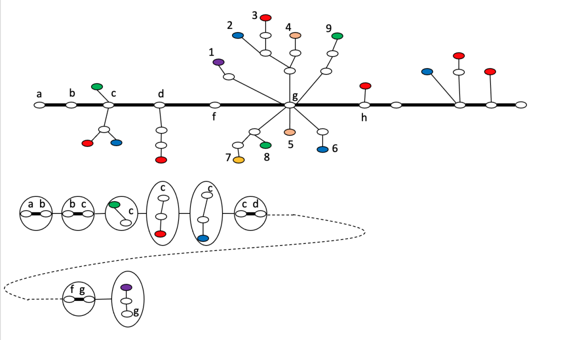

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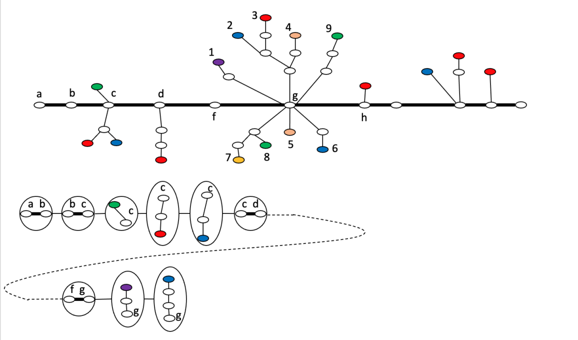

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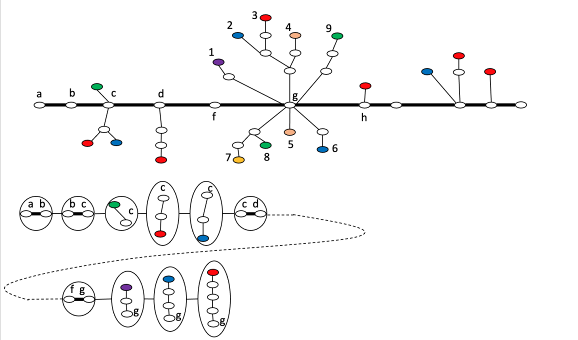

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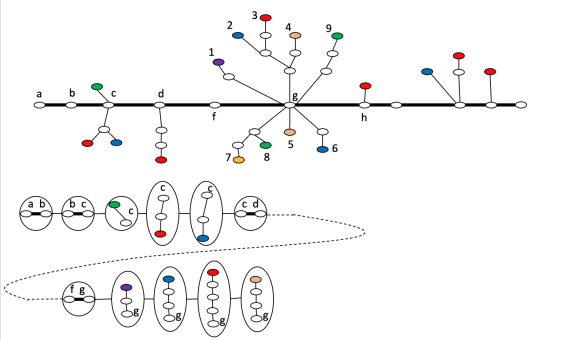

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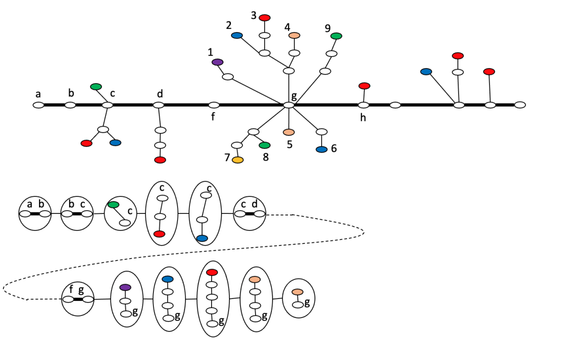

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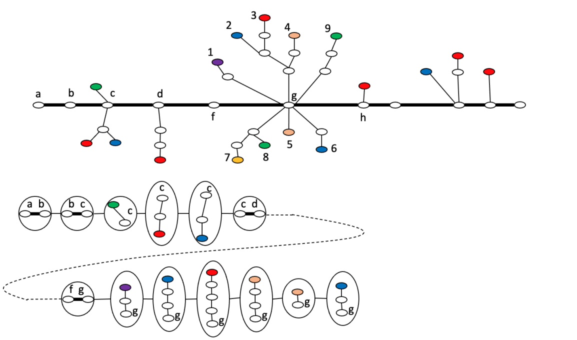

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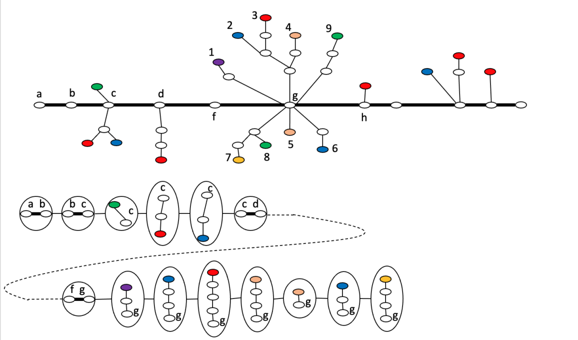

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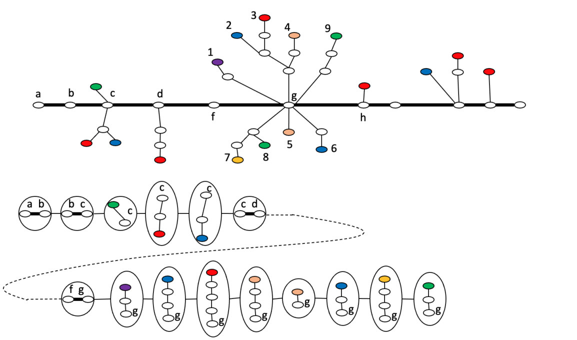

Pathlength of trees

Linear time algorithm for any tree T [Dissaux,N. LATIN 22]

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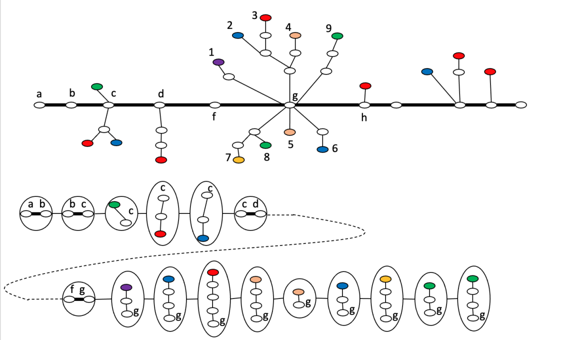

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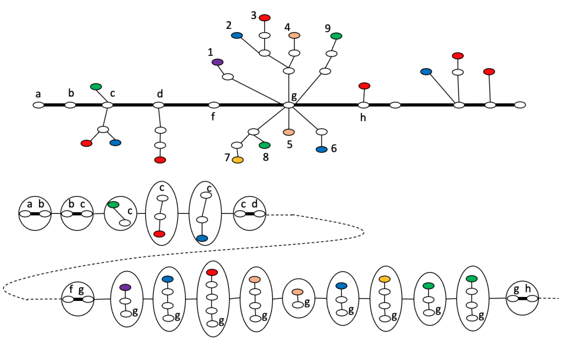

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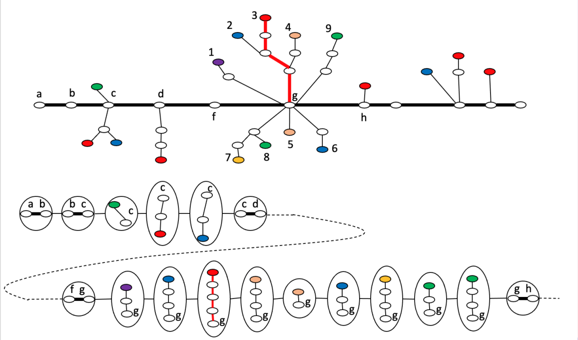

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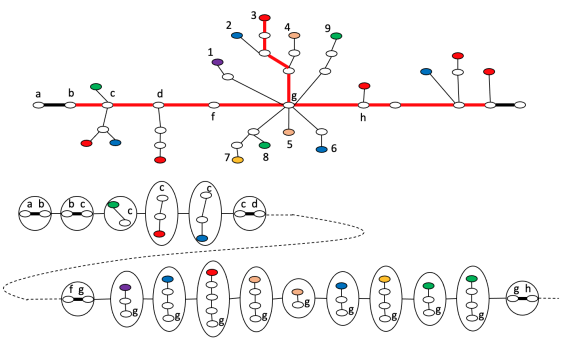

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T contains the star S_k with 3 branches of length k as isometric subgraph.

$p\ell(S_k) = k$ [DG 07]

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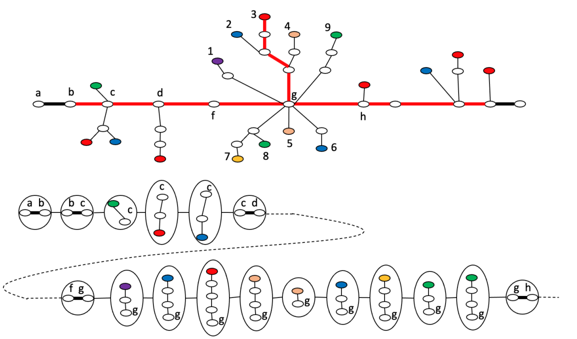

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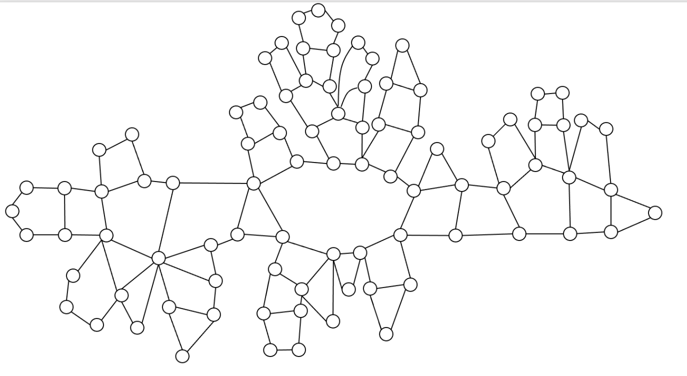
$\Rightarrow p\ell(T) = k$.

Pathlength of Outerplanar graphs

Outerplanar: $K_4, K_{2,3}$ minor-free $\Leftrightarrow \exists$ planar embedding with all vertices on outer-face

Example of 2-connected outerplanar:



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Let $k \geq 0$. There exists an algorithm that: [Dissaux,N. LATIN 22]

given an outerplanar graph G, in time $O(n^3(n+k^2))$,

- either returns a path-decomposition of length $\leq k+1$,
- or states that $p\ell(G) > k$.

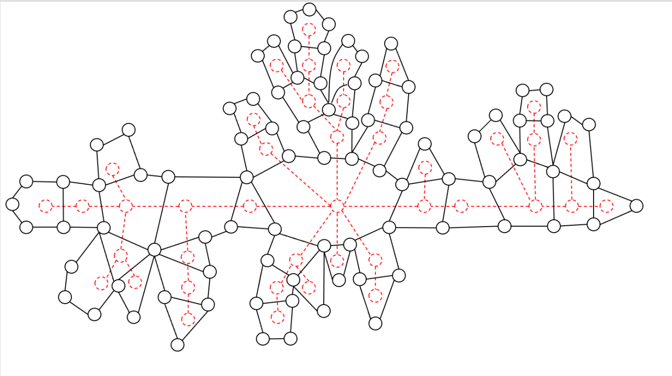
Two steps:

1. Show that, for every $k \geq p\ell(G)$, there exists a path-decomposition of length $\leq k+1$ with "good" properties;
2. Compute such a decomposition in polynomial-time.

Open: Does there exist an exact polynomial-time algorithm?

Pathlength of Outerplanar graphs: use the dual

Weak dual of outerplanar graph G is a tree G^*
Idea: "Mimic" the strategy on trees: follow a diameter, add "branches" in this order.



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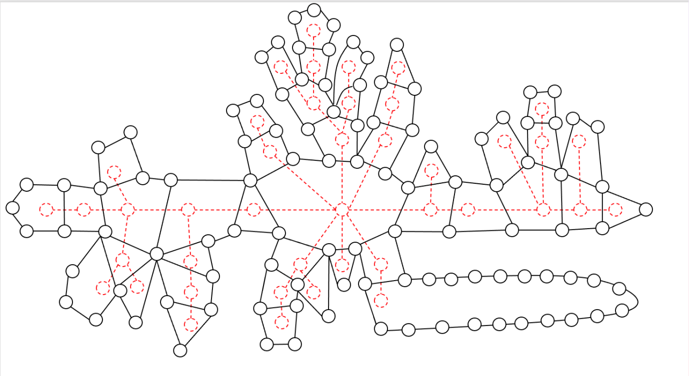

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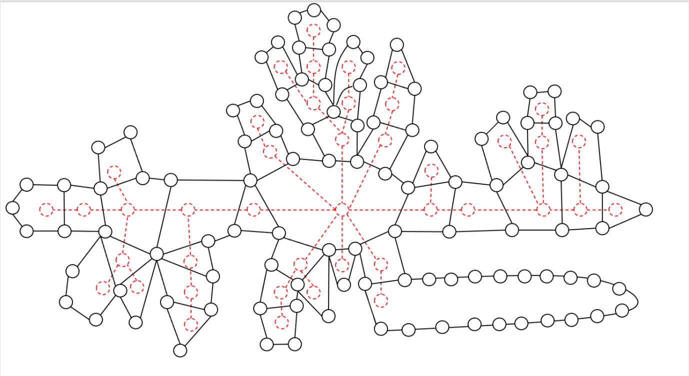
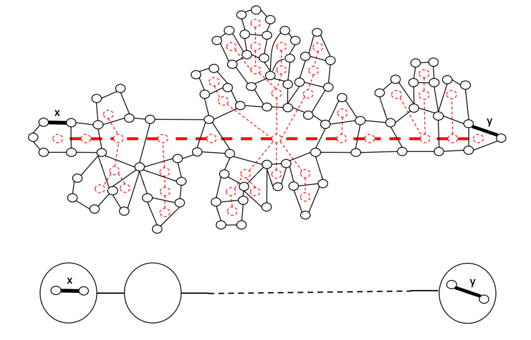
Which "main" path to follow? **We will try them all!**



(x, y)-Path-Decomposition

Let $x, y \in E(G)$. (x, y)-Path-Decomposition: x in the first bag, y in the last bag.
$p\ell(G, x, y)$: minimum length of a (x, y)-Path-Decomposition of G.

Lemma: $p\ell(G) = \min_{x,y \in E(G)} p\ell(G, x, y)$.



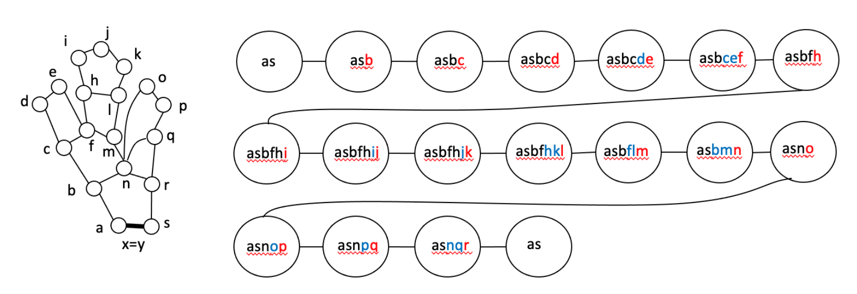
We will try to compute an optimal (x, y)-Path-Decomposition for every $x, y \in E(G)$.
("only" $O(n^2)$ possibilities).

(x, x)-Path-Decomposition: greedy algorithm

Computation of (x, y)-Path-Decomposition: Case $x = y = \{a, s\}$.

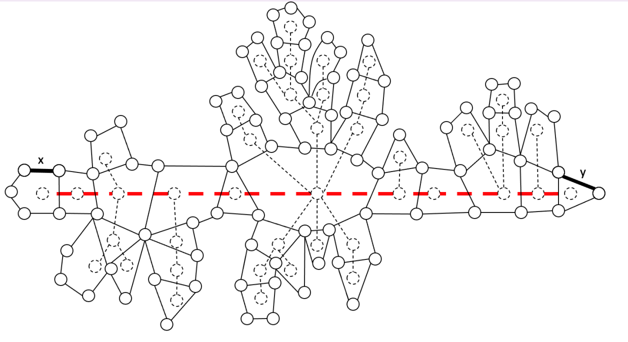
Greedy algorithm: add the vertices in the path-decomposition P in a DFS ordering (from x) guided by the outer-face.

Lemma: $length(P) = \max_{v \in V(G)} \max\{dist(a, v), dist(s, v)\} \leq p\ell(G, x, x)$.



(x, y)-Path-Decomposition: x, y not in a same face

$x \neq y \in E(G) \Rightarrow$ define a path in the dual



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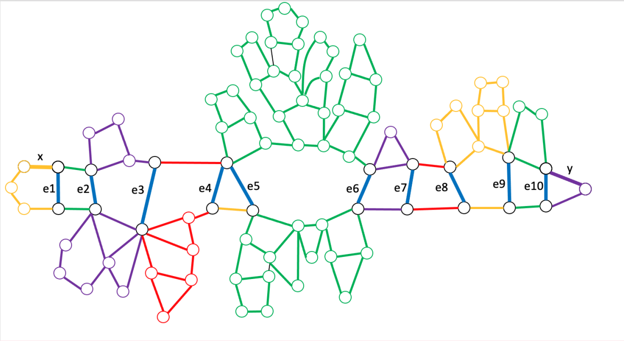
Case $x \neq y \in E(G)$, and there are $e_1, \cdots, e_q$ edge separators of x and y.



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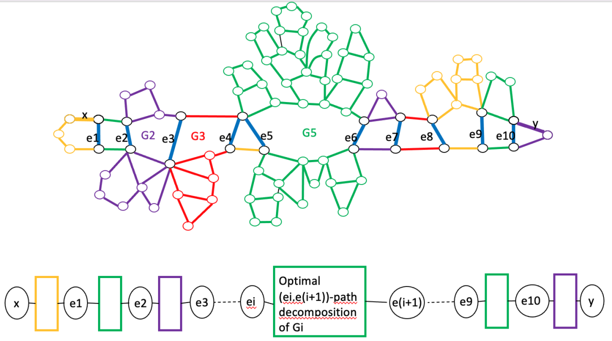
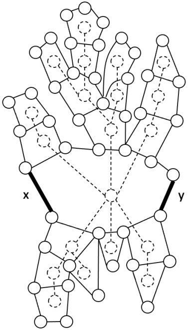
Lemma: If $x, y \in E(G)$ not in the same face, there exists an (x, y)-Path-Decomposition with length $p\ell(G, x, y)$ which is "well separated".



(x, y)-Path-Decomposition: x, y in a same face

$x \neq y \in E(F)$ for some face F.
Components of $G \setminus F$: "branches".



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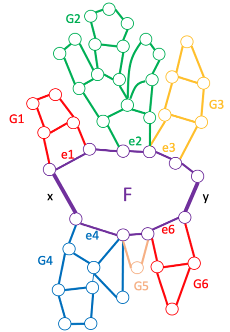

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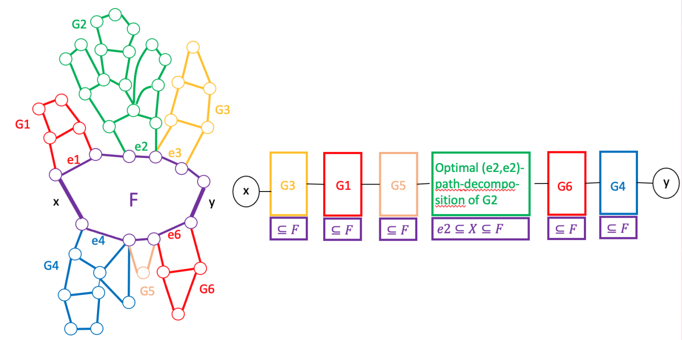
Moreover, for each branch G_i, it is a greedy (e_i, e_i)-path-decomposition.

Can we guess the ordering of the "branches"?



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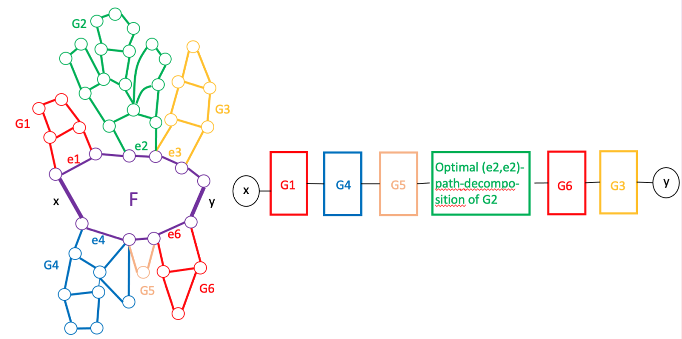

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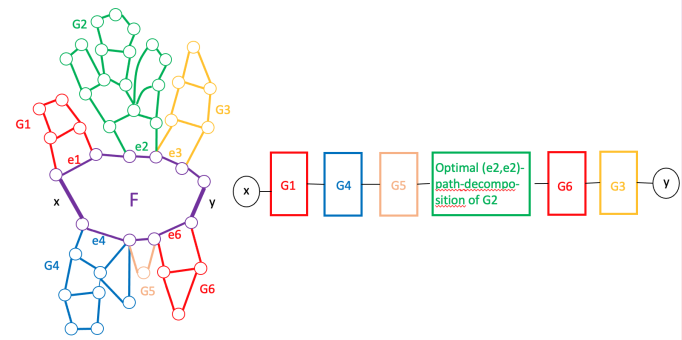
By dynamic programming, in time $O(n + |F|^2) \leq n + is(G)^2 = O(n + p\ell(G)^2)$.



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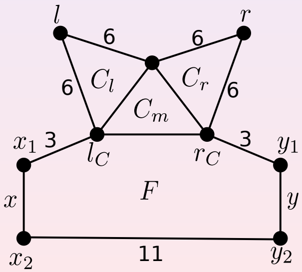
Remark: the $+1$ cannot be avoided in our algorithm.



Further work

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- New algorithmic applications?

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