Nondeterministic Graph Searching: From Pathwidth to Treewidth

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MFCS 05, September 2*nd*, 2005

Graph Searching

Goal

In an undirected simple graph,

- omniscient and arbitrary fast fugitive ;
- a team of searchers ;

We want to find a strategy that catch the fugitive
using the fewest searchers as possible.

Motivation

- game related to famous graphs'parameters : treewidth and pathwidth ;
- we introduce a parametrized version of treewidth.

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Search Strategy, Parson. [GTC,1978]

Sequence of two basic operations,. . .

1. Place a searcher at a vertex of the graph ;
2. Remove a searcher from a vertex of the graph.

. . . that must result in catching the fugitive

An edge is cleared when both its ends are occupied by a searcher.

We want to minimize the number of searchers.

Let $s(G)$ be the smallest number of searchers needed to catch a fugitive in a graph G.

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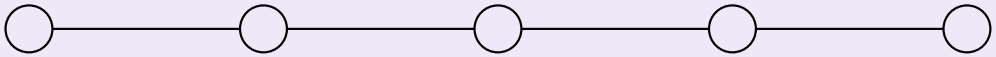
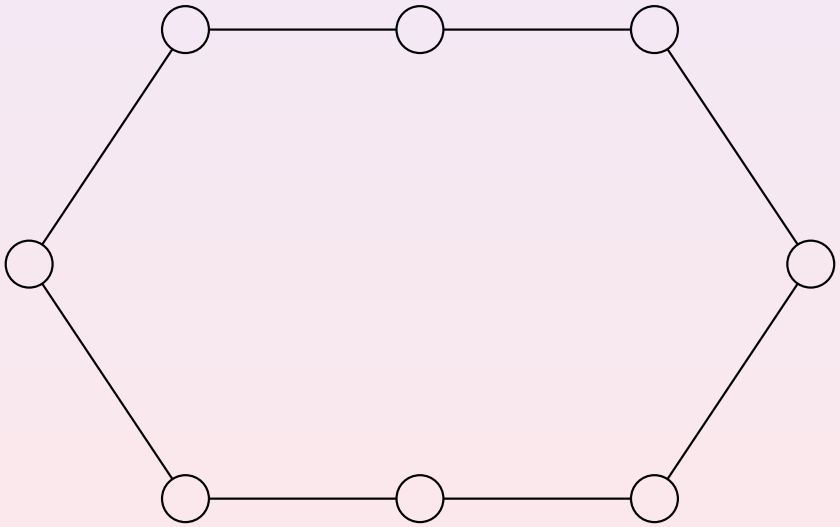
Simple Examples : Path and Ring





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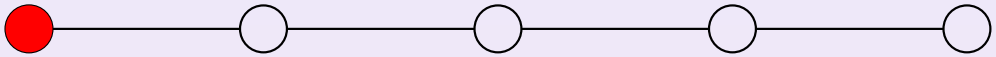

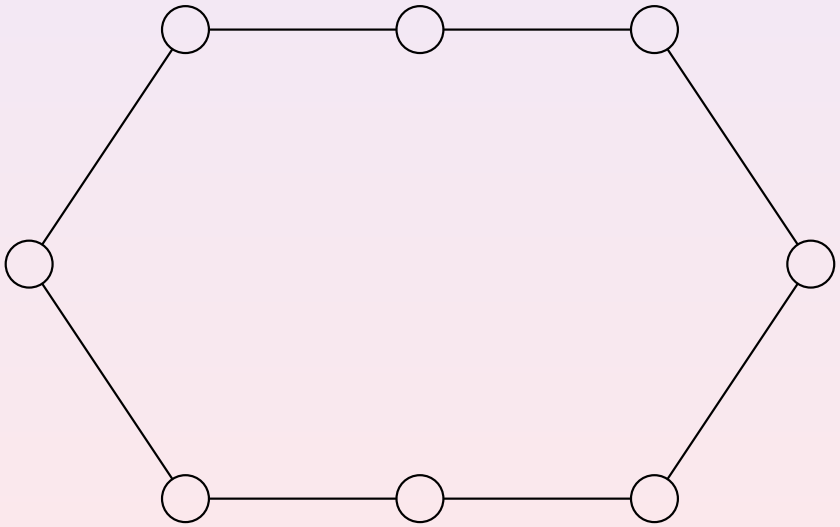

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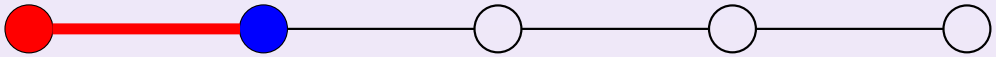

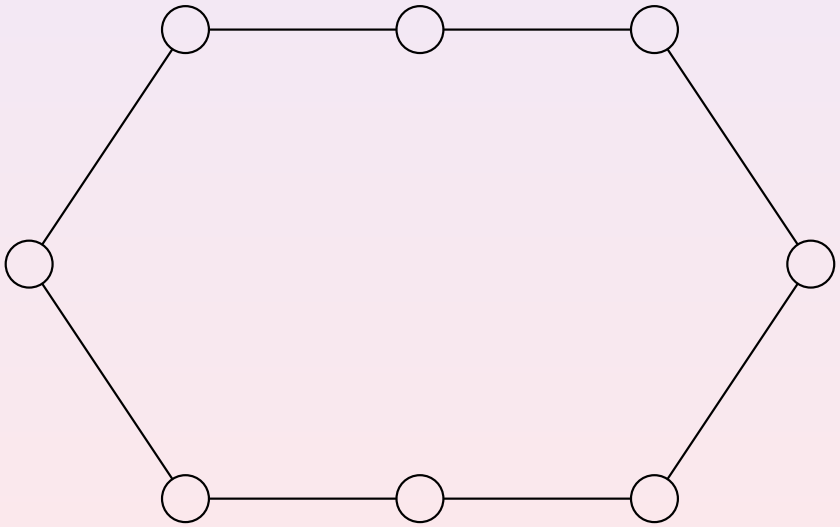

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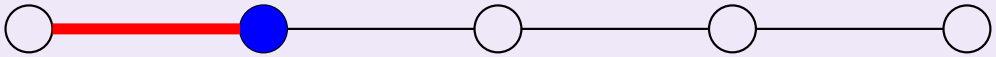

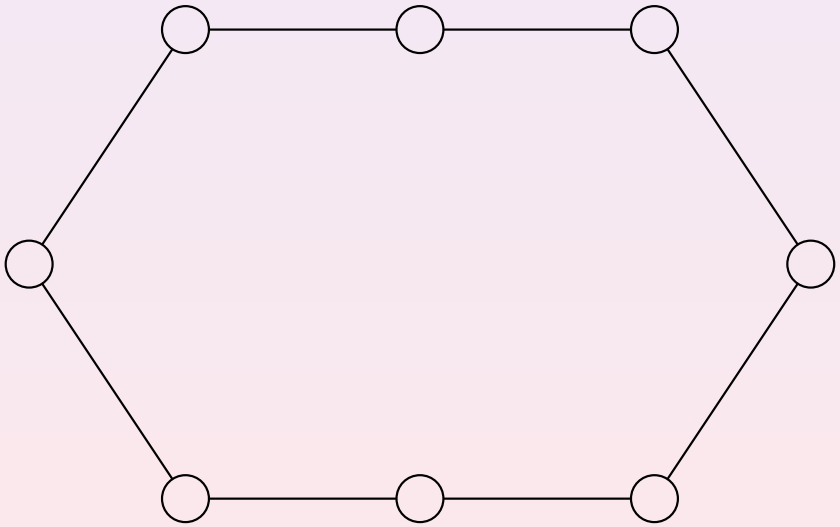

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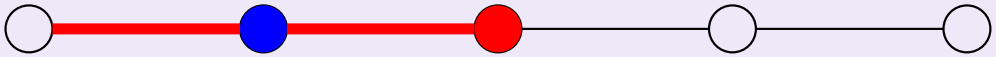

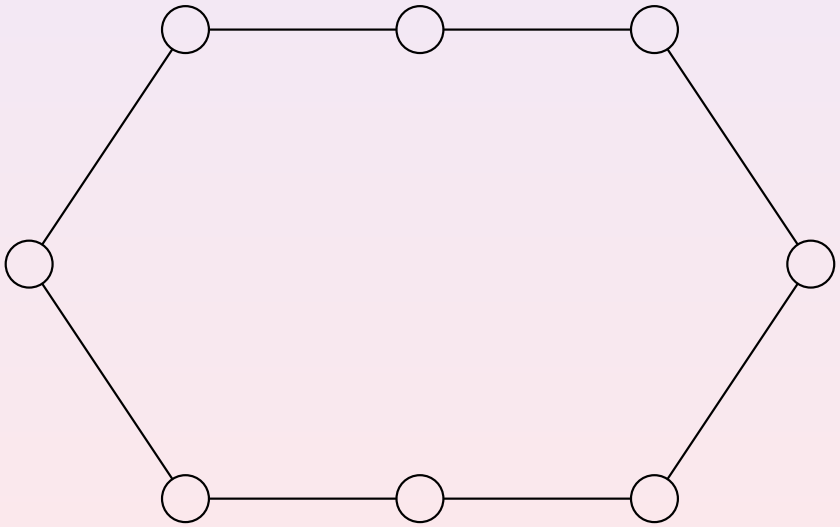

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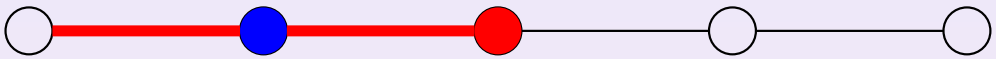
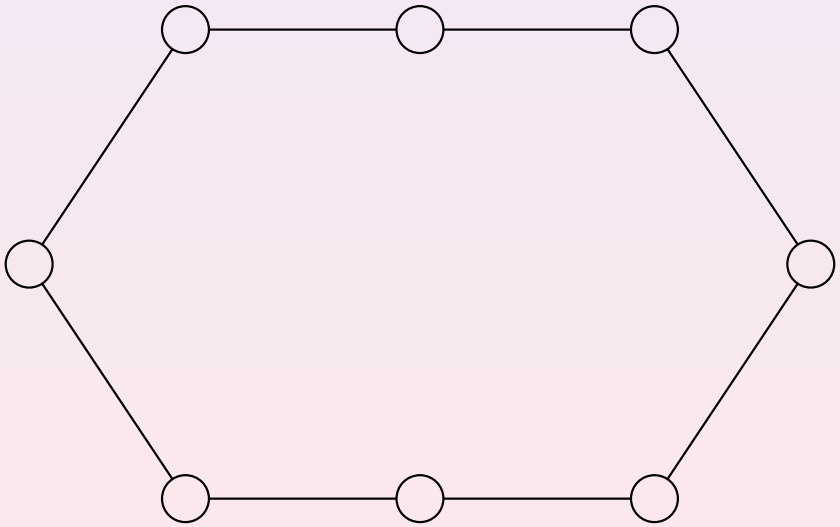


s(Path)=2



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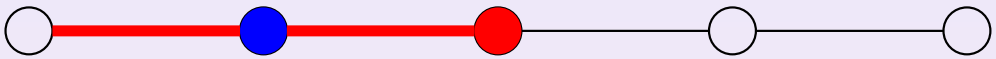

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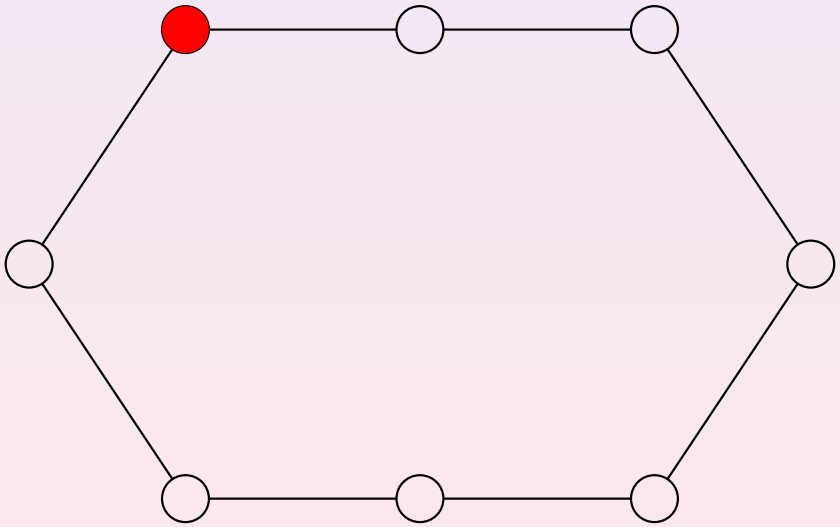

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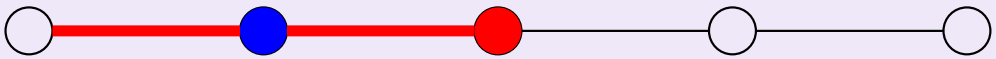

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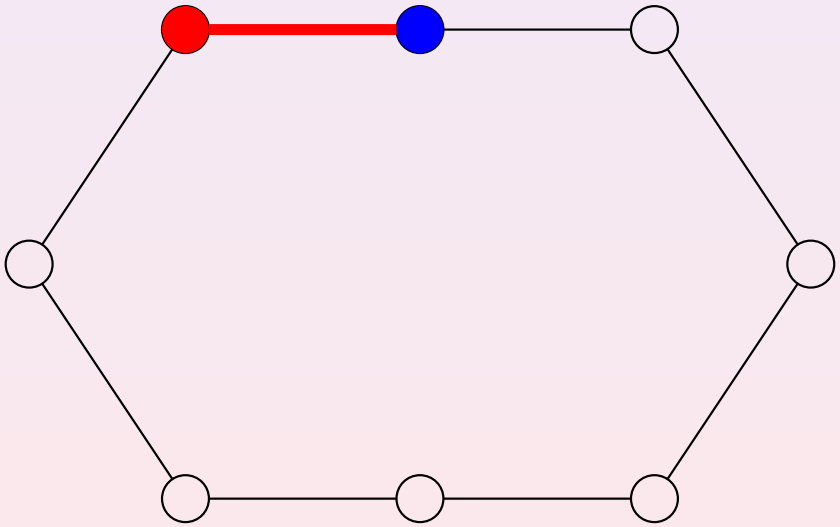

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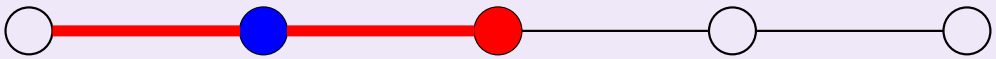

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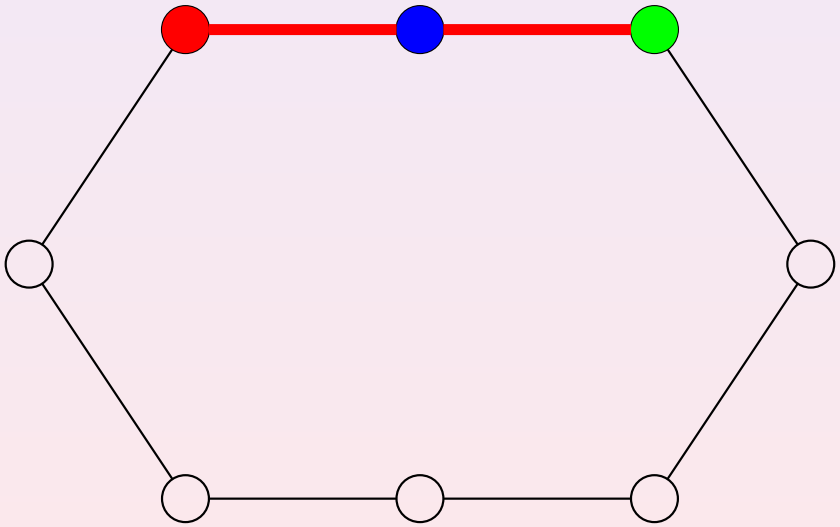

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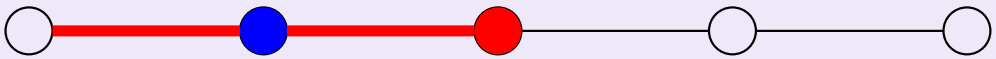

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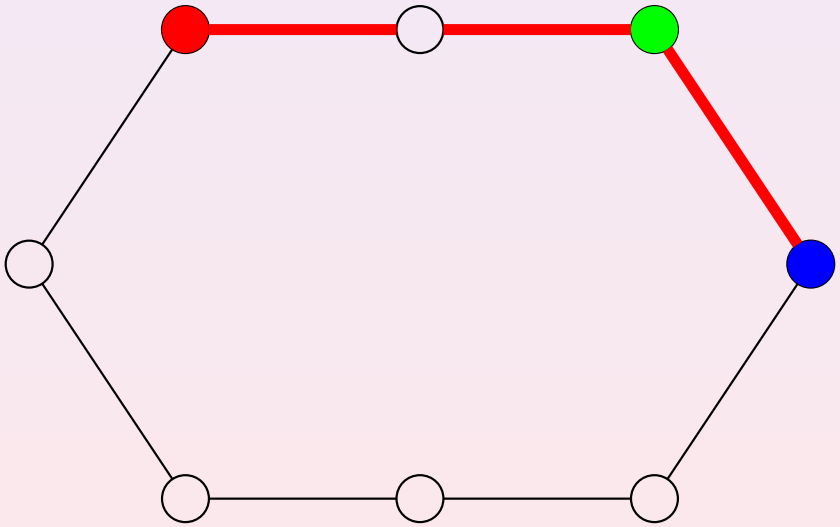

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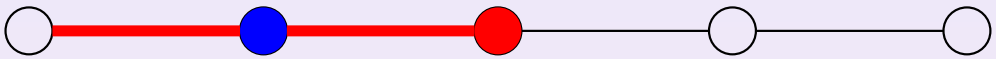

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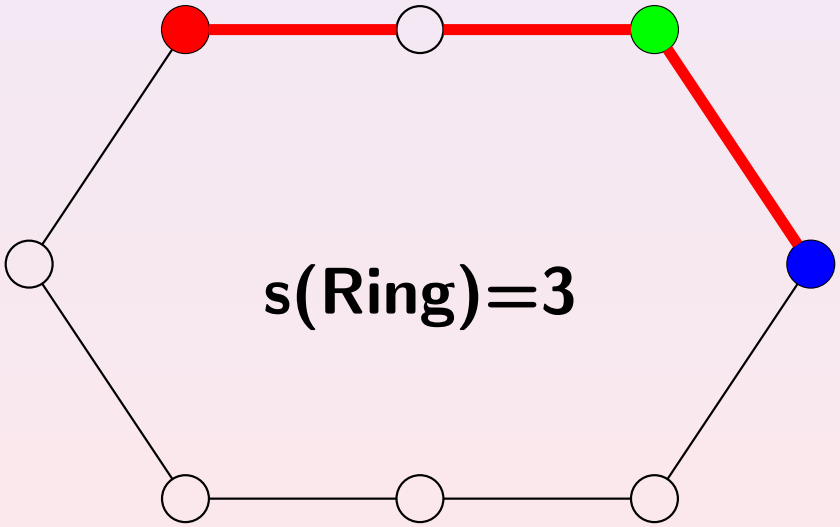
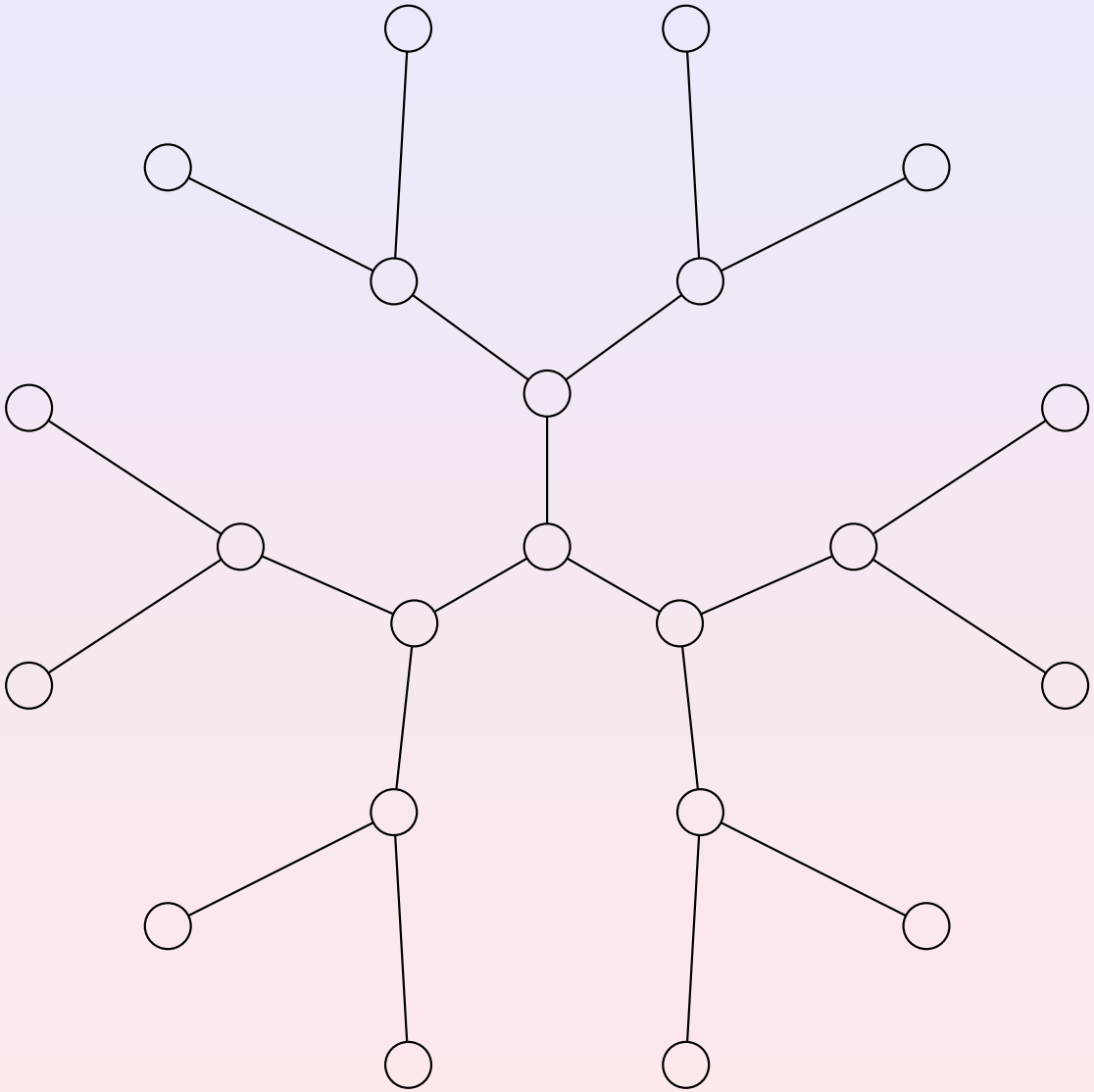


s(Ring)=3

Graph searching in a tree



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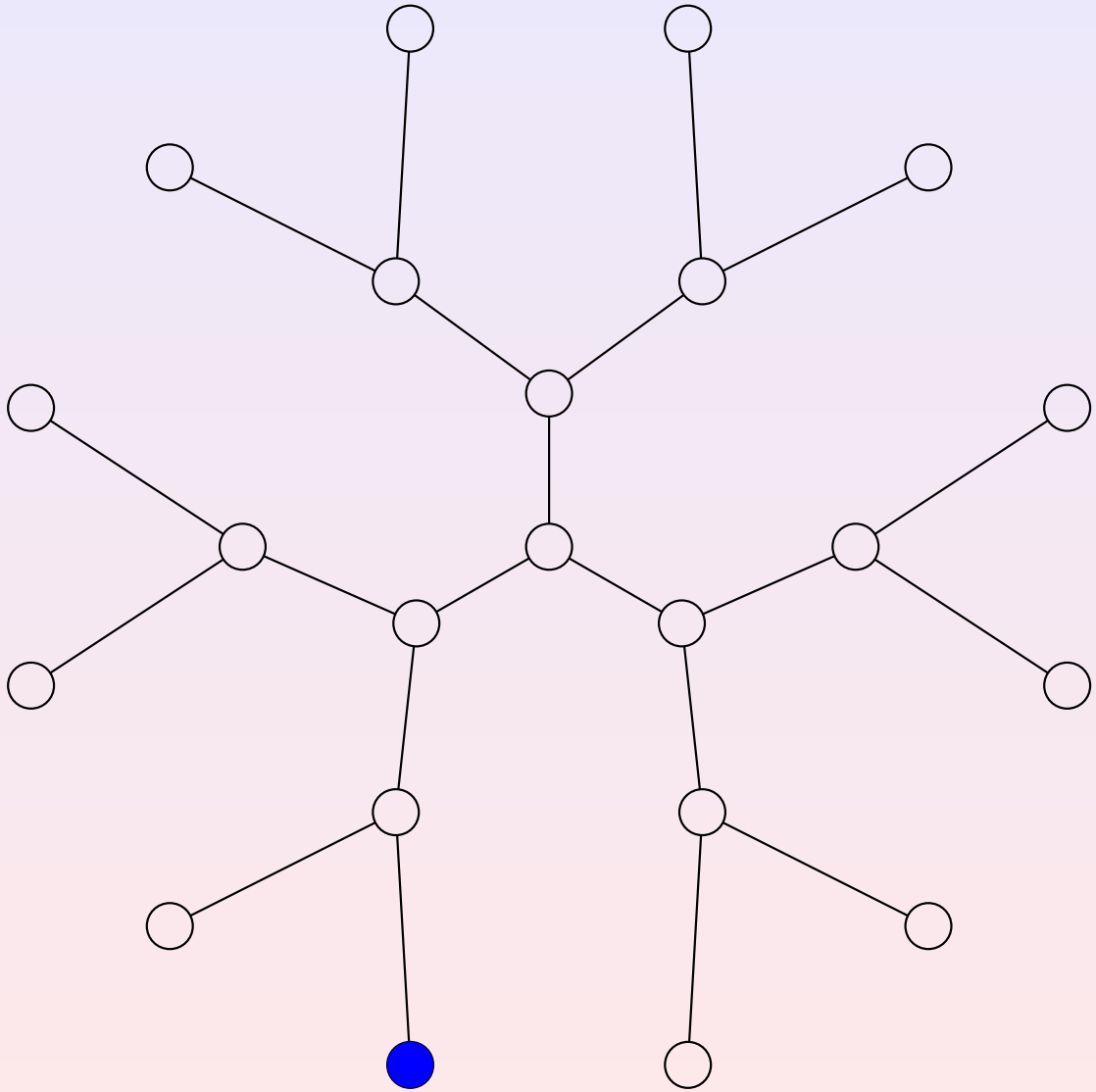

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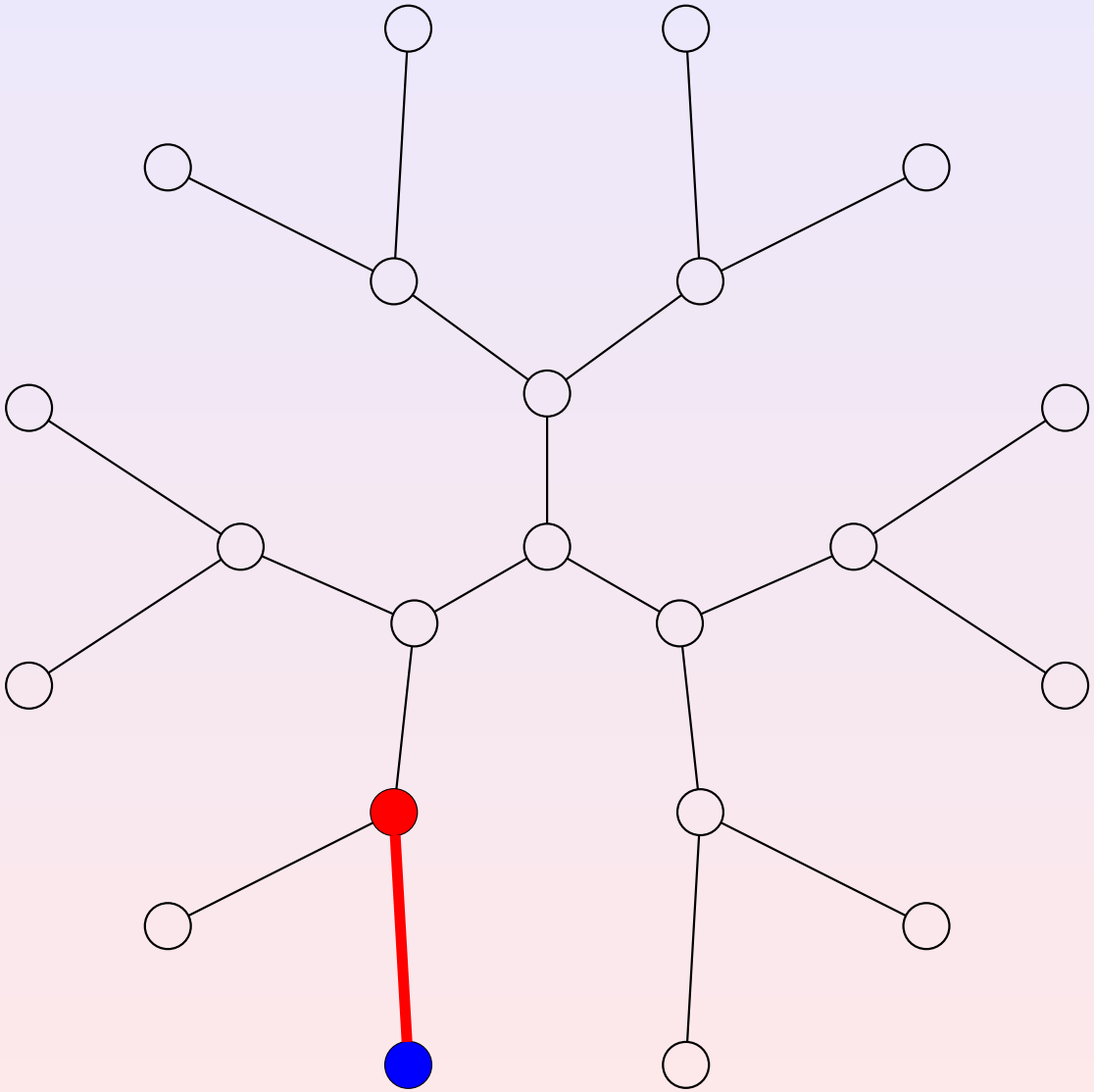

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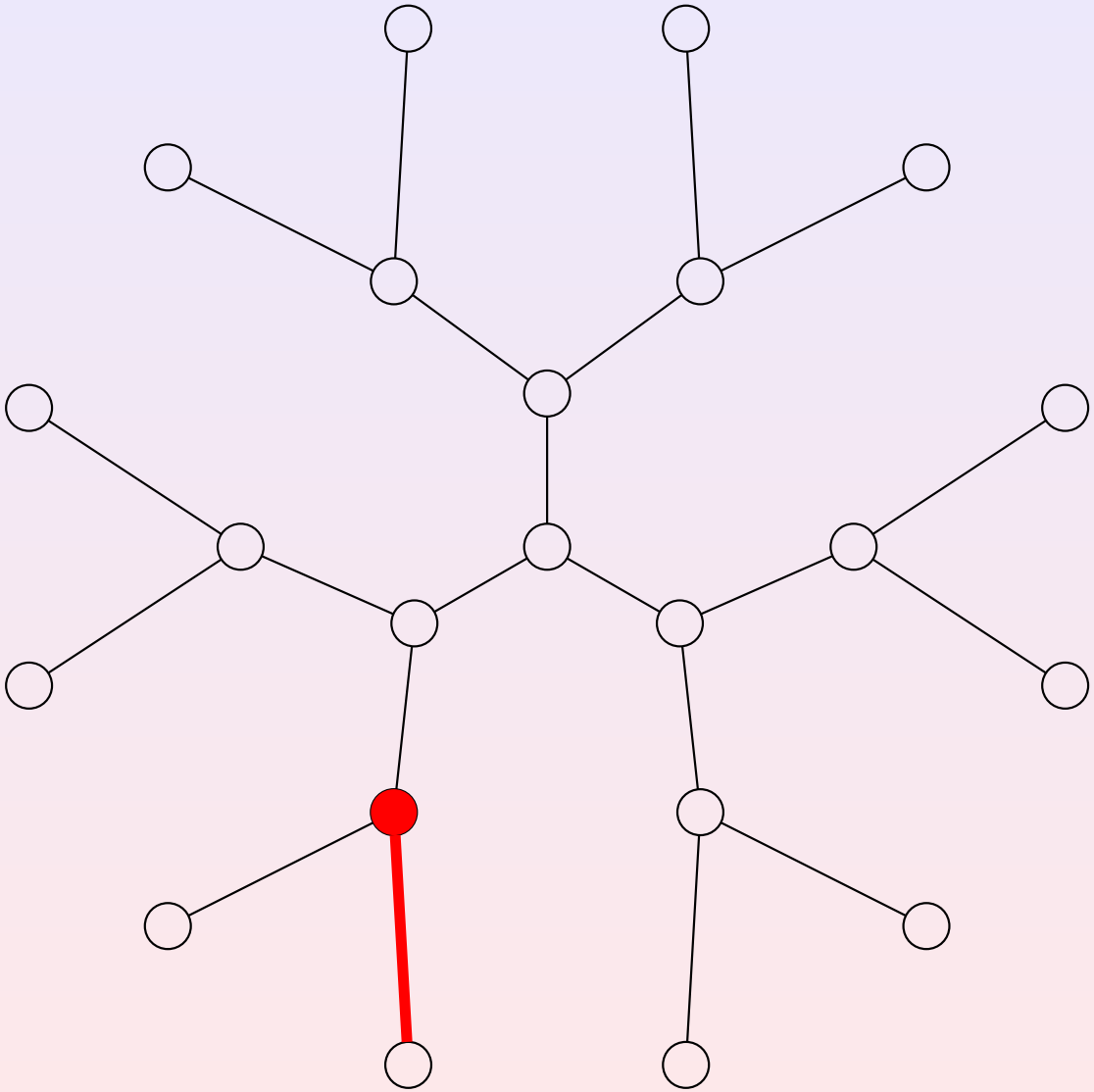

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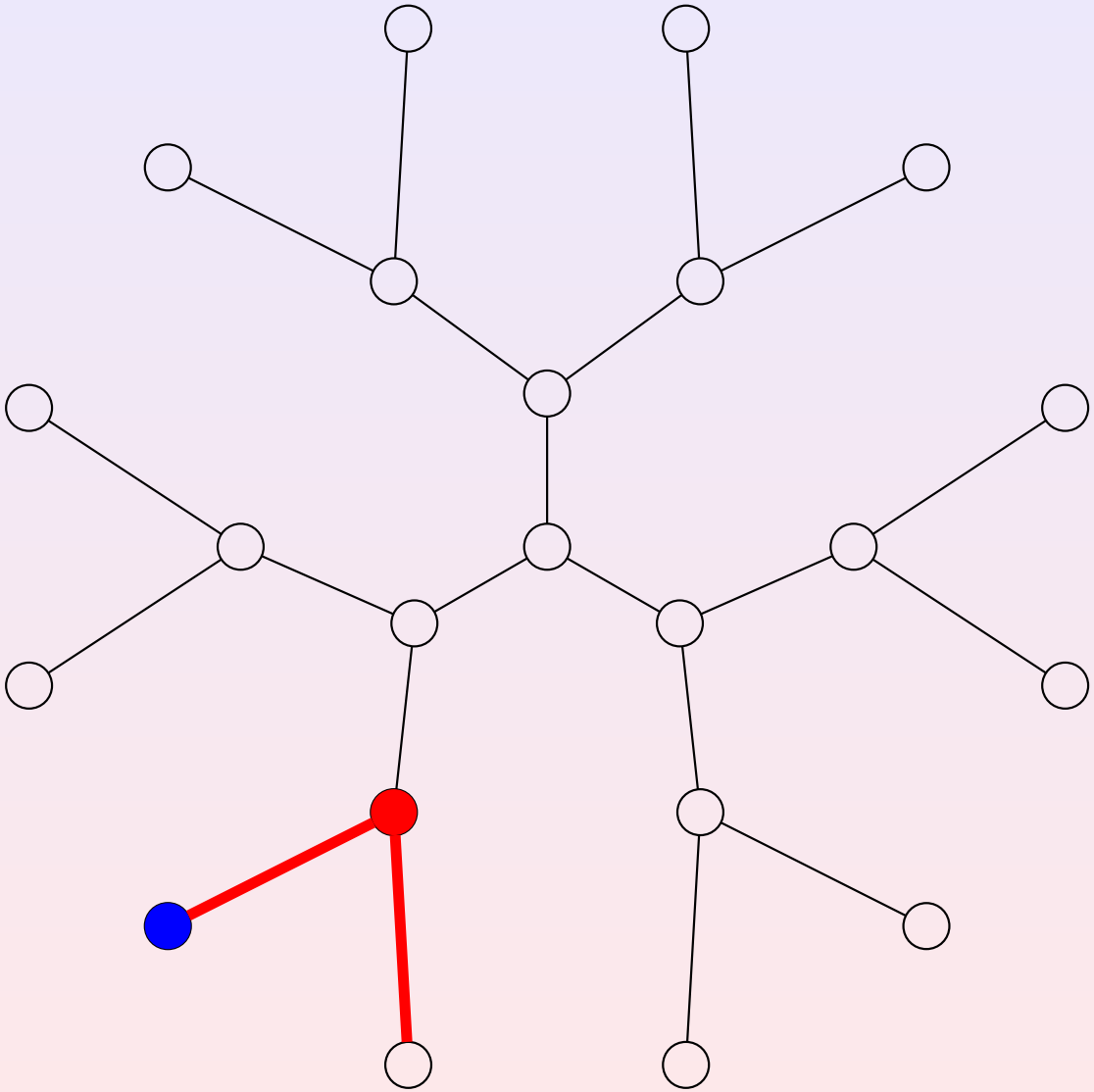

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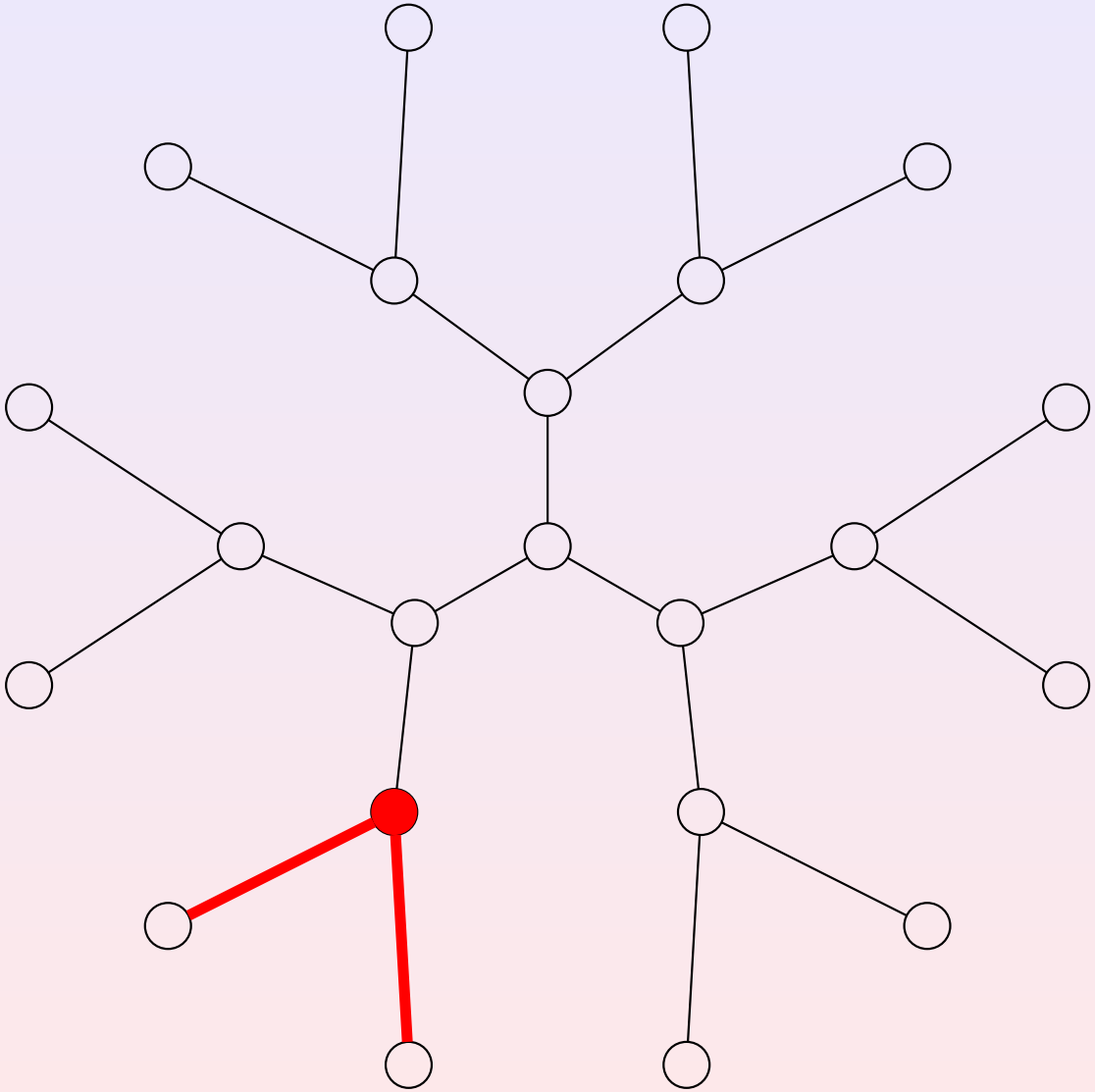

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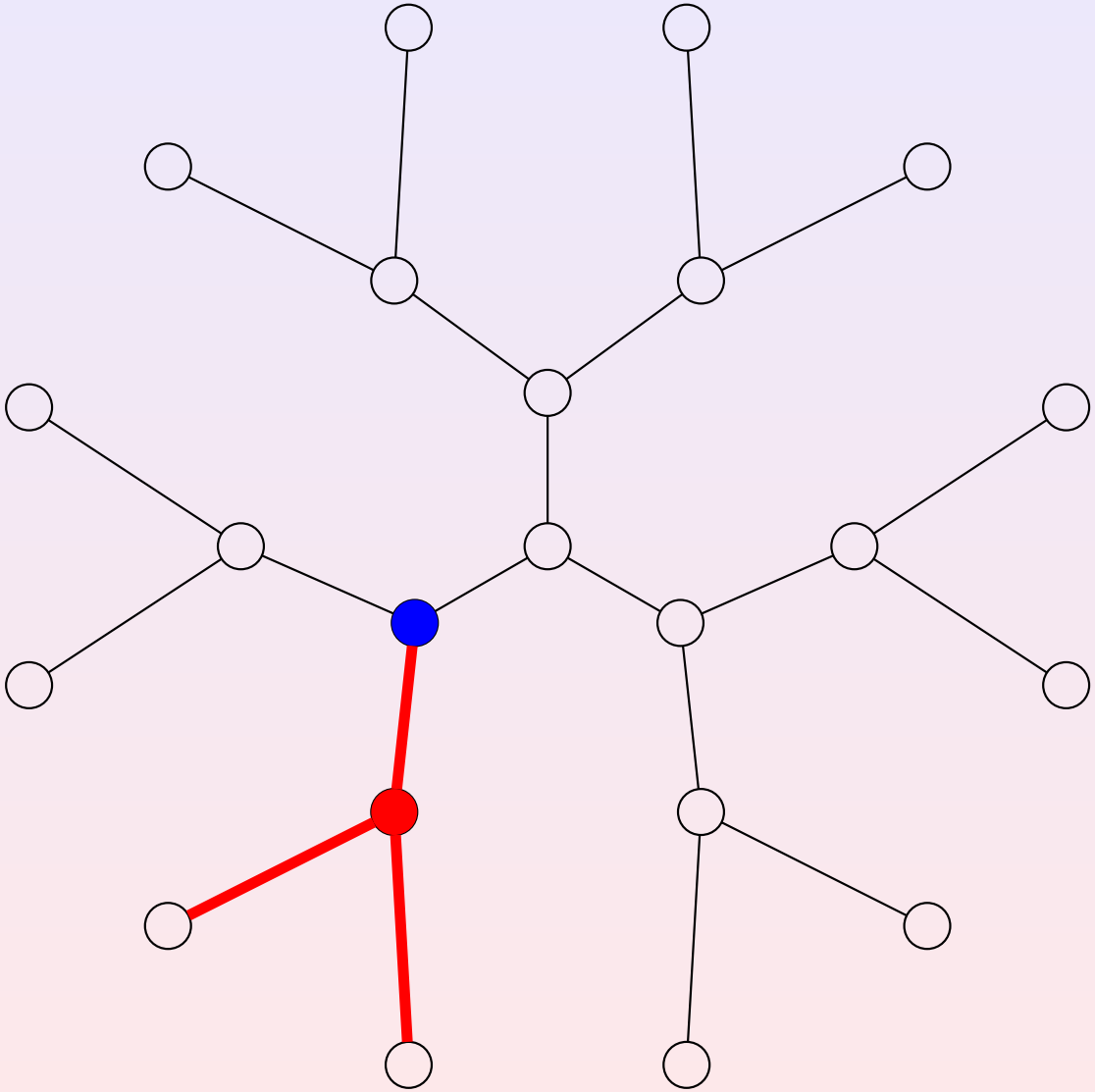

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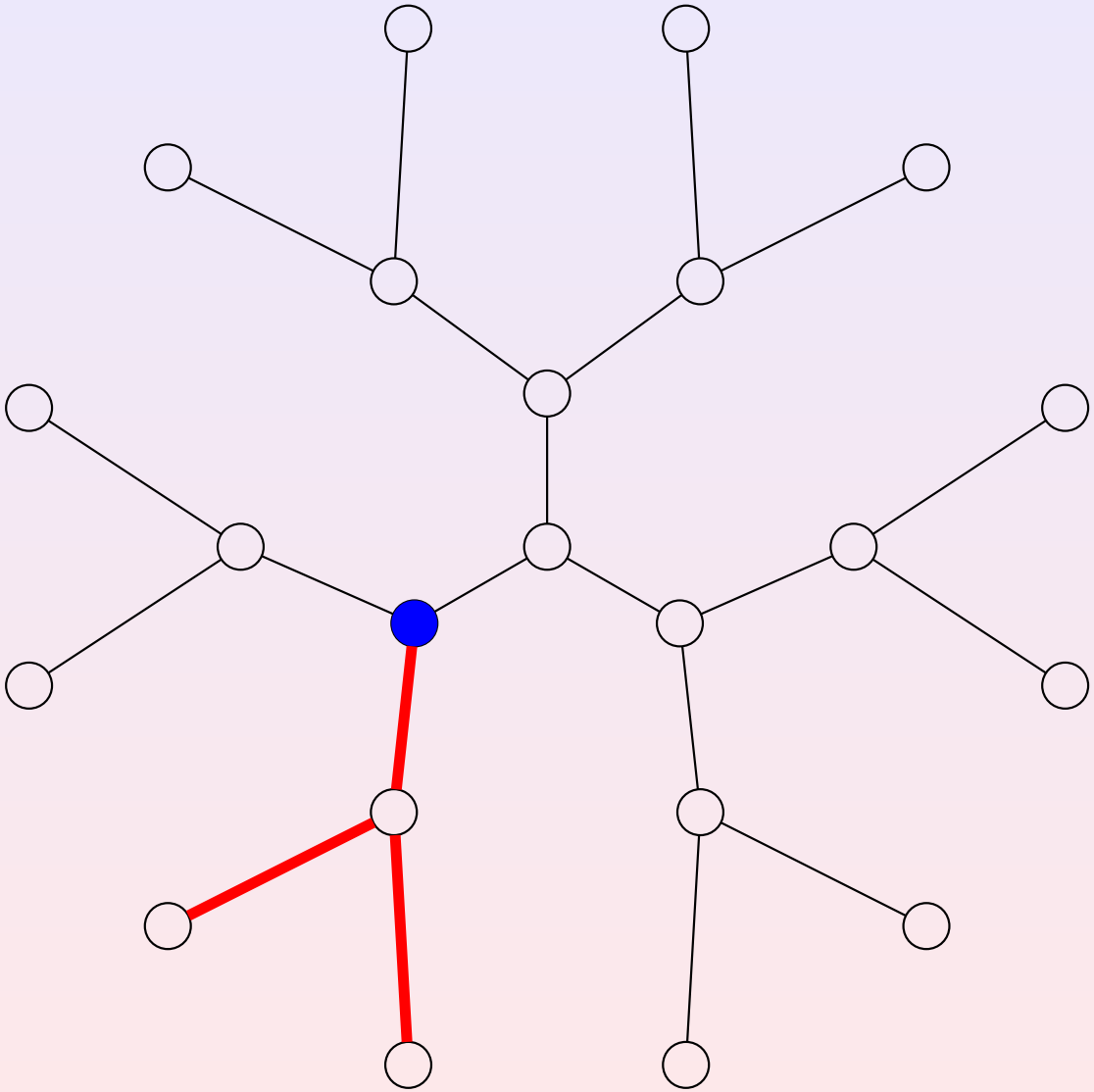

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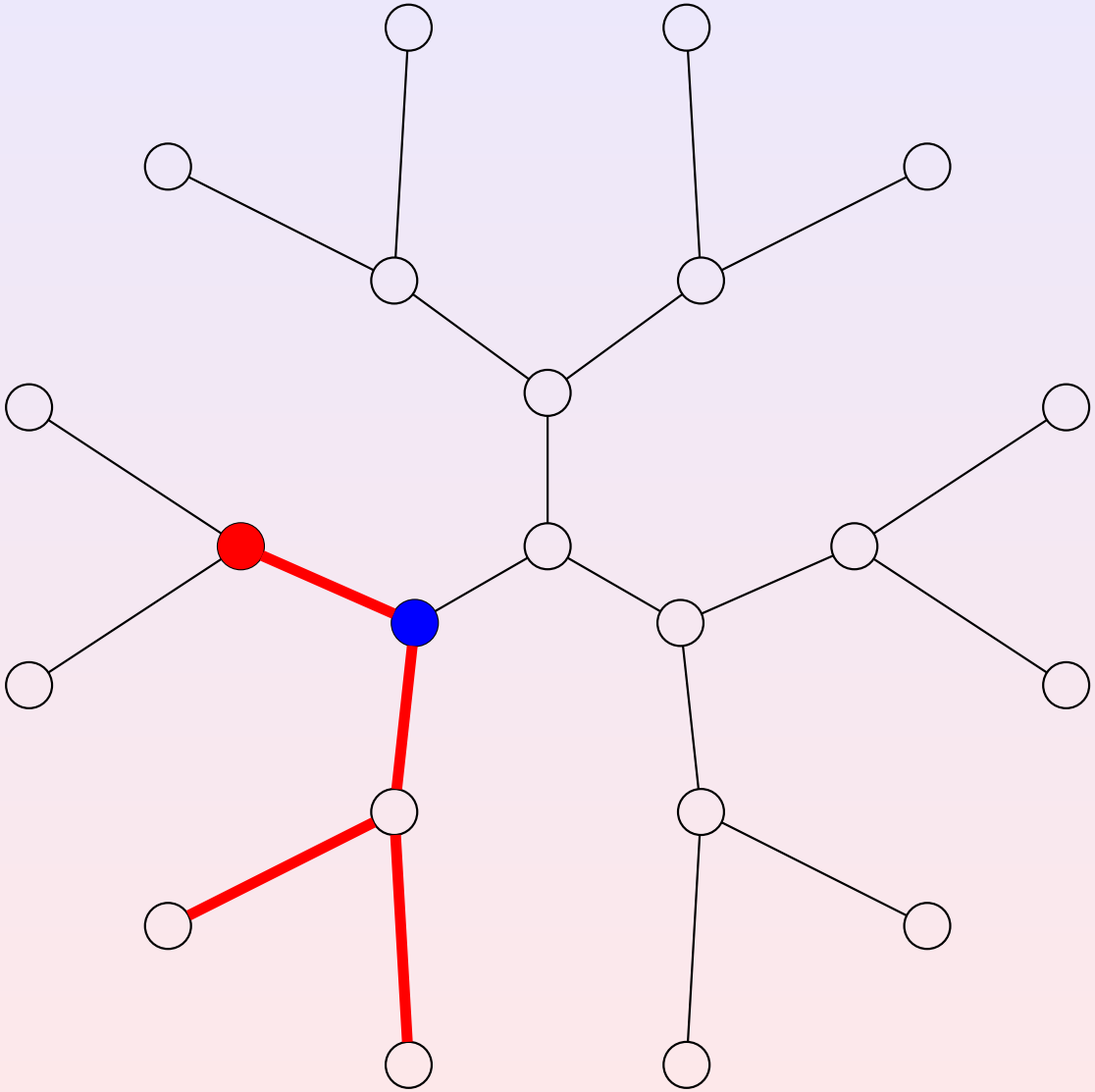

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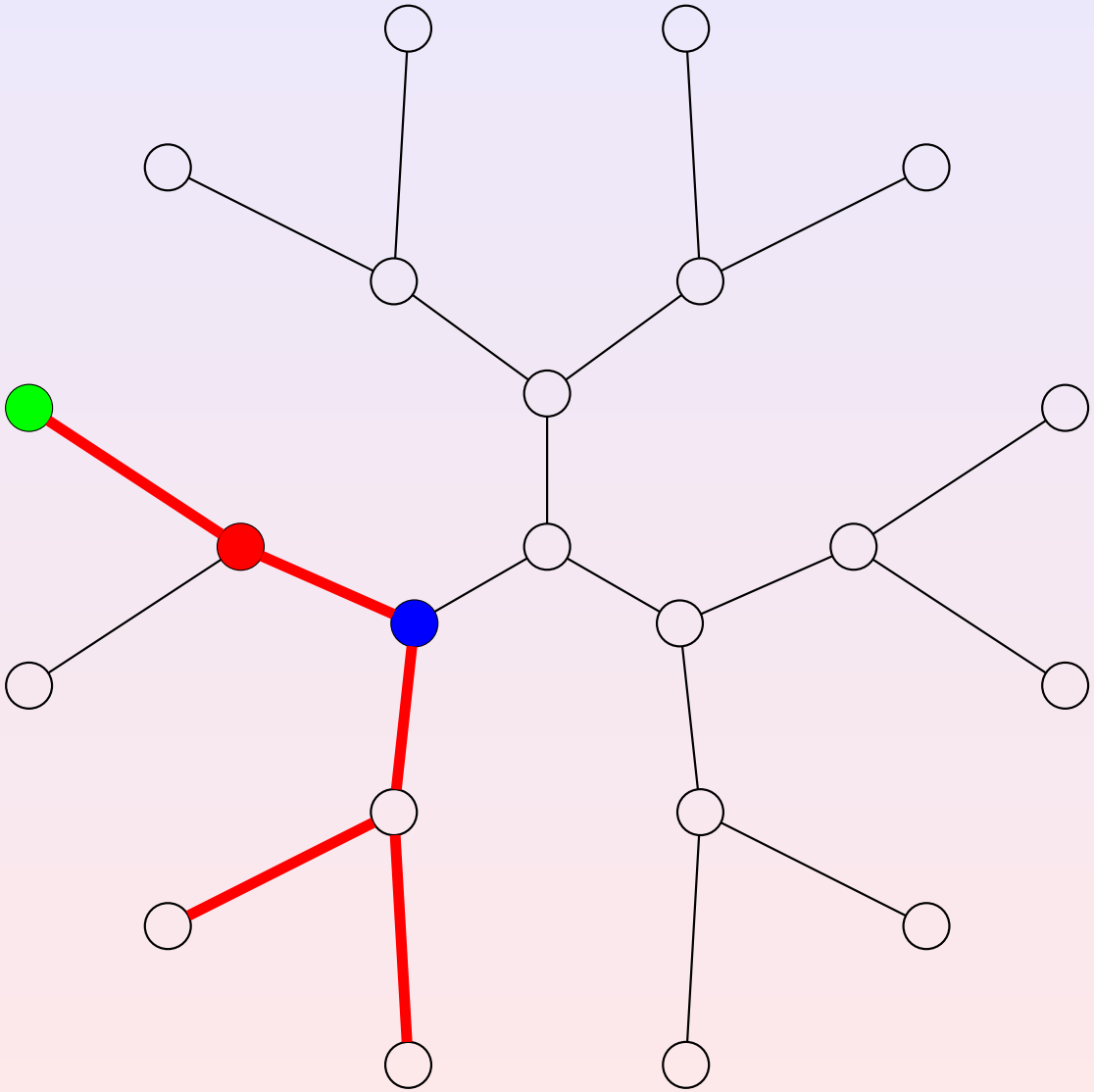

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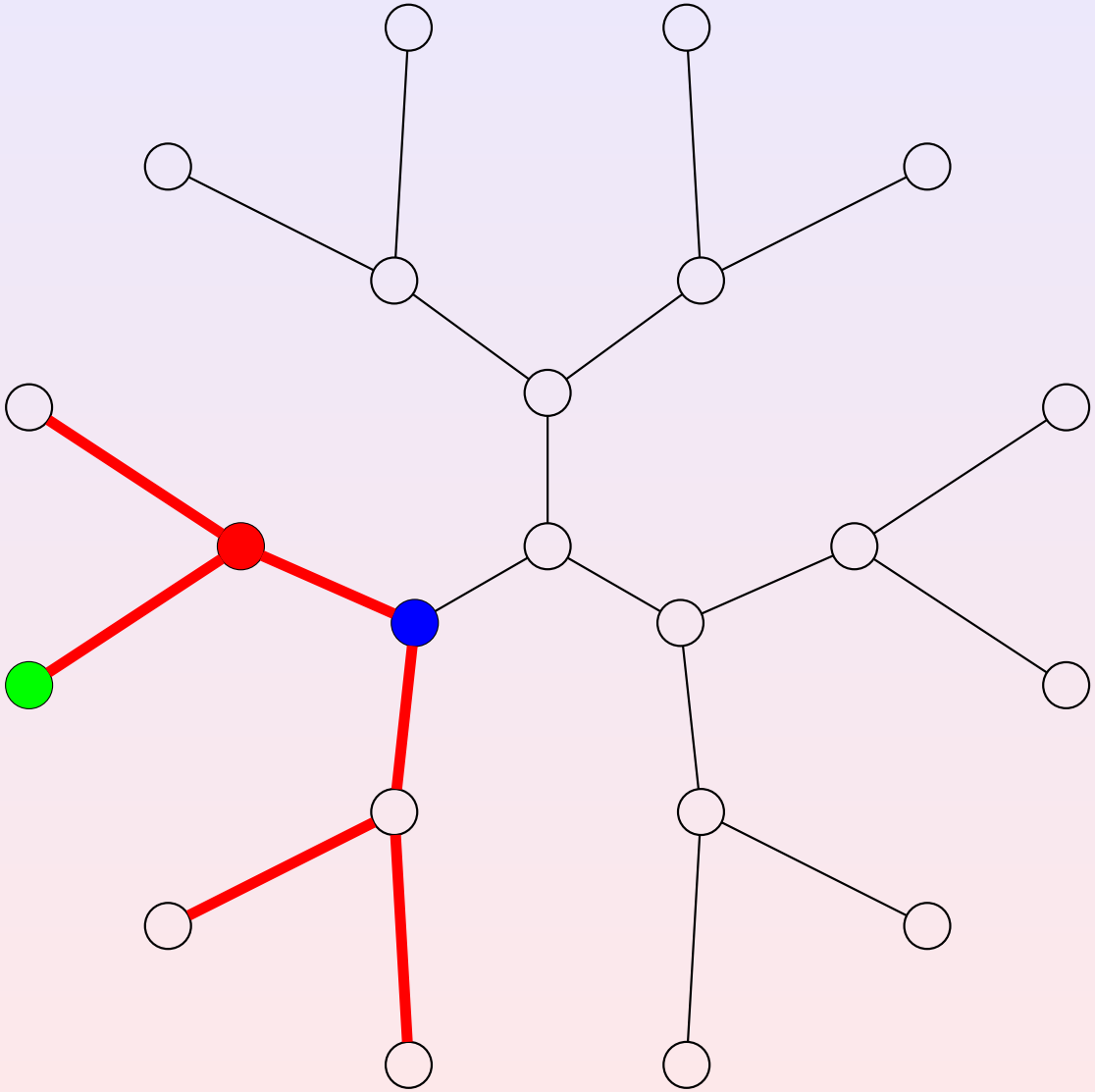

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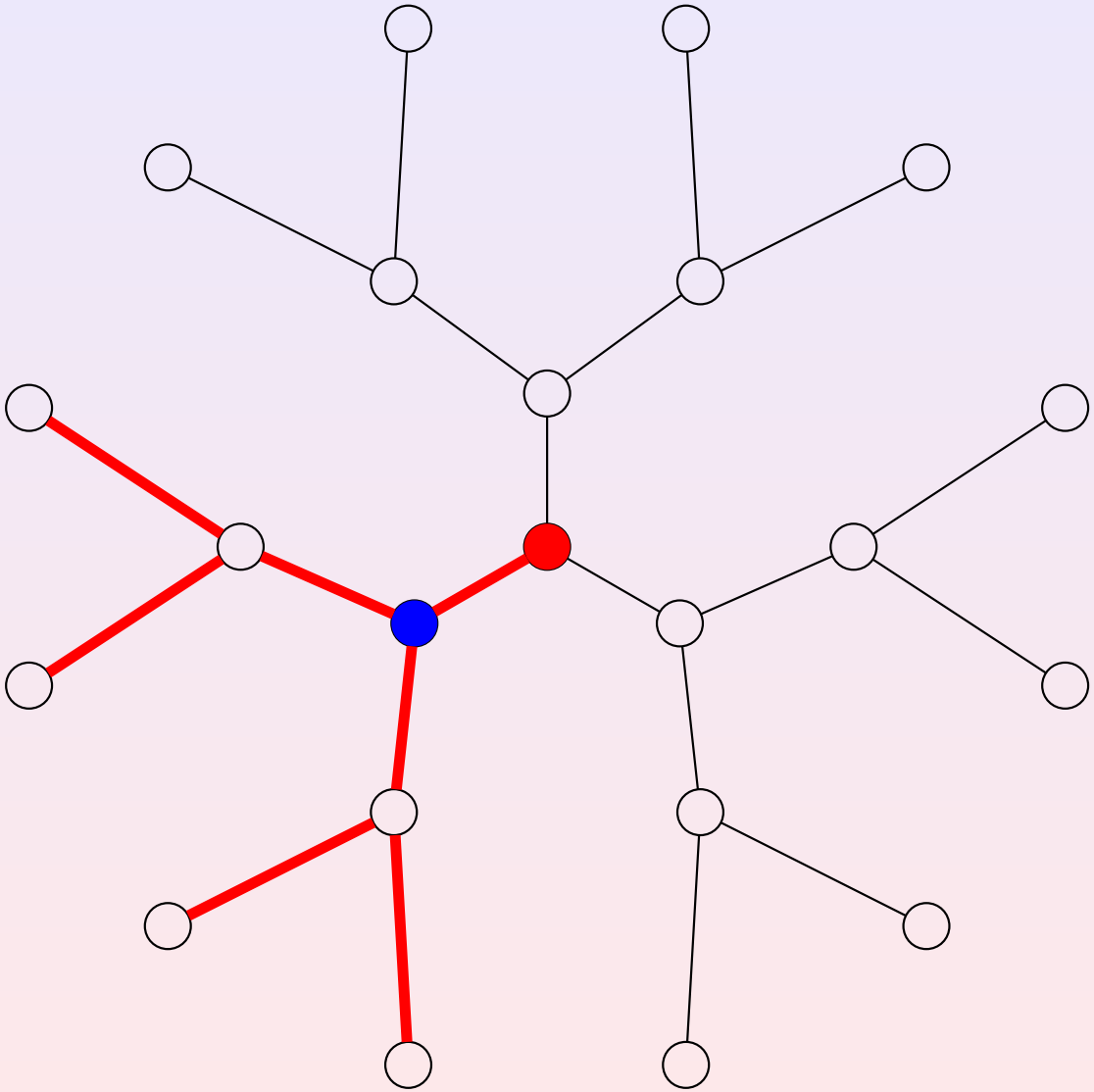

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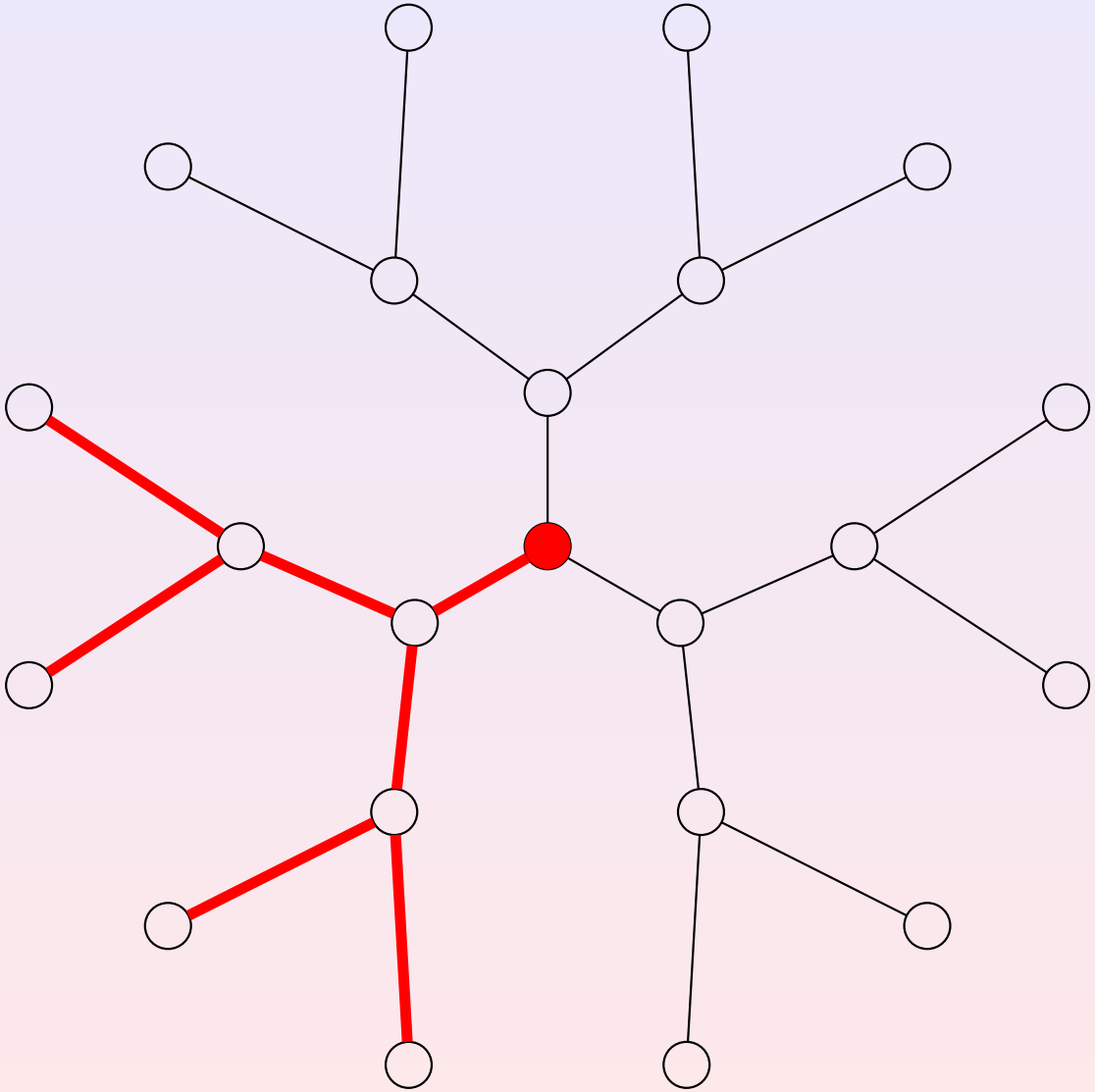

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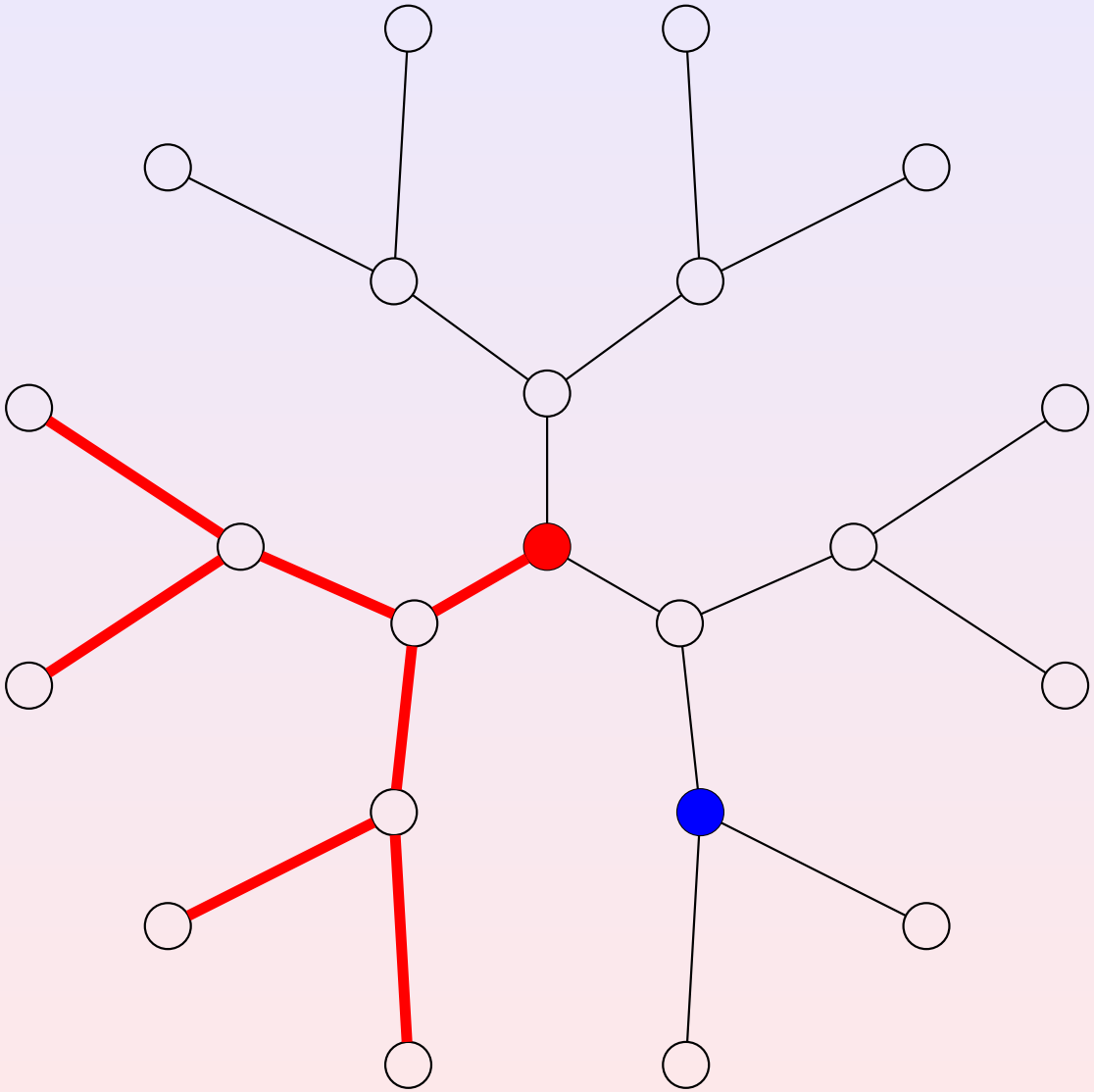

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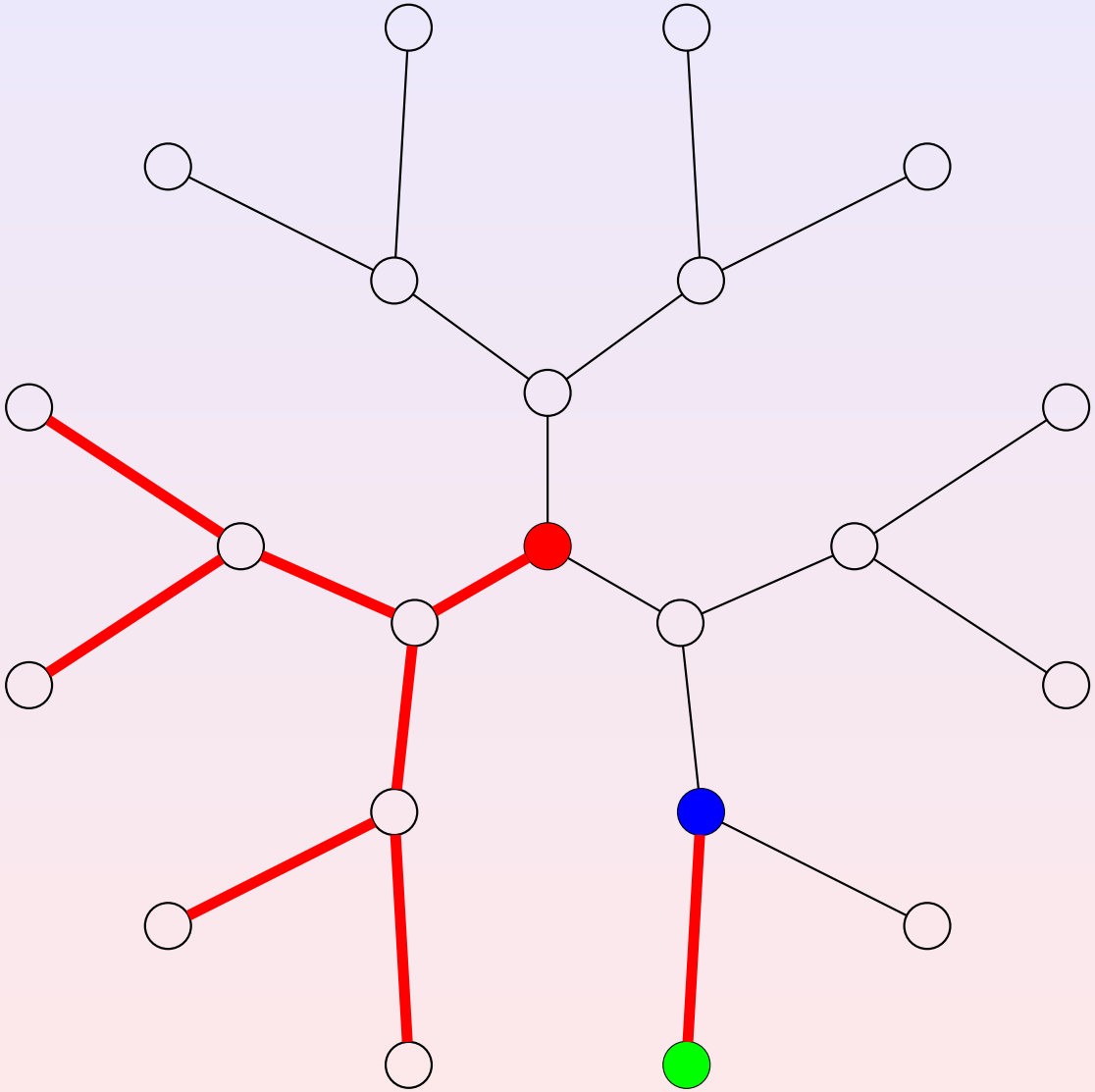

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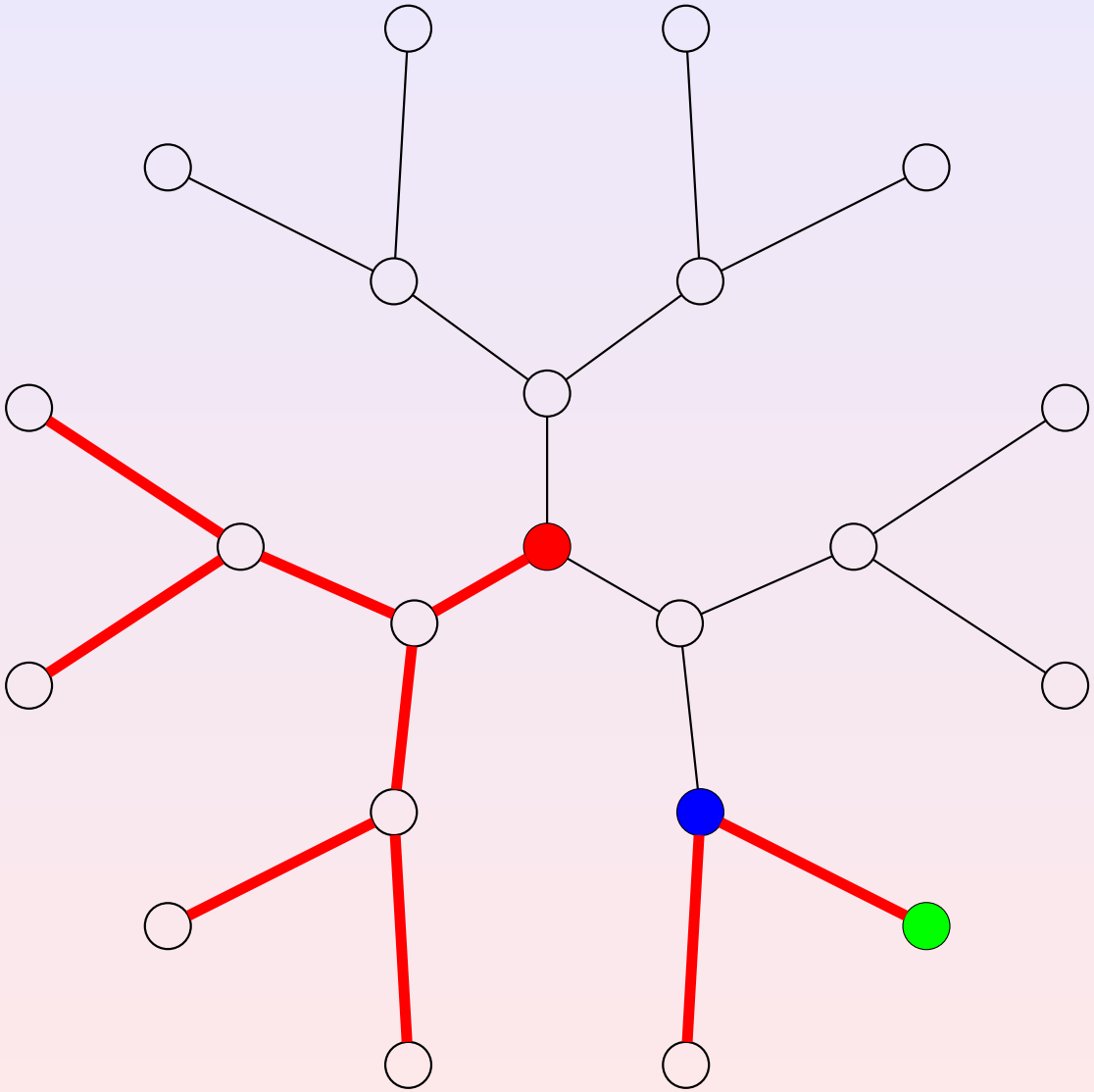

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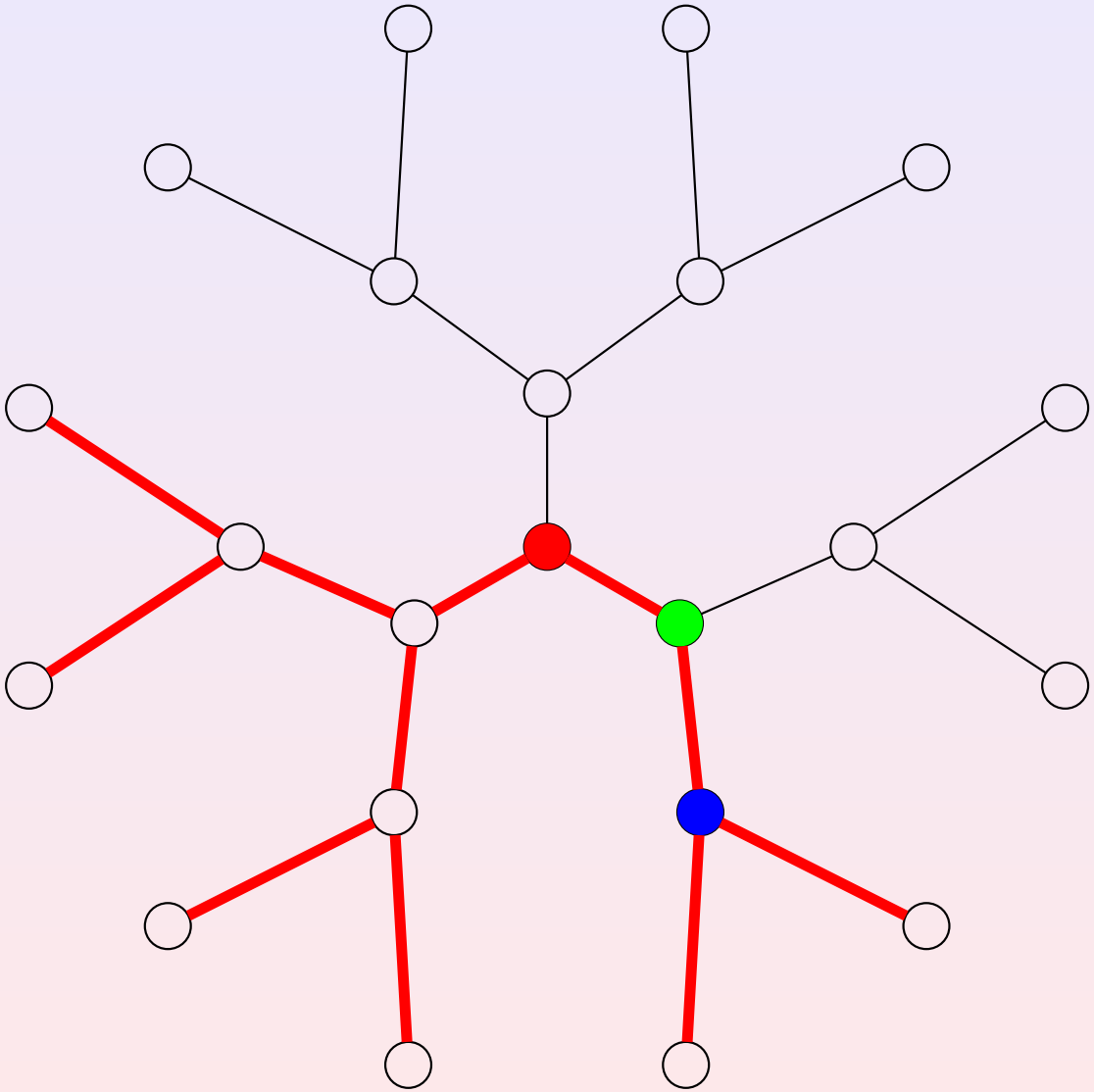

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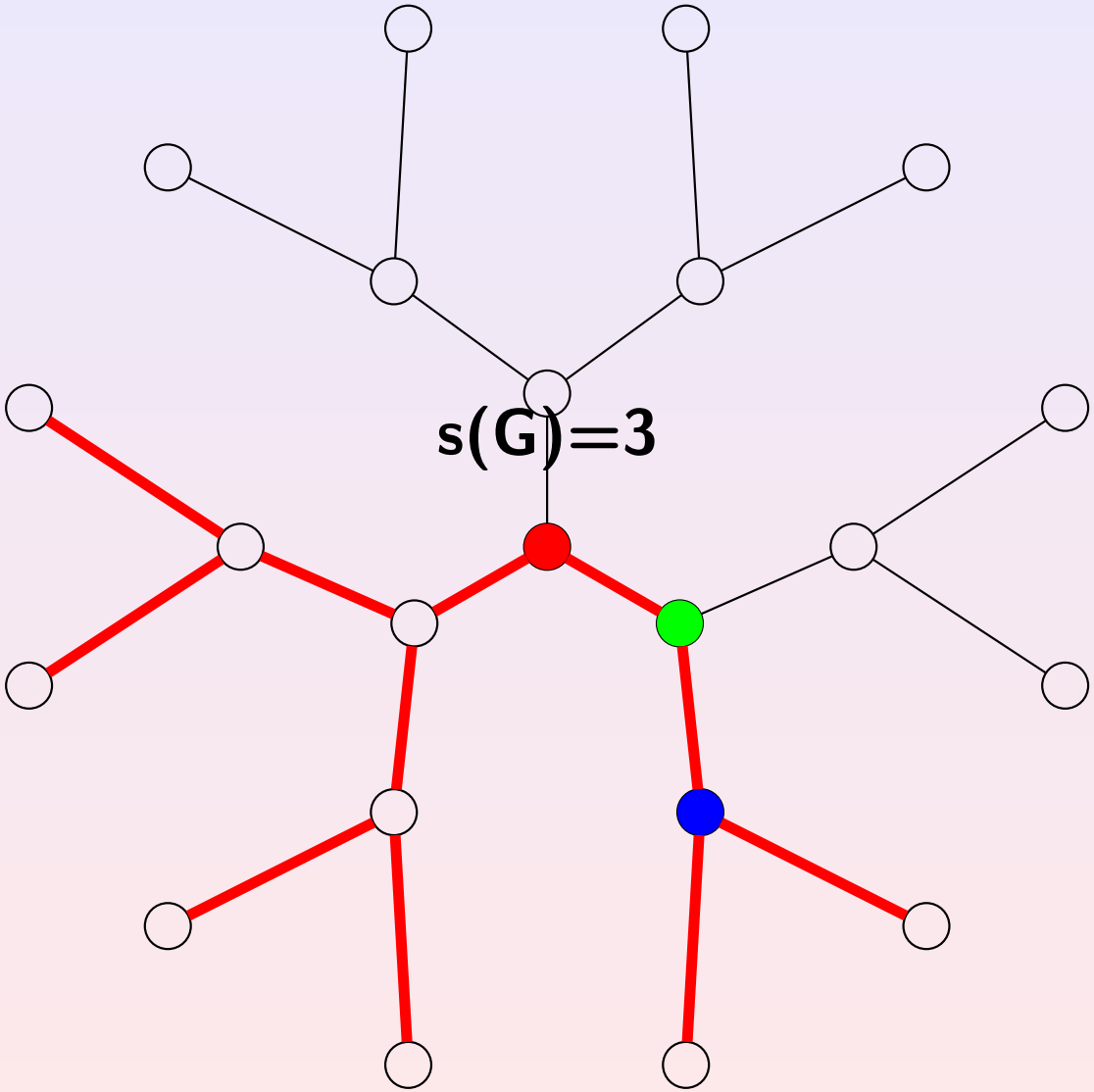

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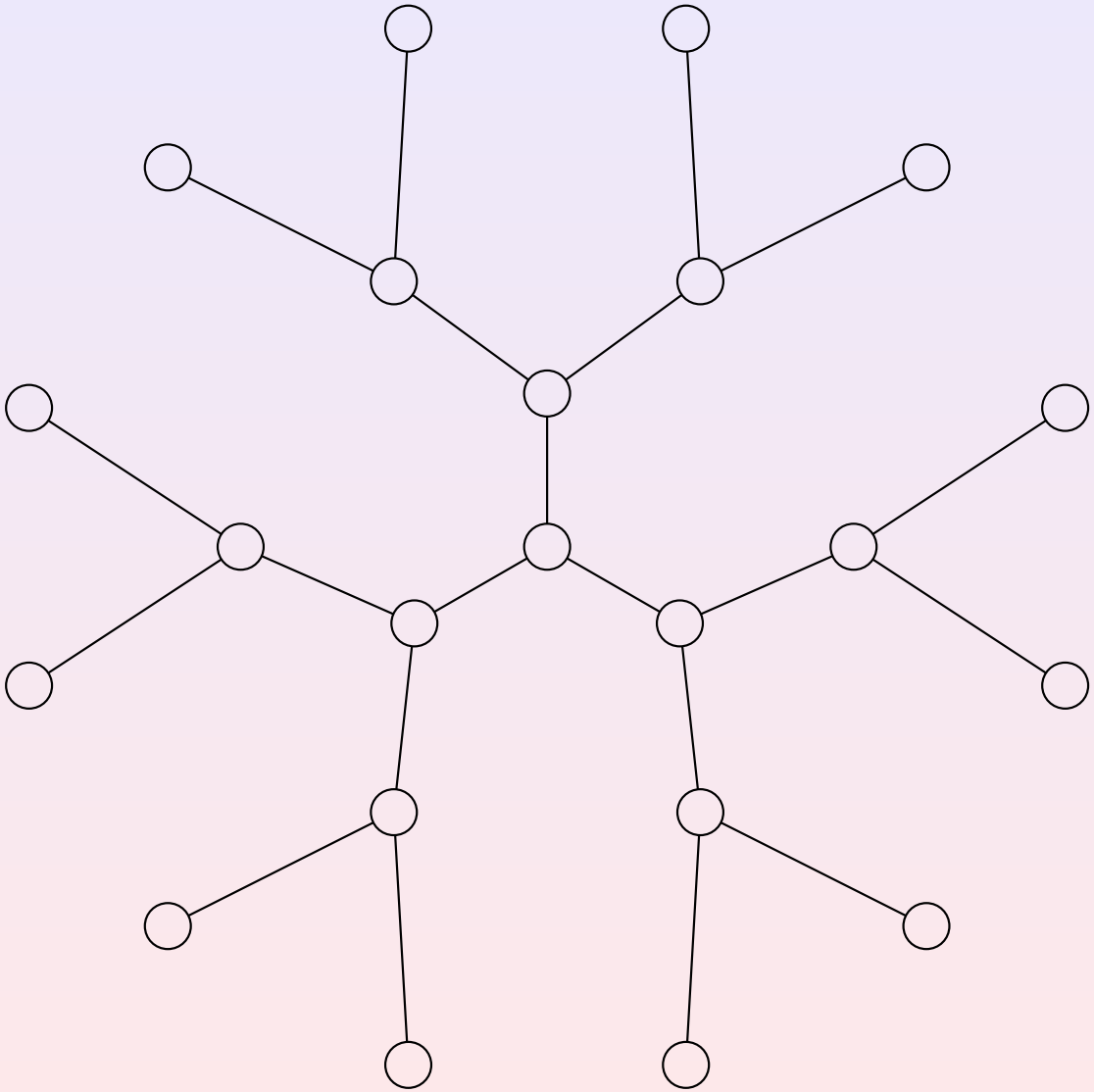

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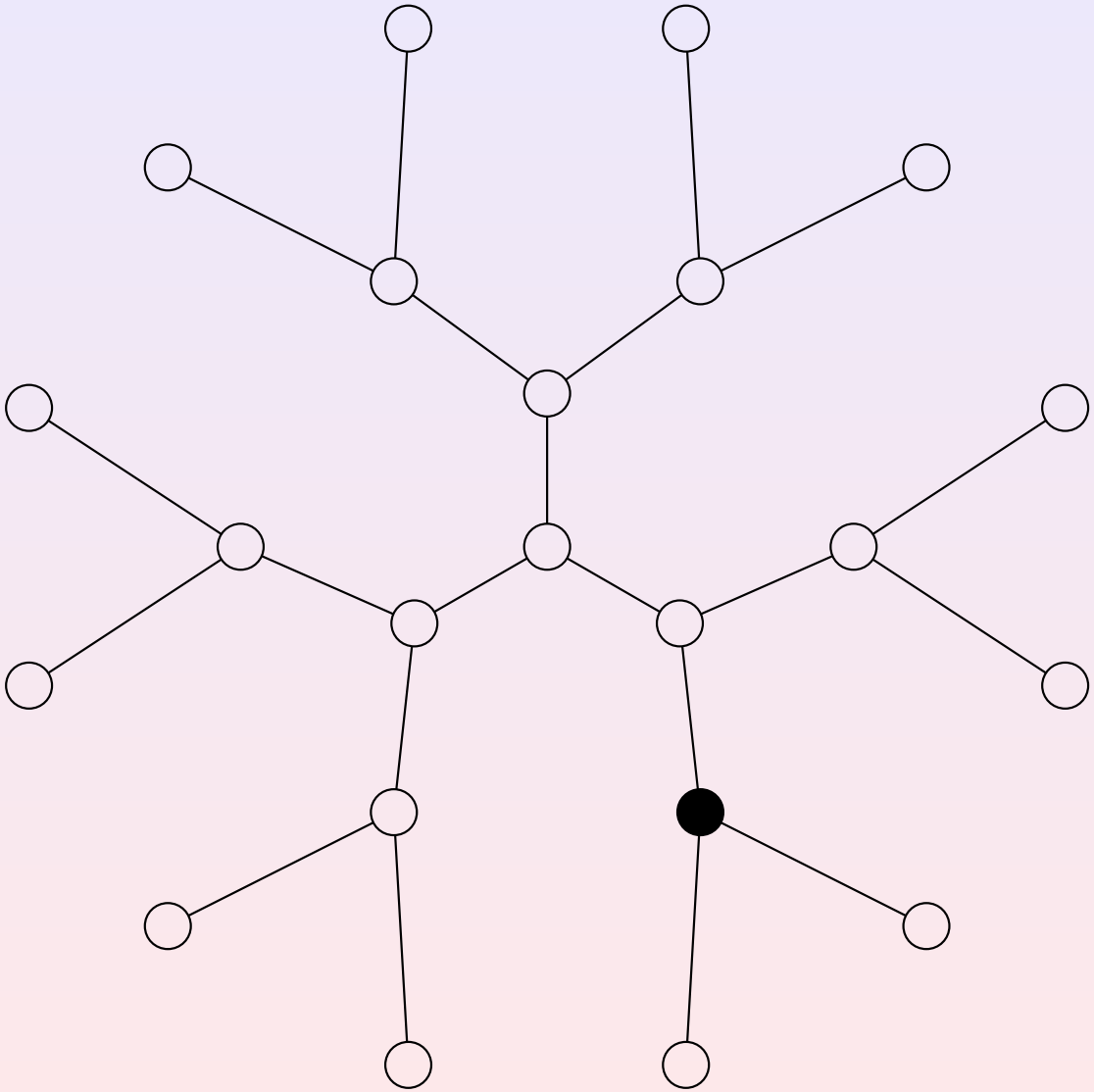

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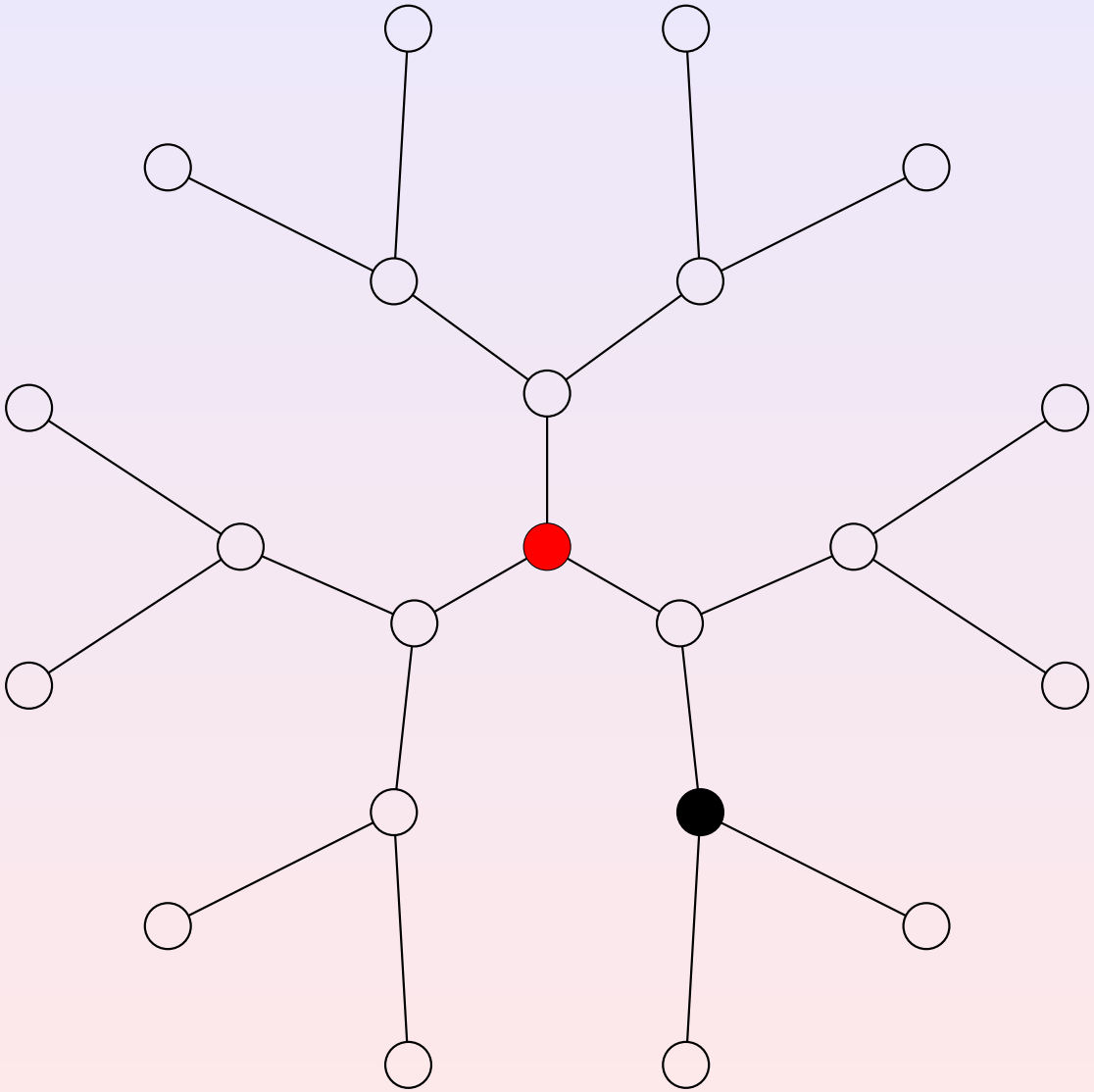

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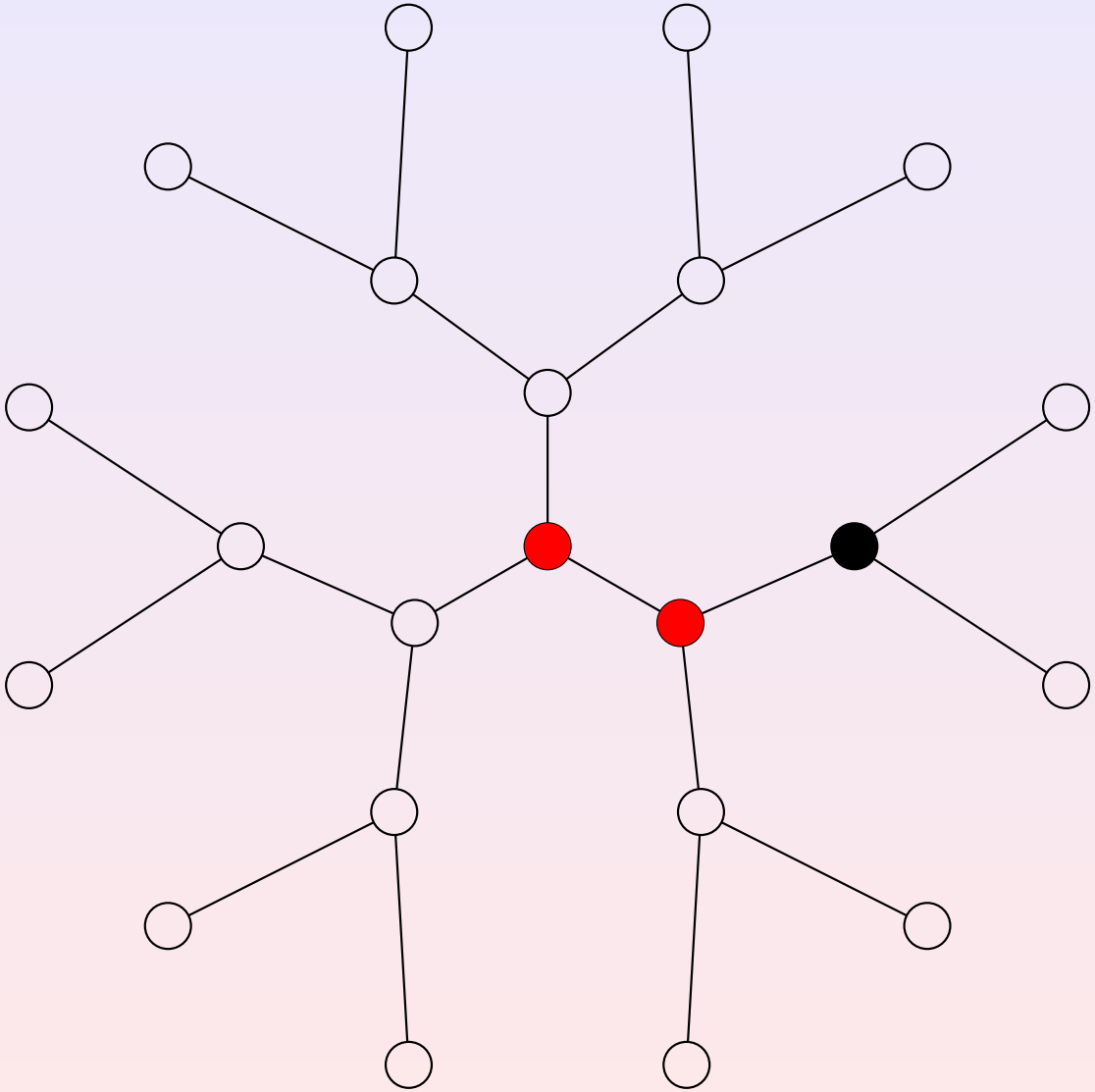

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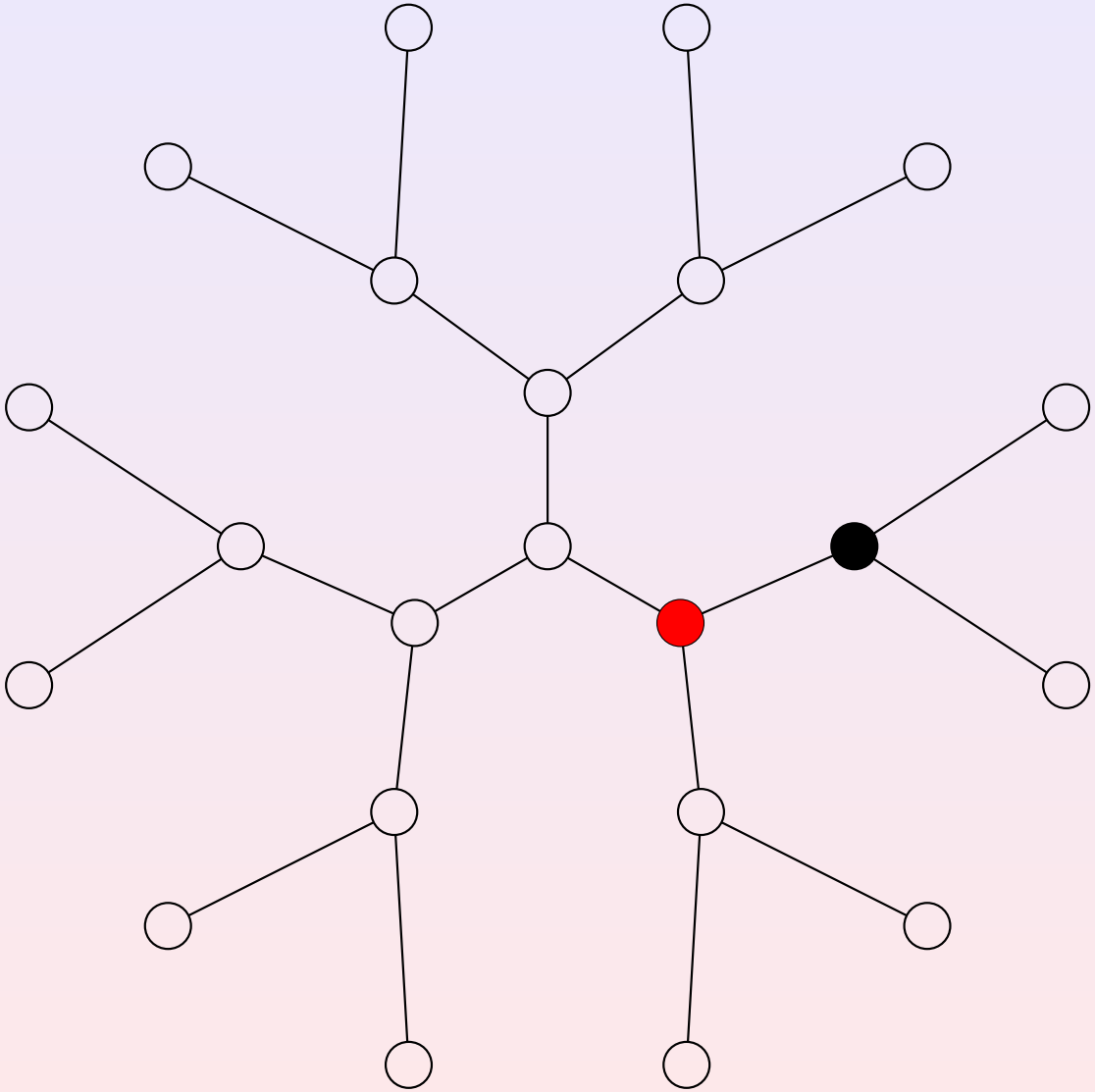

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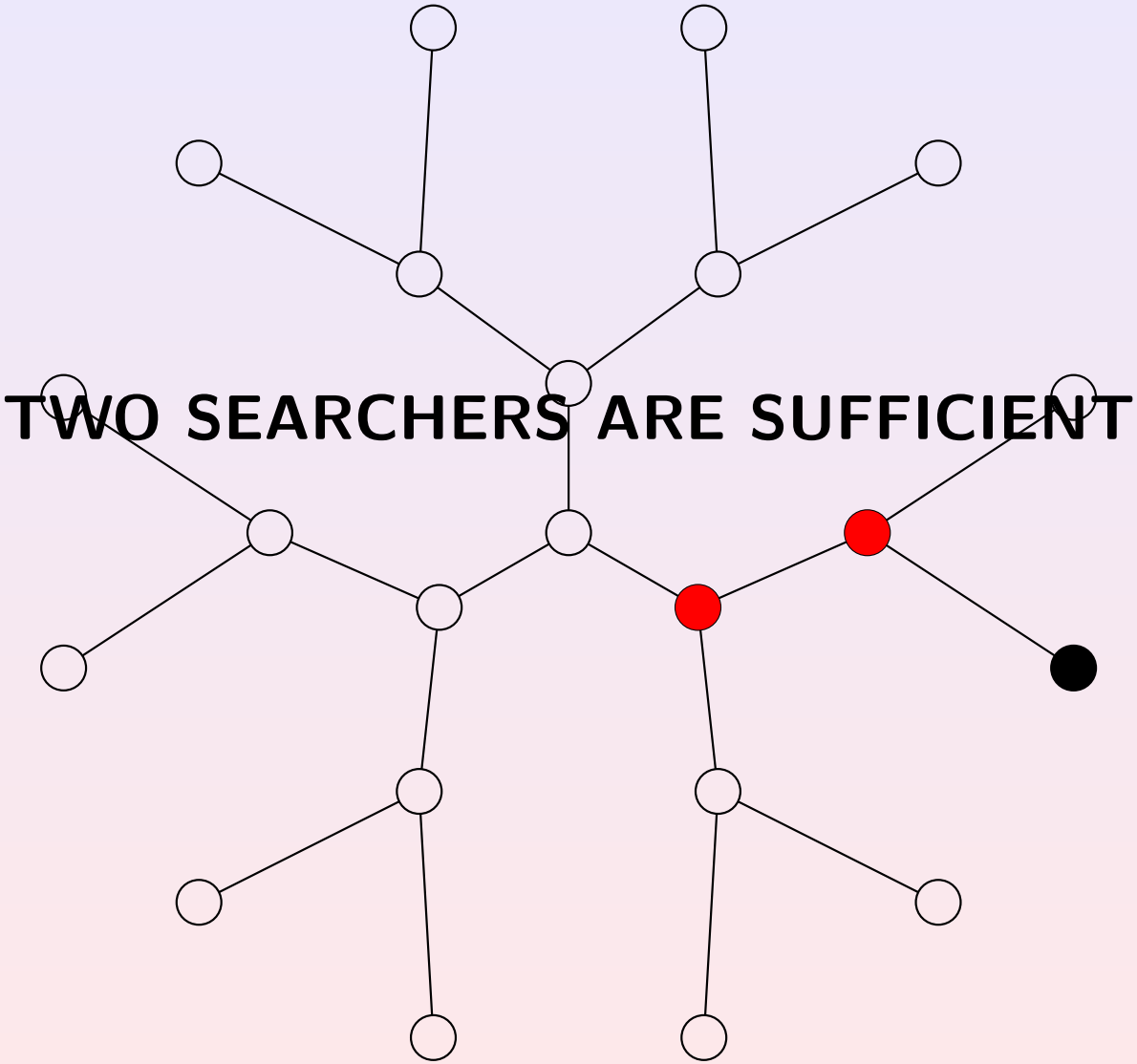
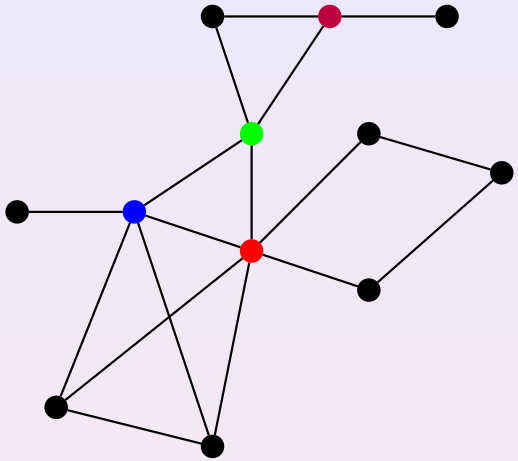


Tree and Path Decompositions



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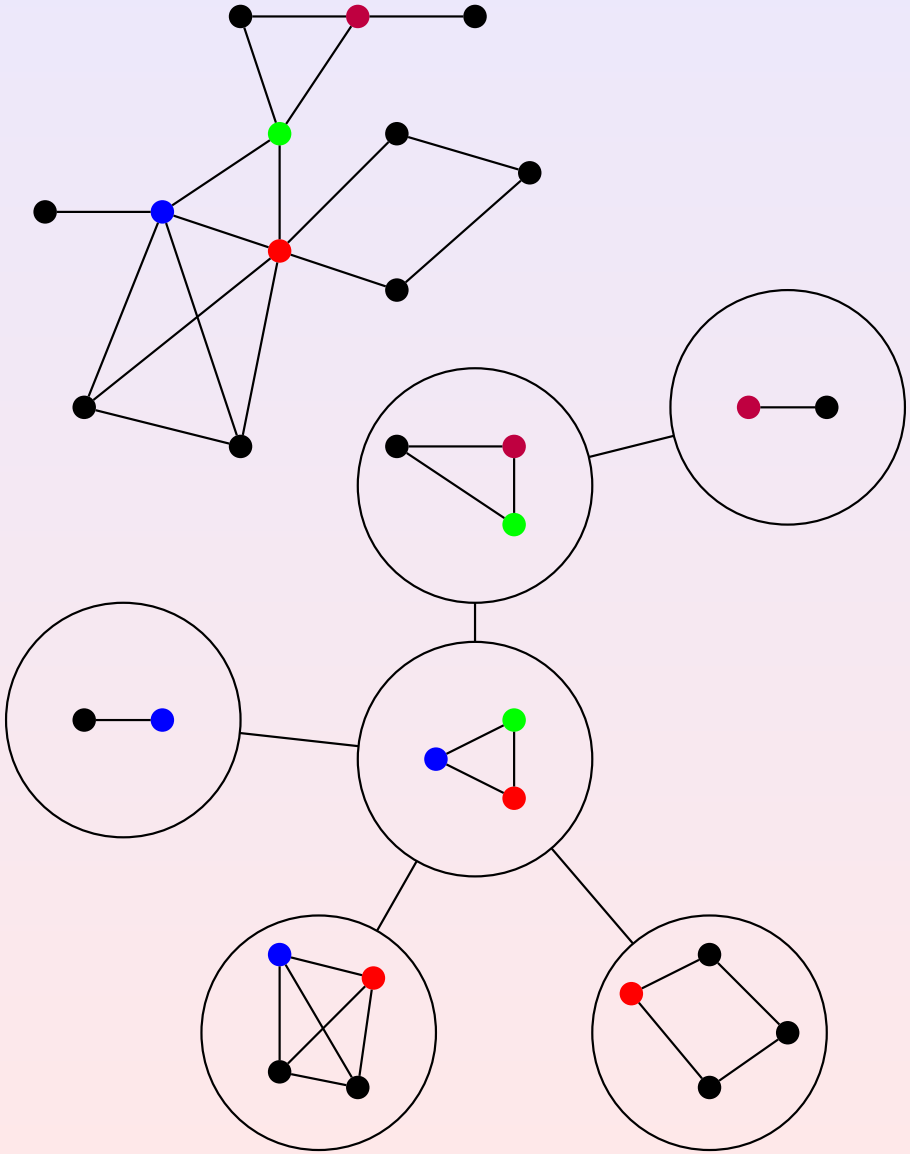

a tree T and bags $(X_t)_{t \in V(T)}$

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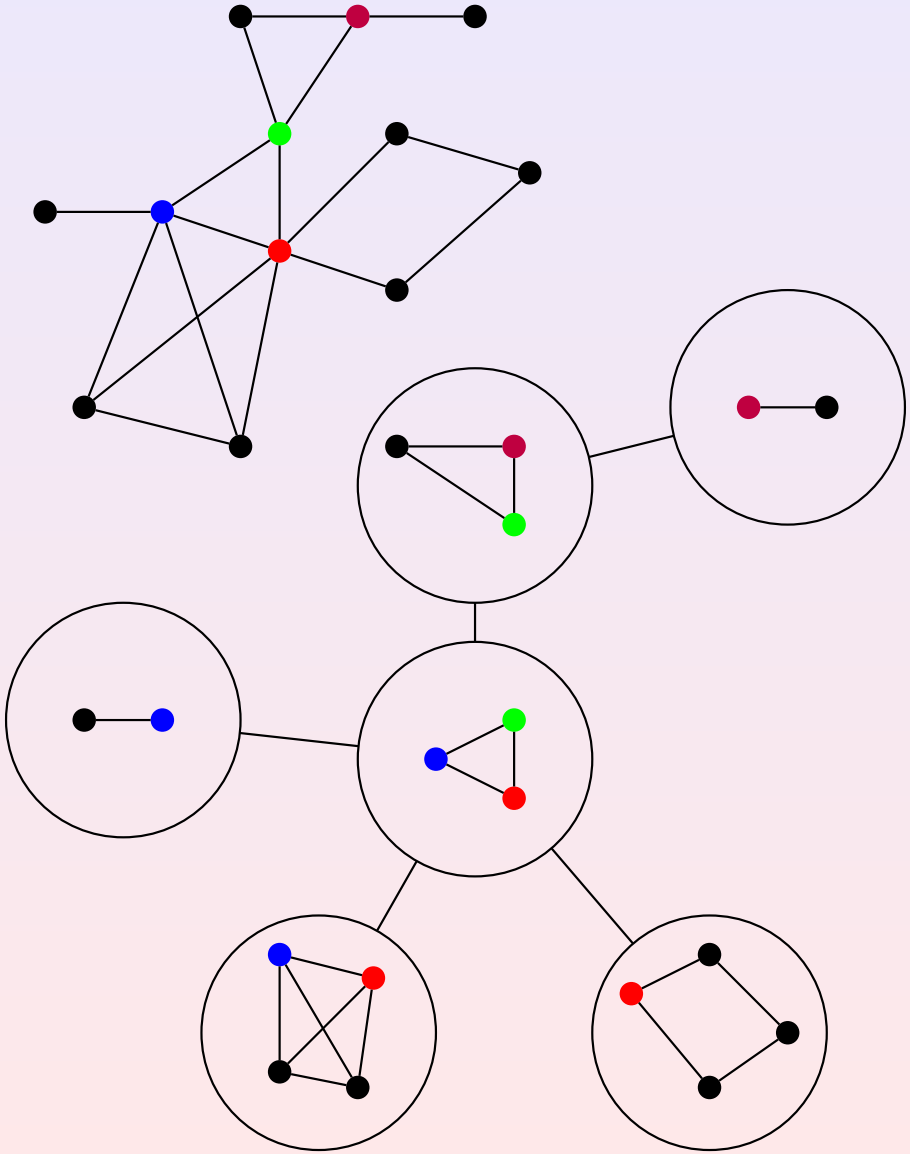

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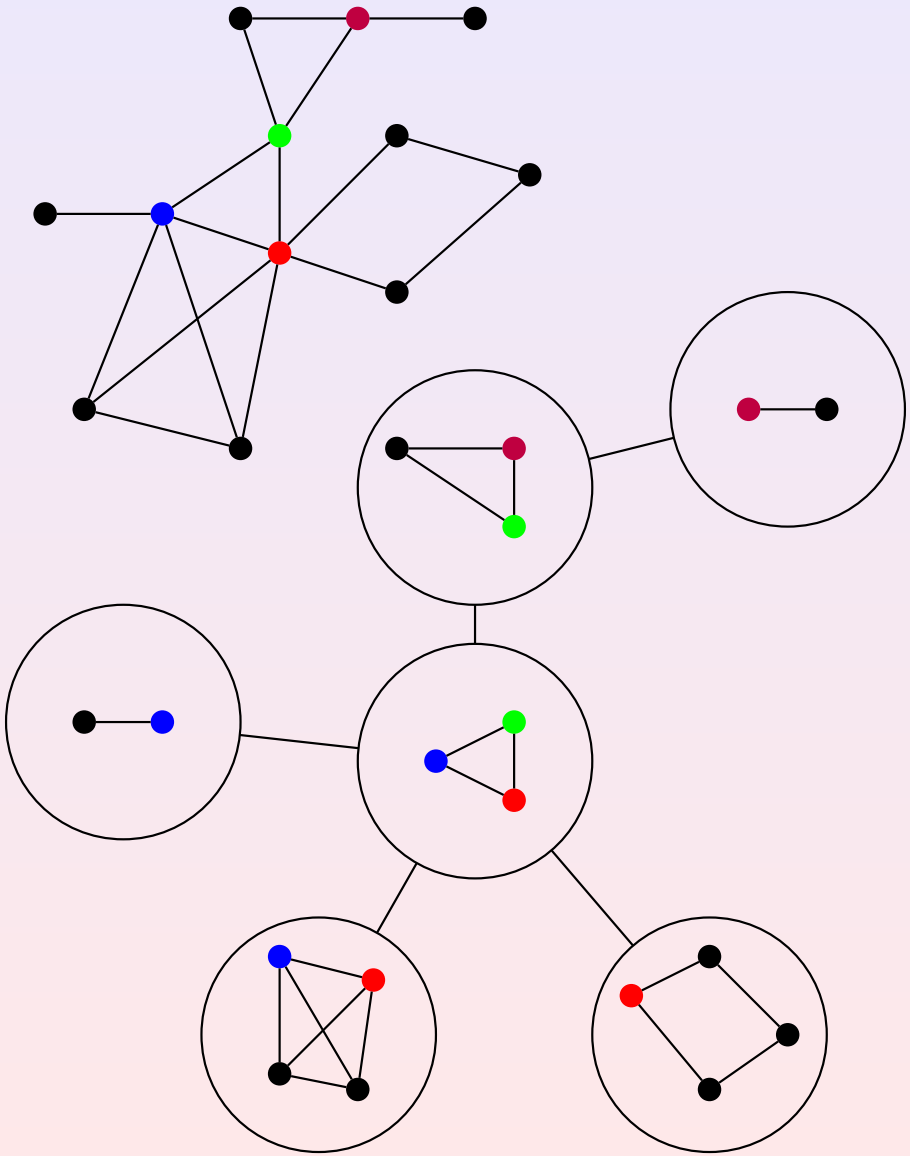

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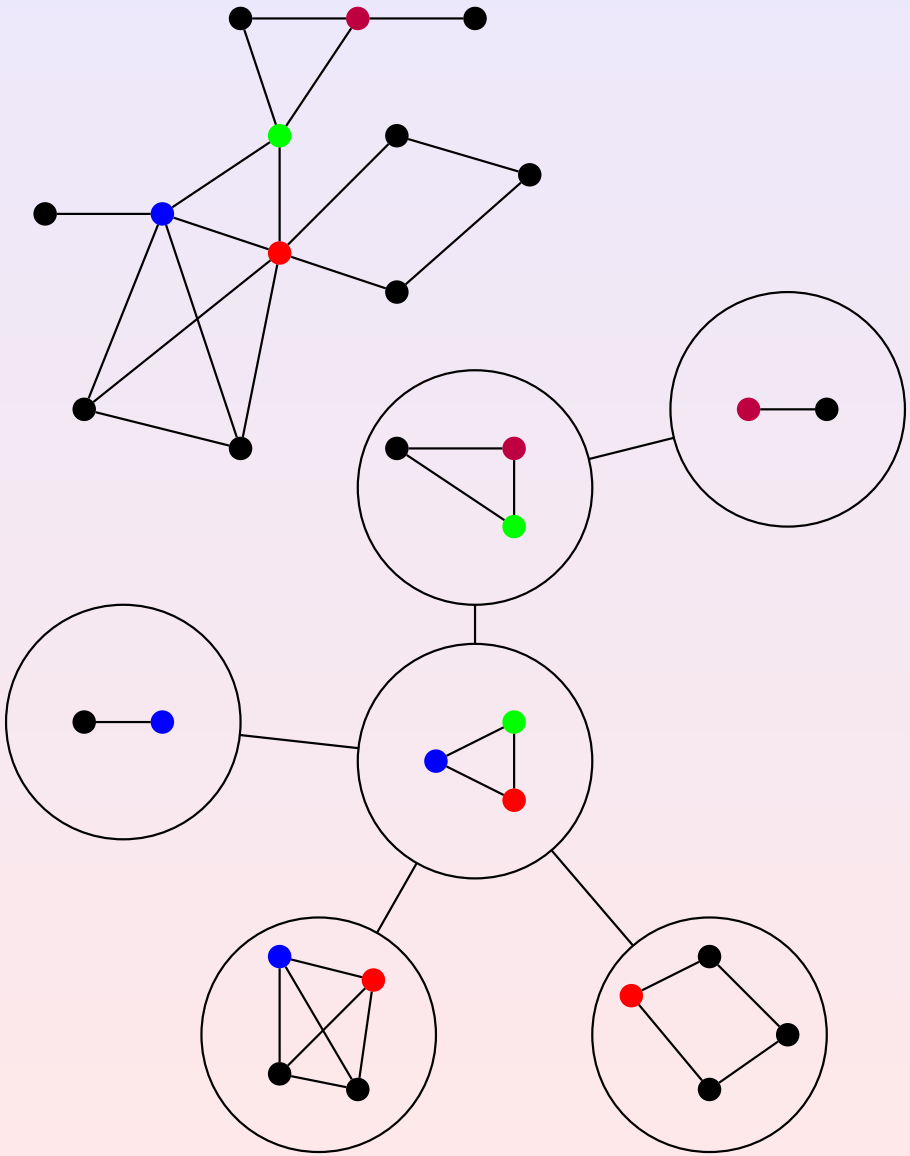

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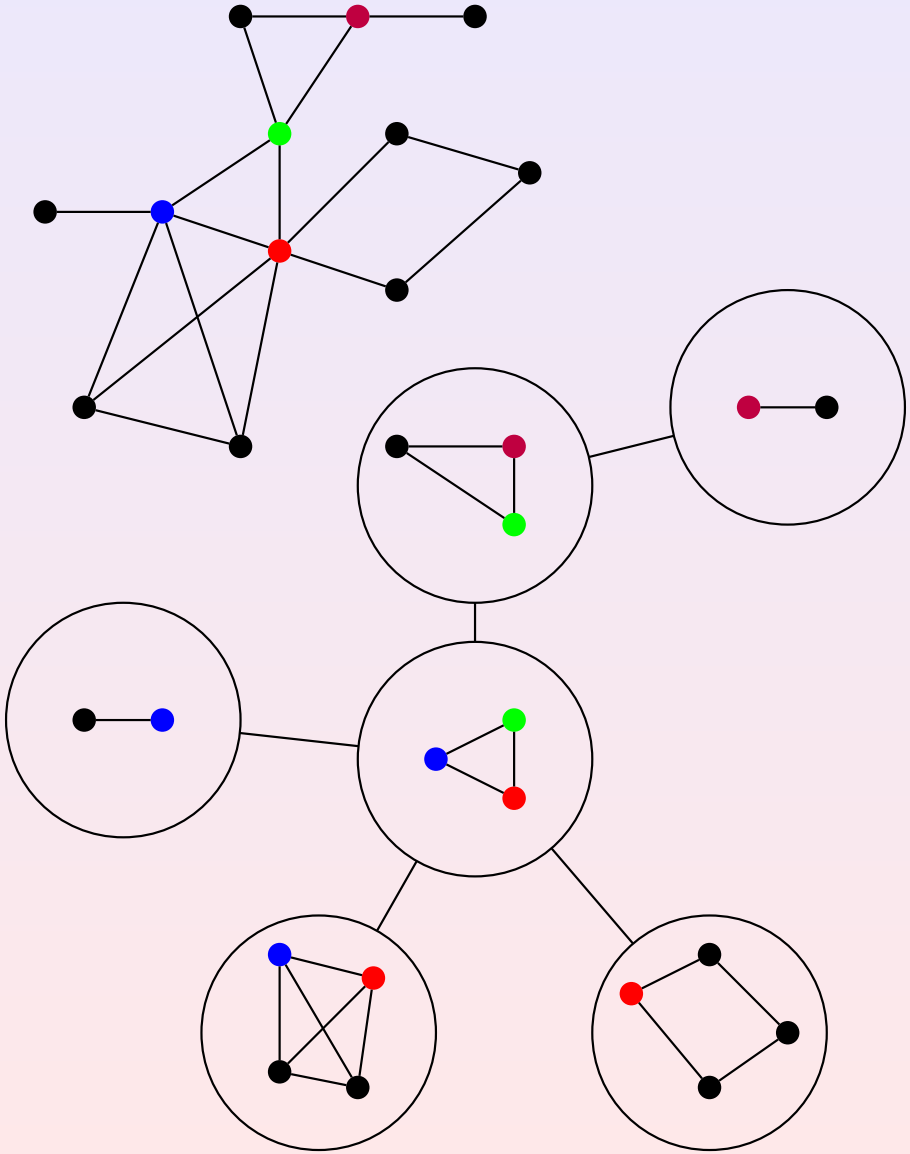

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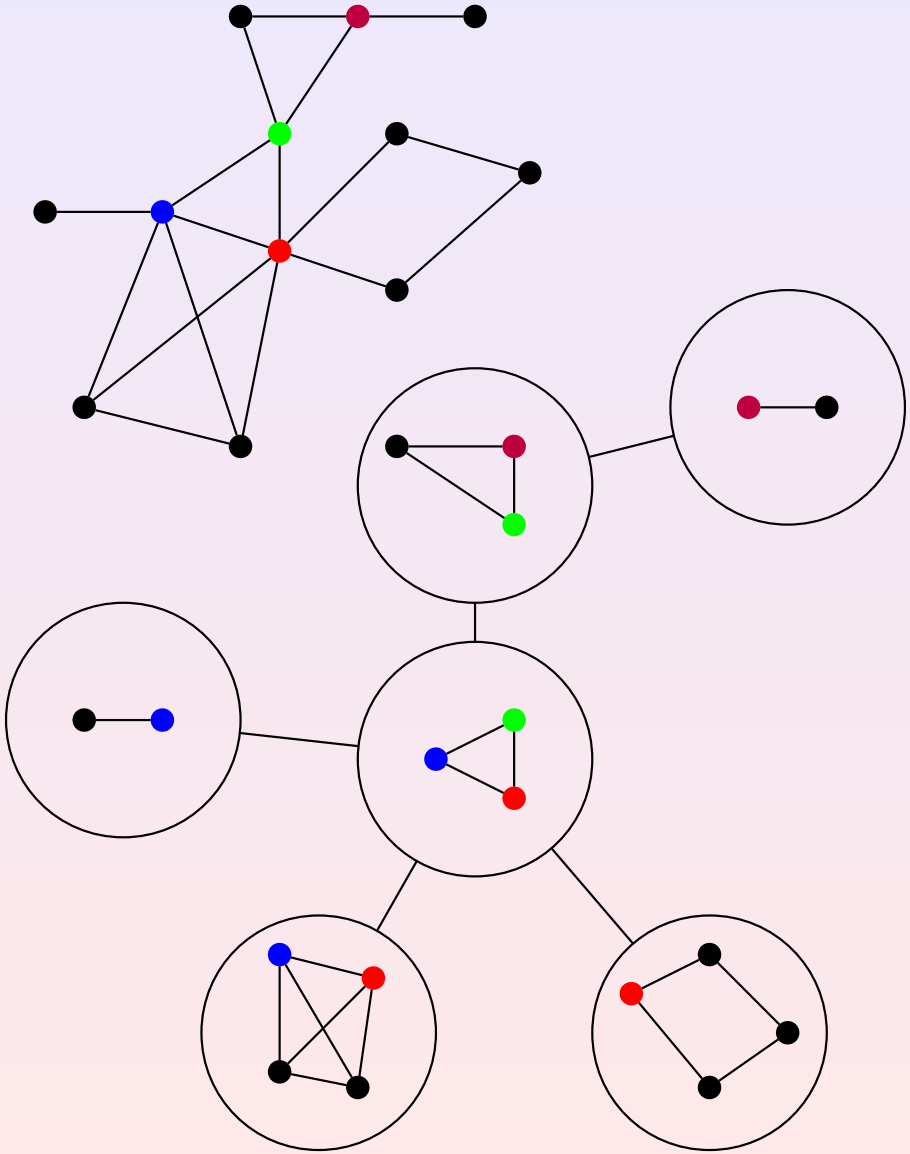
Width = Size of larger Bag -1

Tree and Path Decompositions



a tree T and bags $(X_t)_{t \in V(T)}$

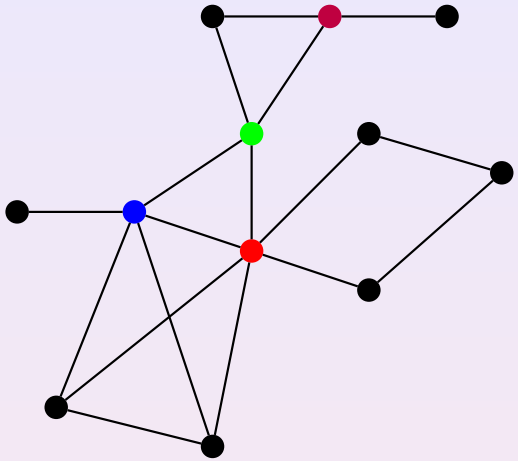
- every **vertex** of G is at least in one bag ;
- both ends of an **edge** of G are at least in one bag ;
- Given a vertex of G, all bags that contain it, form a **subtree**.

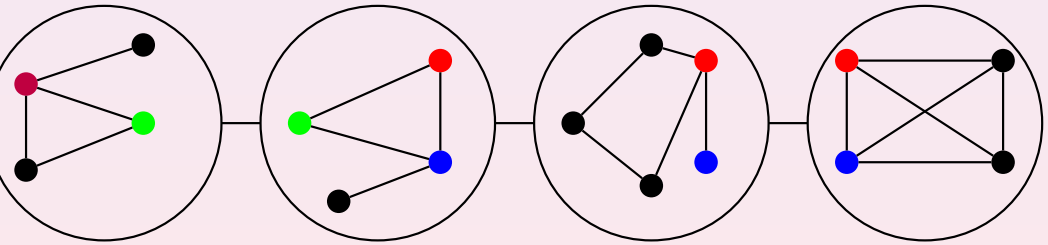
Width = Size of larger Bag -1

treewidth of G

tw(G), minimum width among any tree decomposition

Tree and Path Decompositions





a path P and bags $(X_t)_{t \in V(P)}$

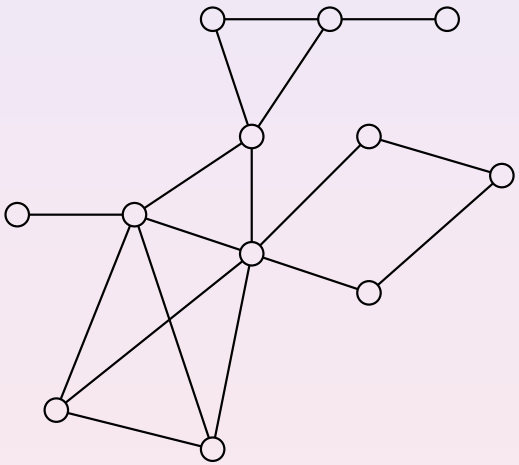
- every **vertex** of G is at least in one bag ;
- both ends of an **edge** of G are at least in one bag ;
- Given a vertex of G, all bags that contain it, form a **subpath**.

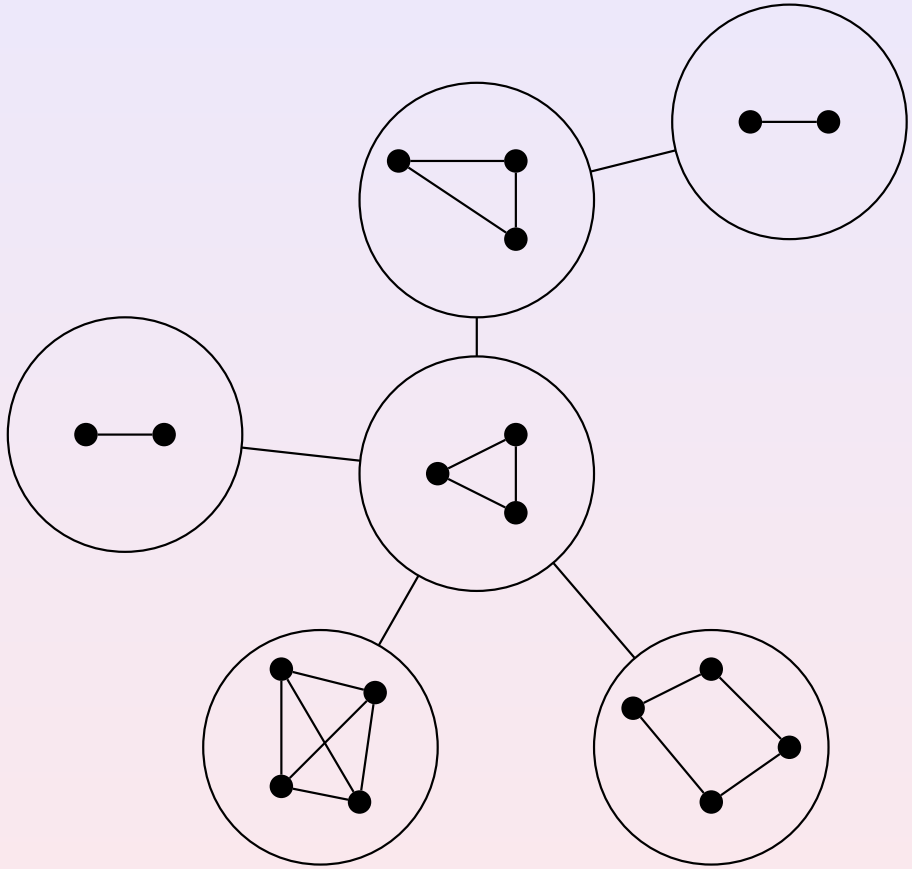
Width = Size of larger Bag -1

pathwidth of G

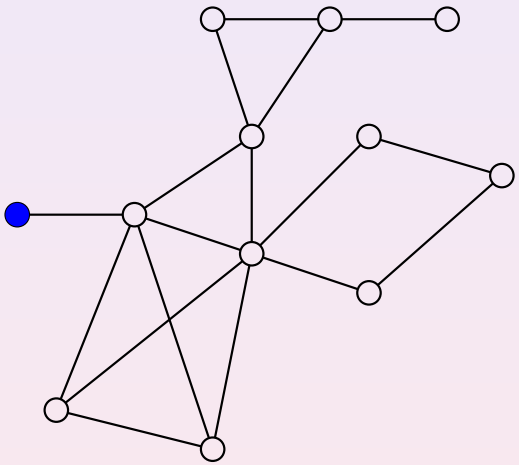
pw(G), minimum width among any path decomposition

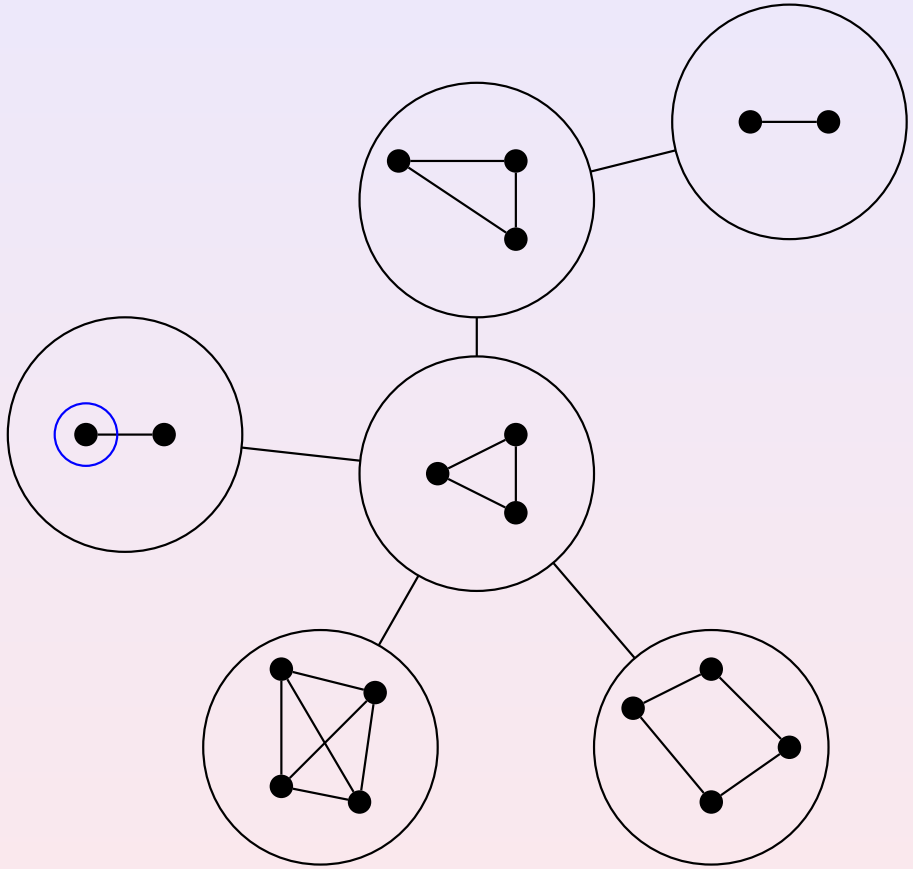
Tree Decomposition and Visible Search





Tree Decomposition and Visible Search





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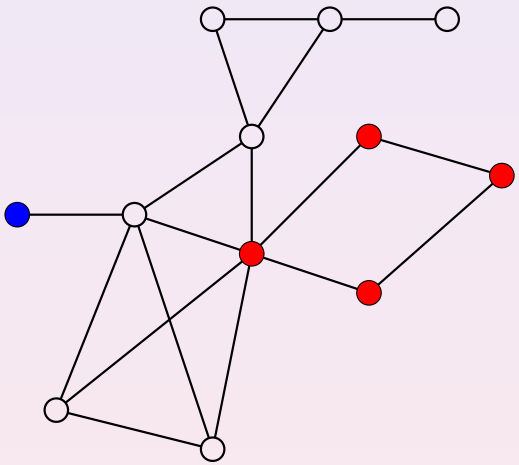

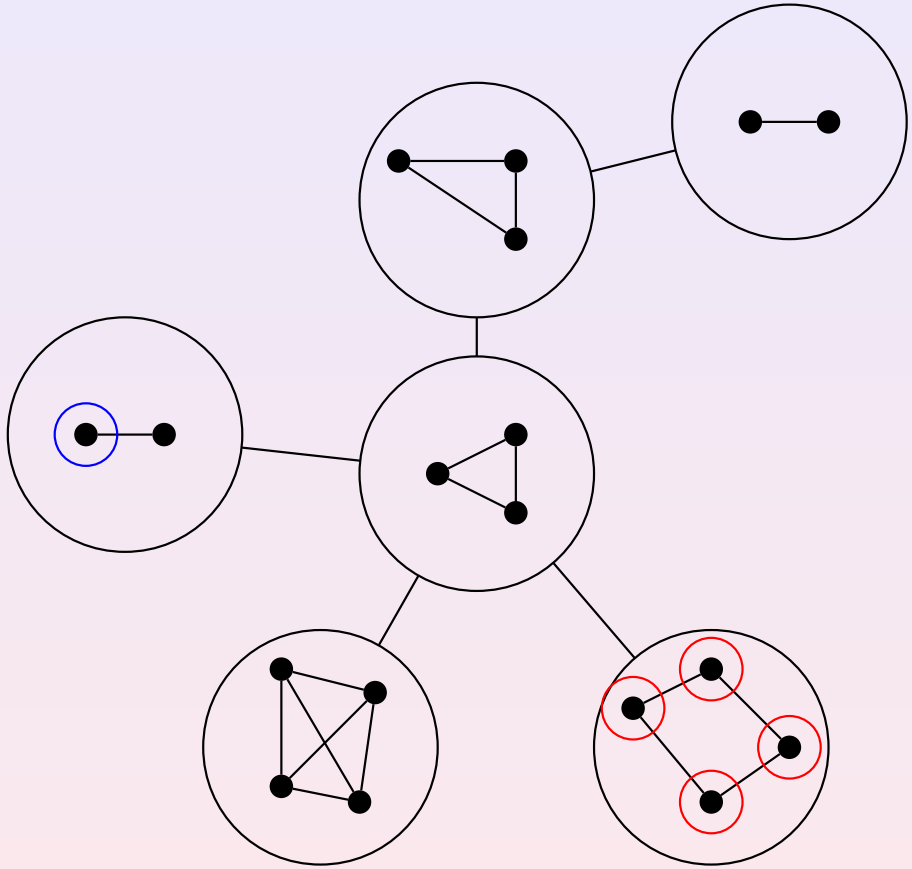

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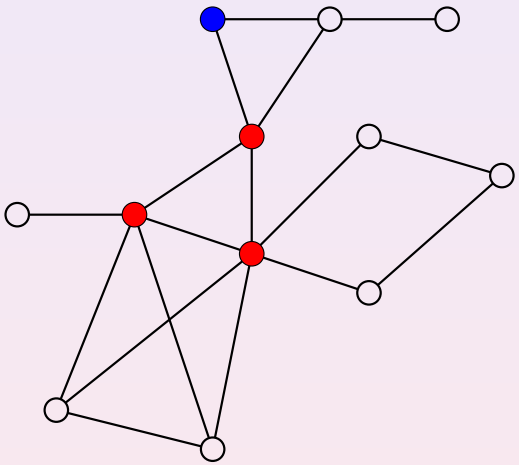

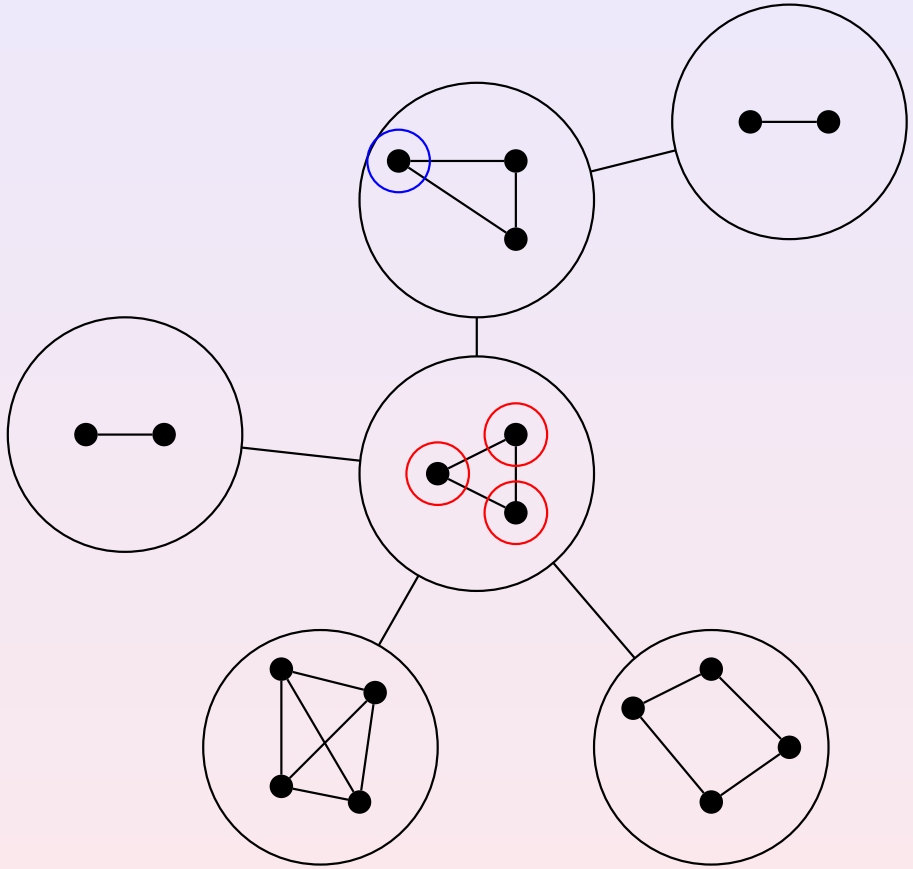

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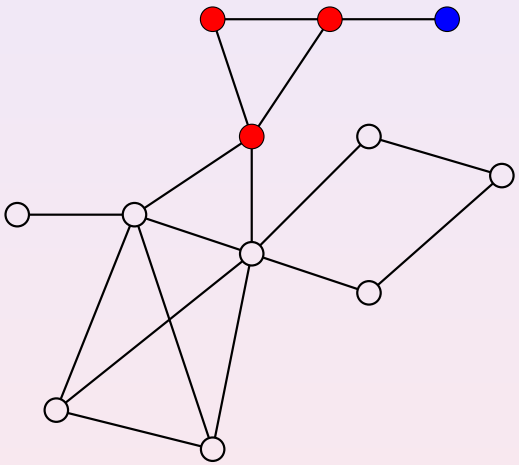
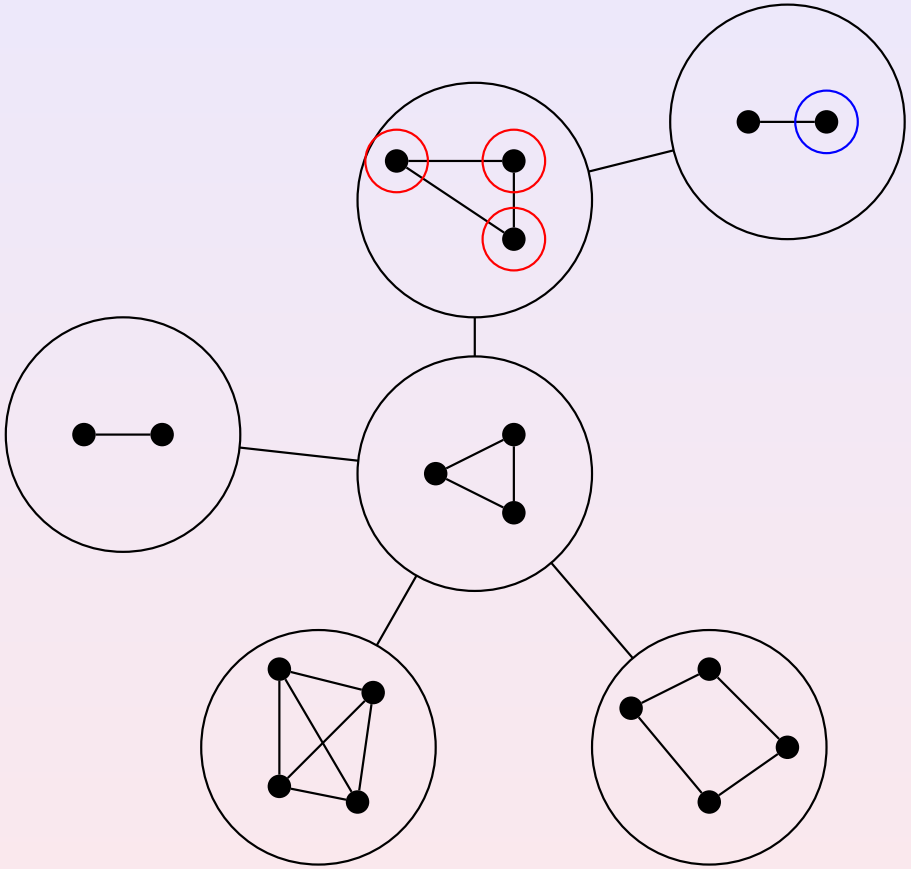
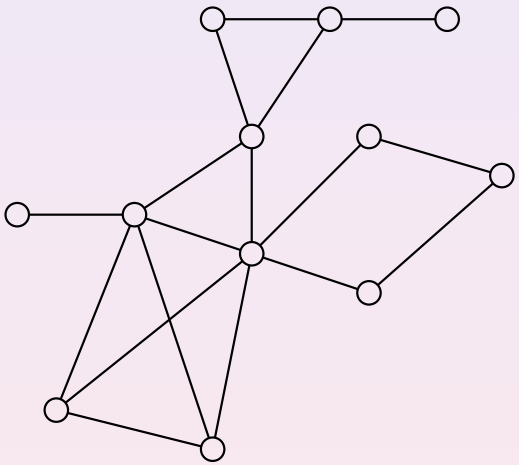
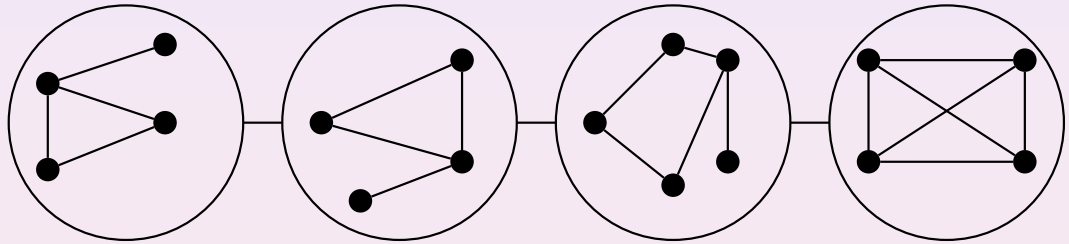




Path Decomposition and Invisible Search





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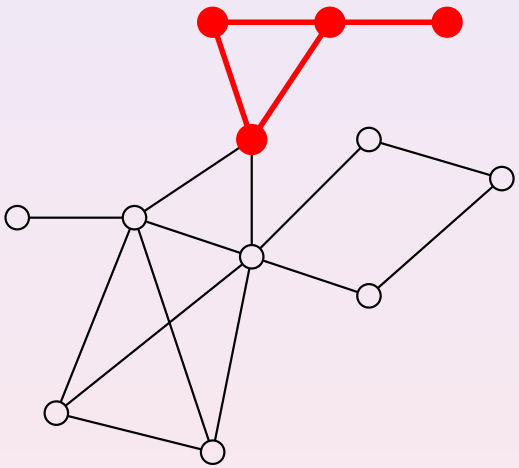

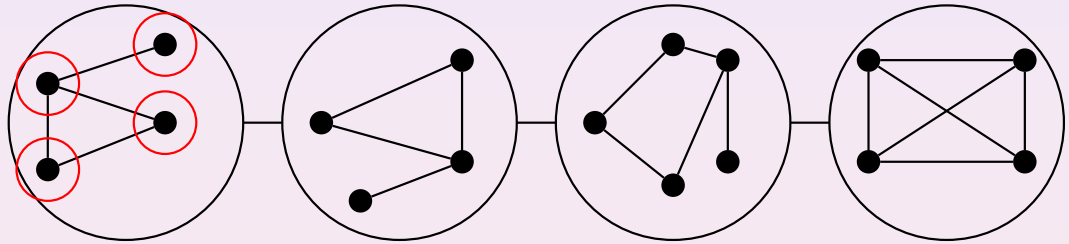

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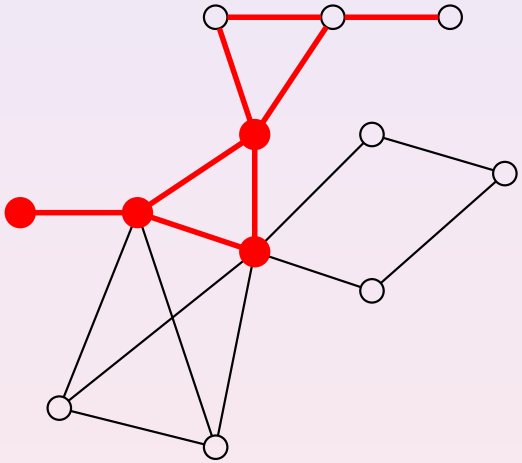

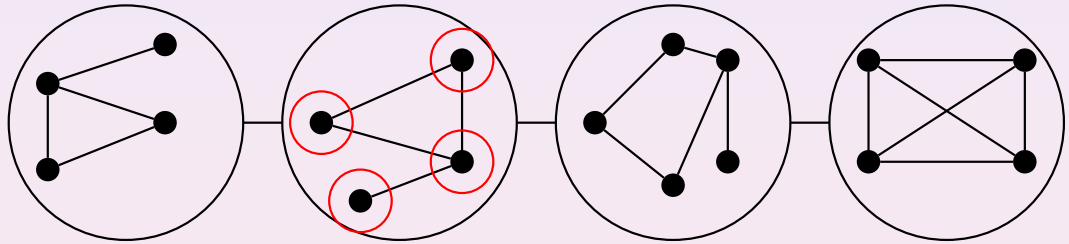

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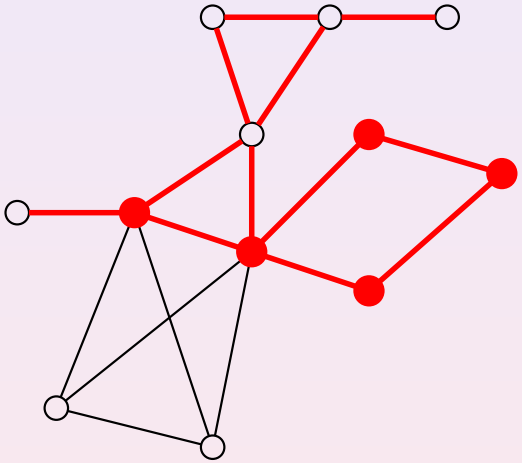
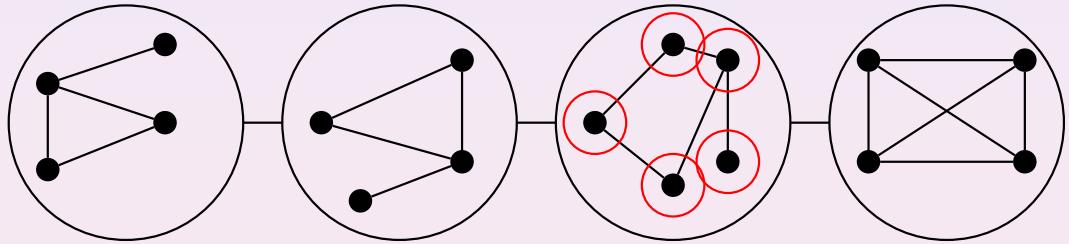




Related Work

$s(G) = pw(G) + 1$

- **Kirousis and Papdimitriou**, Theor. Comp. Sc., 1986
  Searching and Pebbing.
- **Ellis, Sudborough and Turner**. Inf. and Comp., 1994
  The vertex separation and search number of a graph

$s_{visible}(G) = tw(G) + 1$

- **Seymour and Thomas**, J. Combin. Theory, 1993
  Graph searching and min-max theorem for treewidth.

Nondeterministic Graph Searching

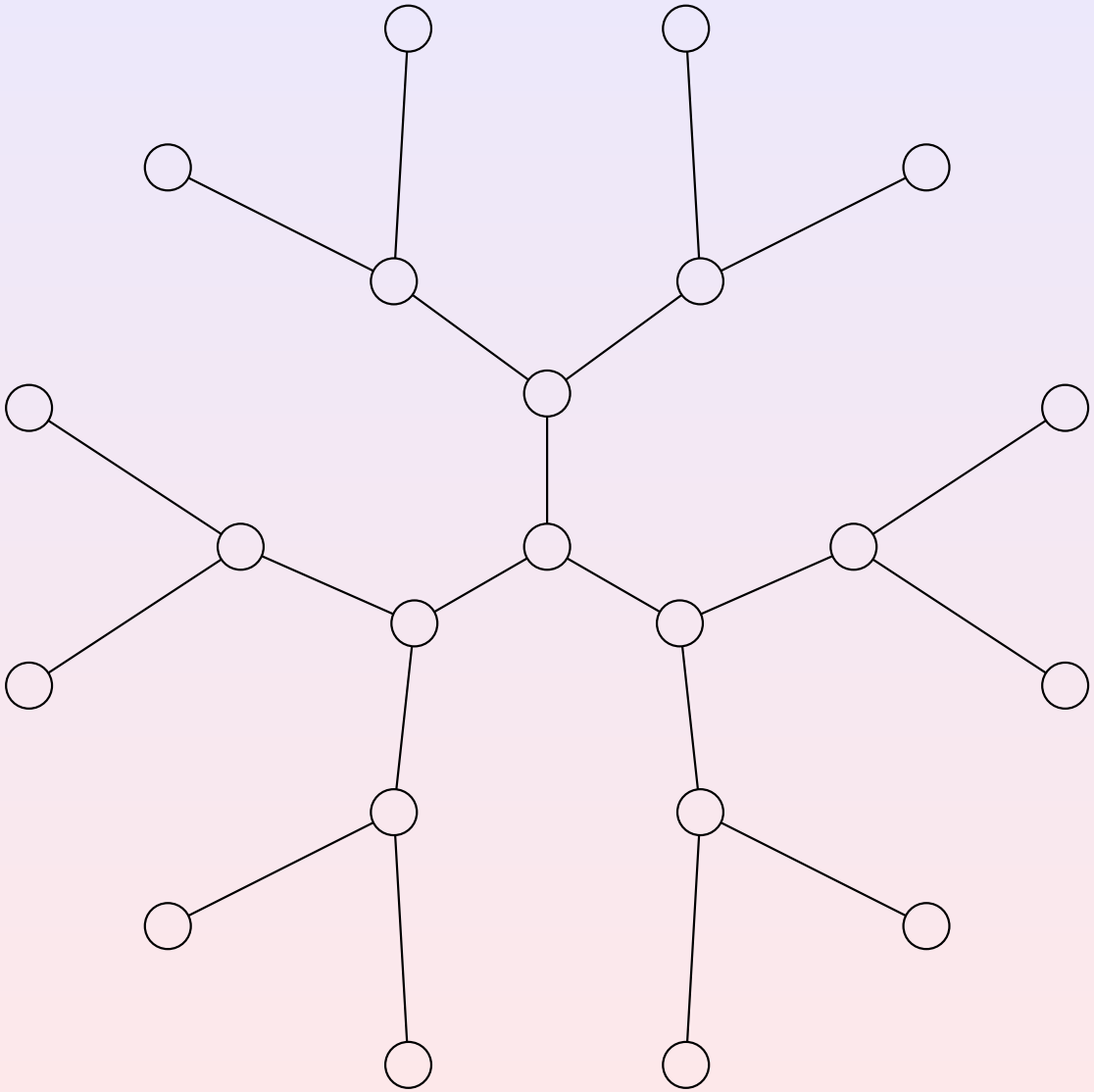
Query operation

- oracle ;
- tradeoff between number of searchers and number of query steps ;
- q-limited nondeterministic search number, $s_q(G)$.

Link with graph searching

- $s(G) = s_0(G)$;
- $s_{visible}(G) = s_\infty(G)$.

Limited Graph searching with 2 queries



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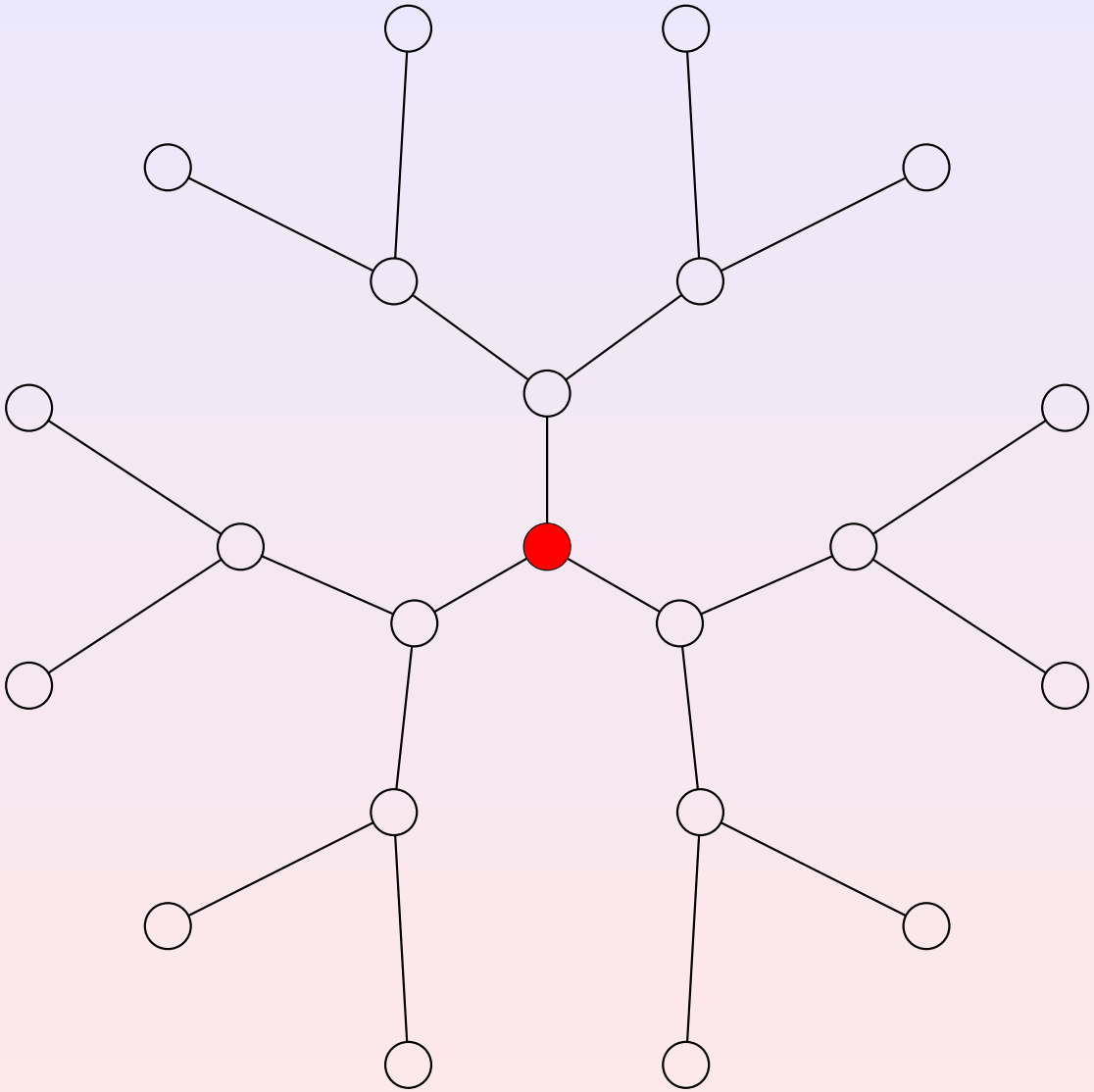

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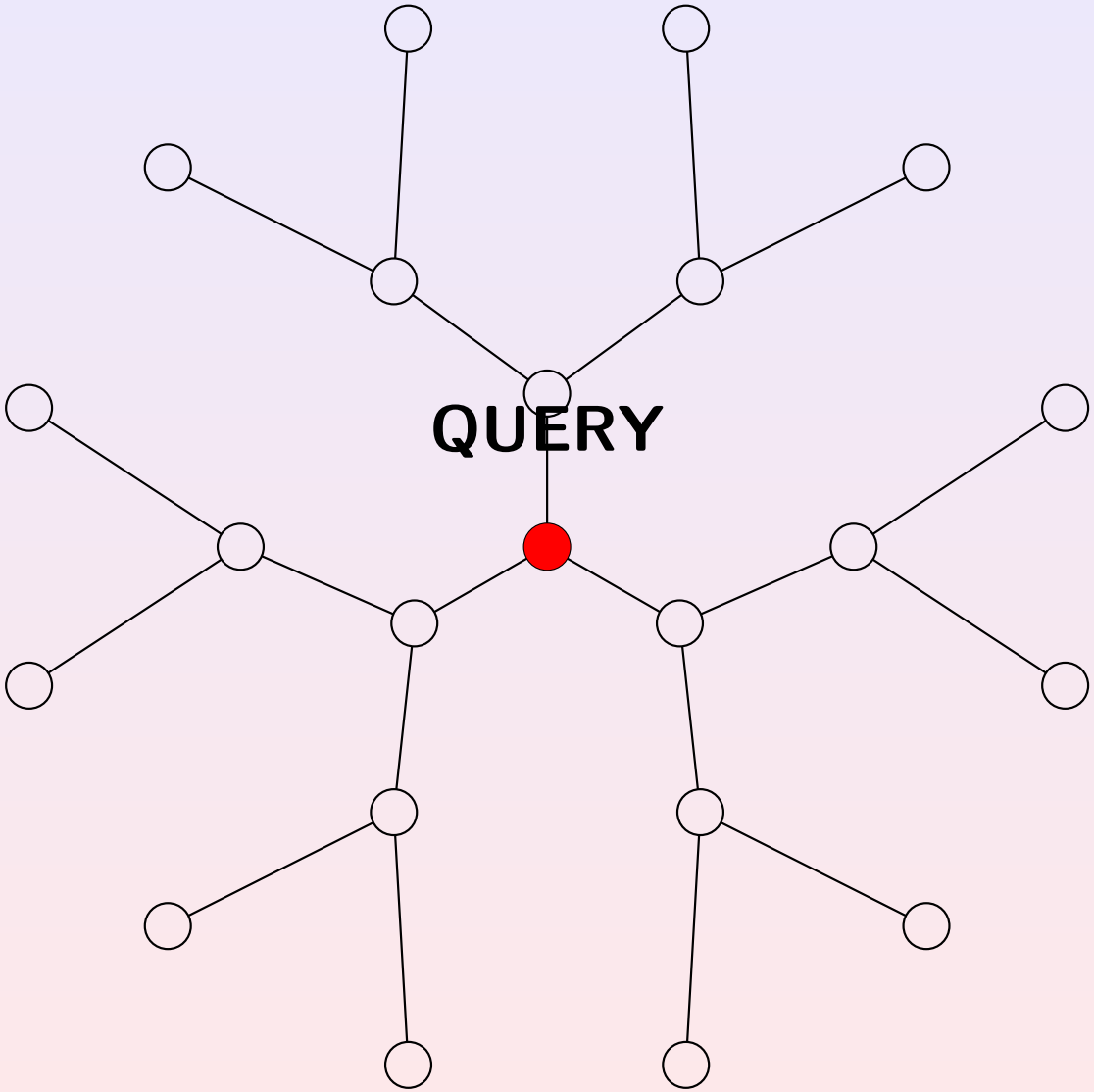

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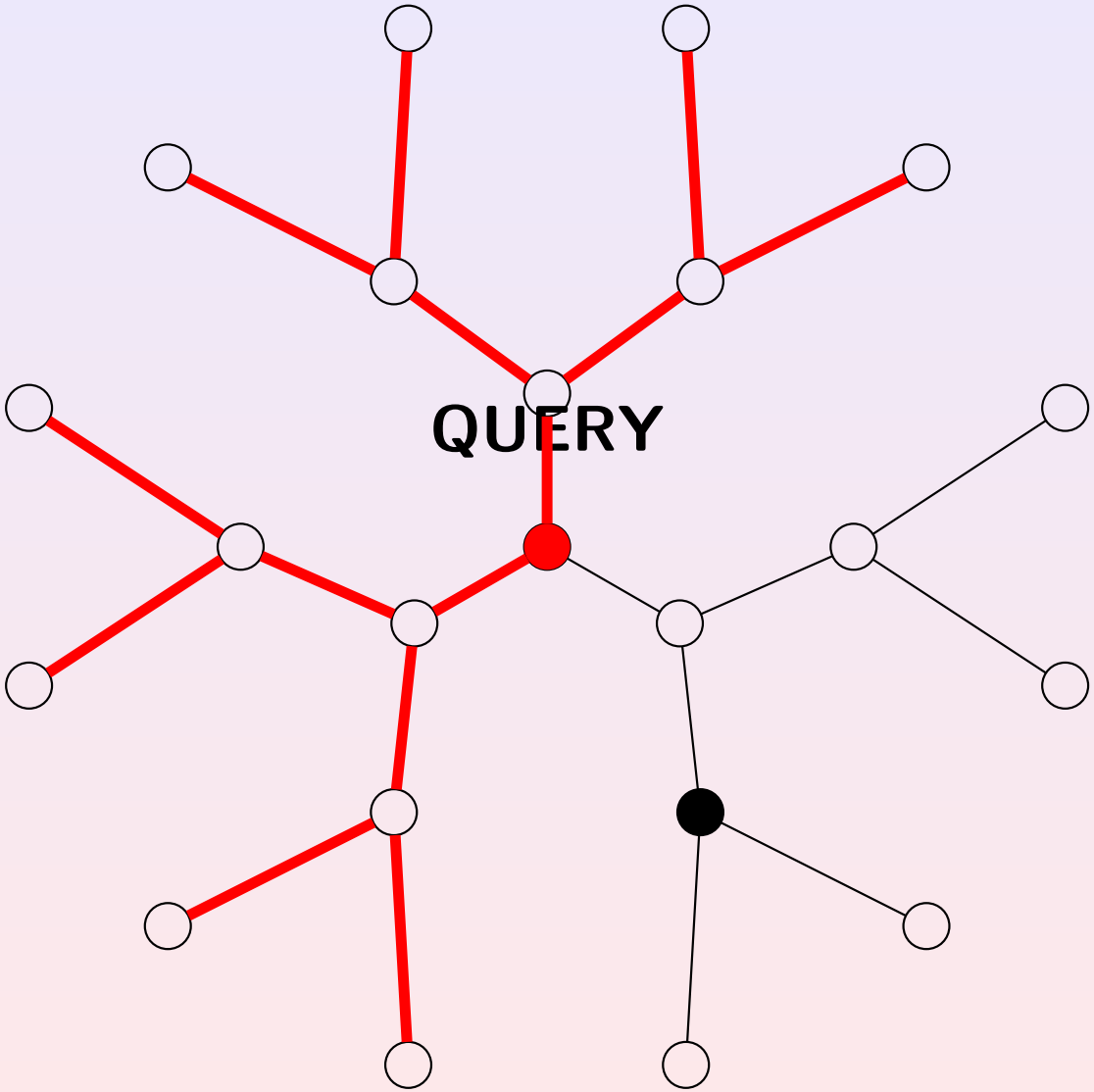

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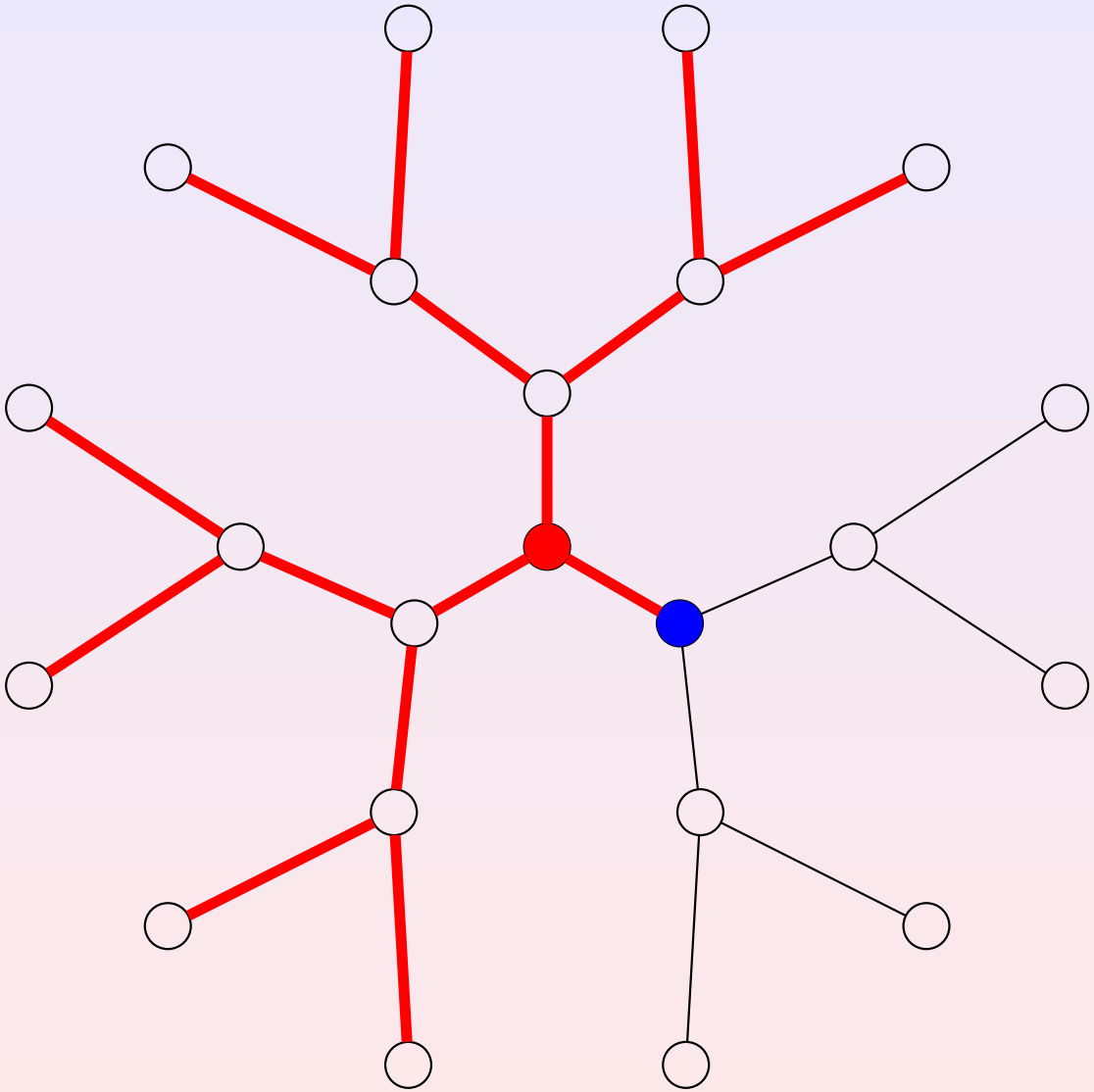

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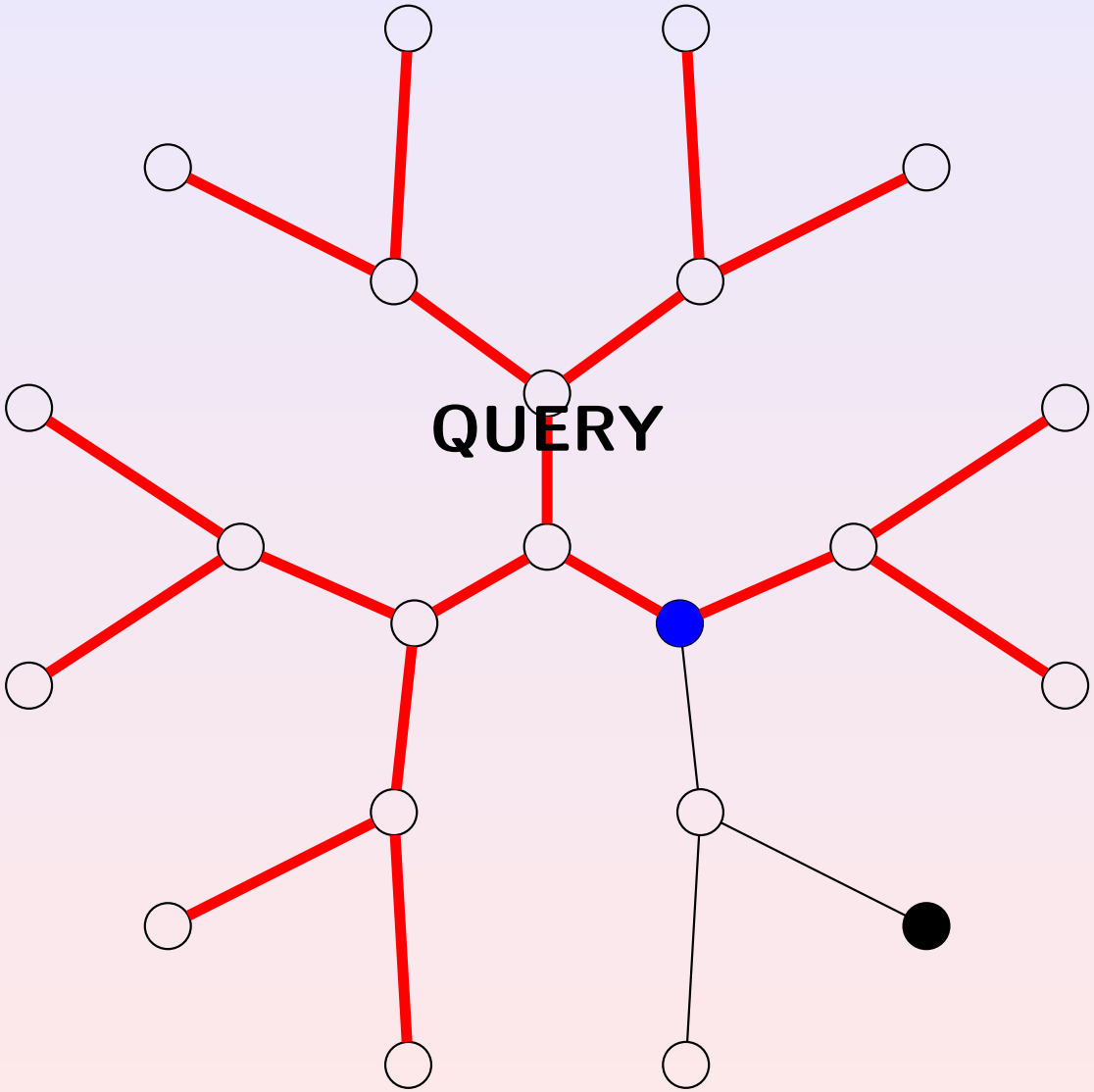

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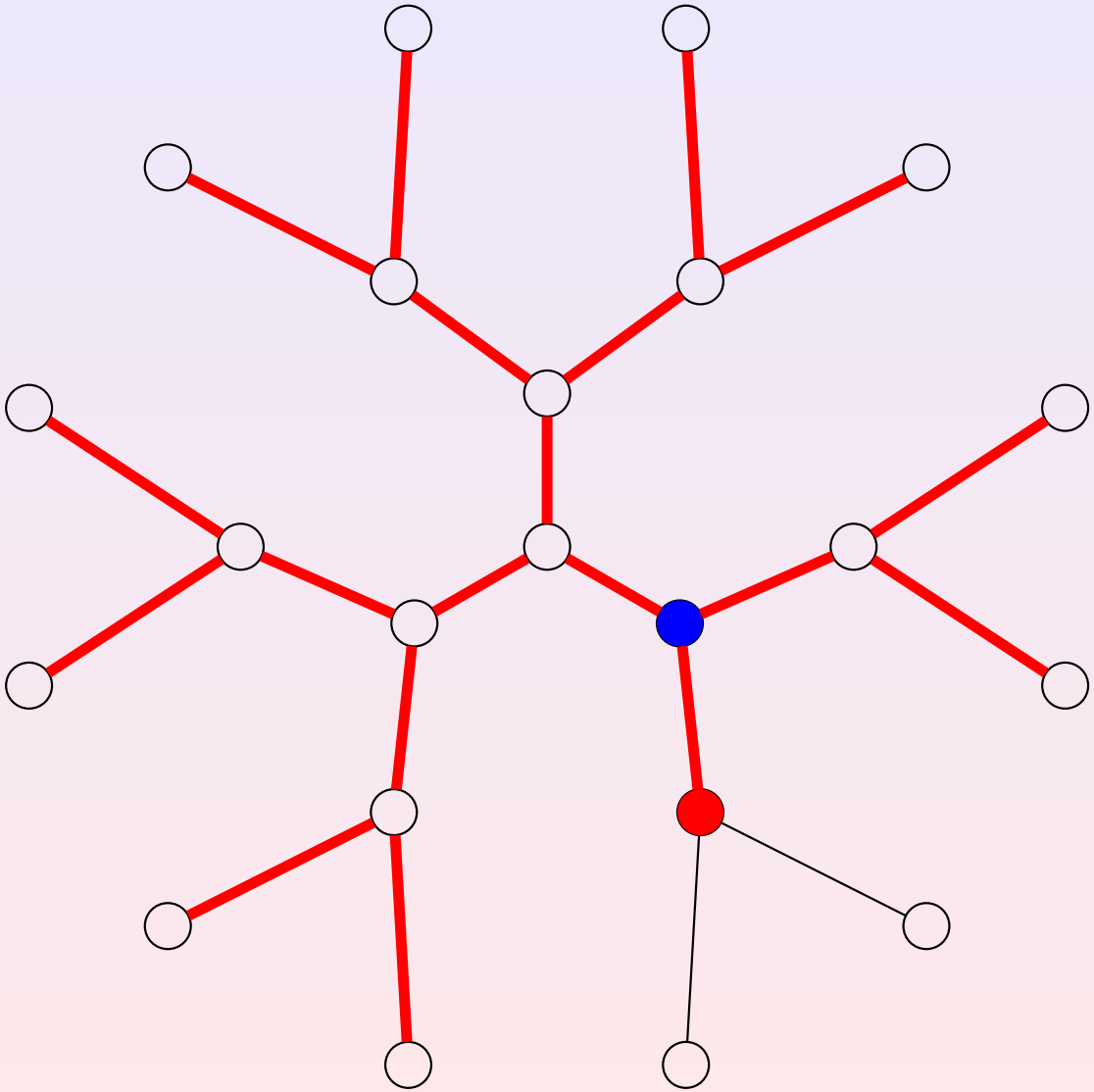

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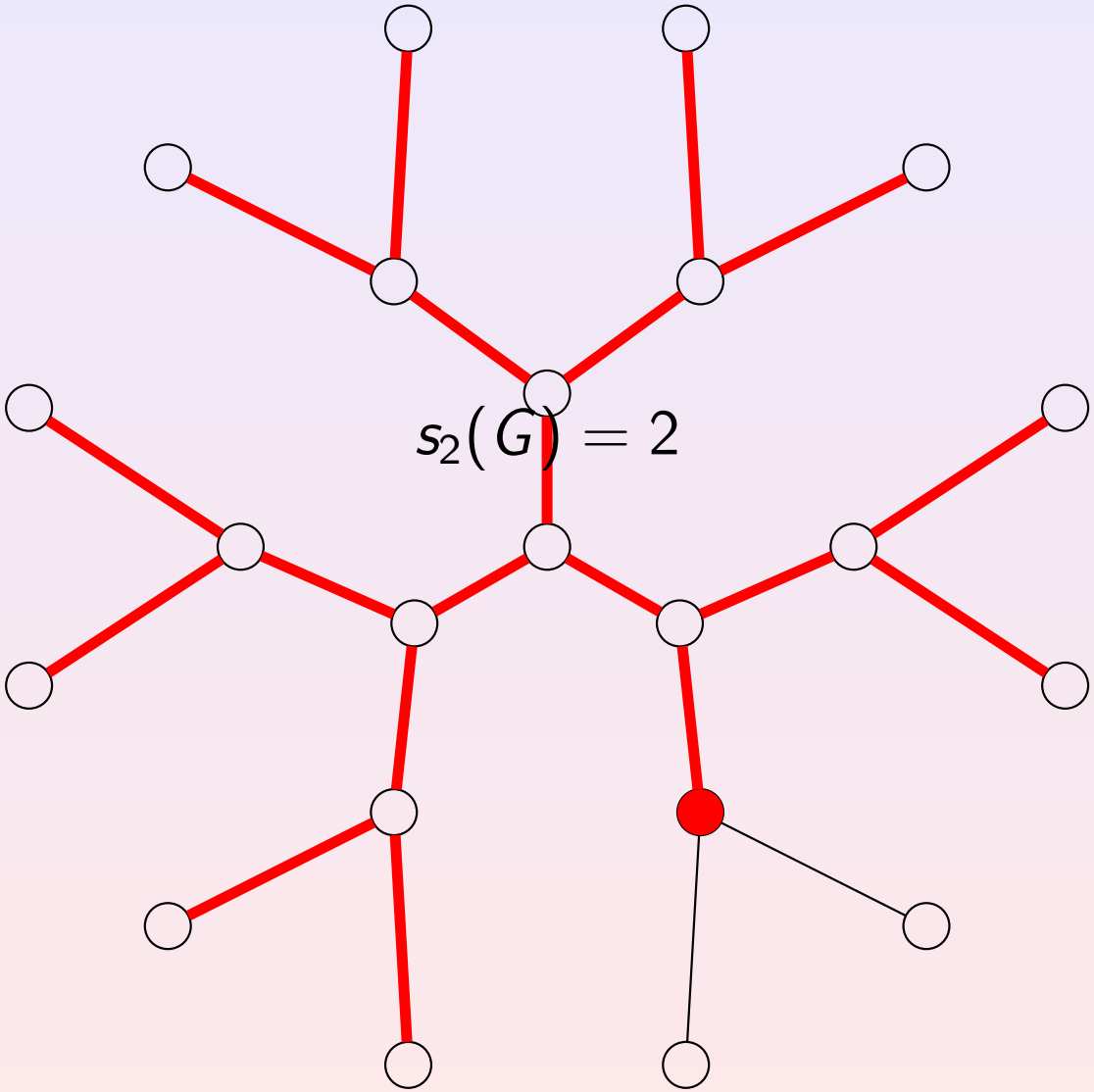
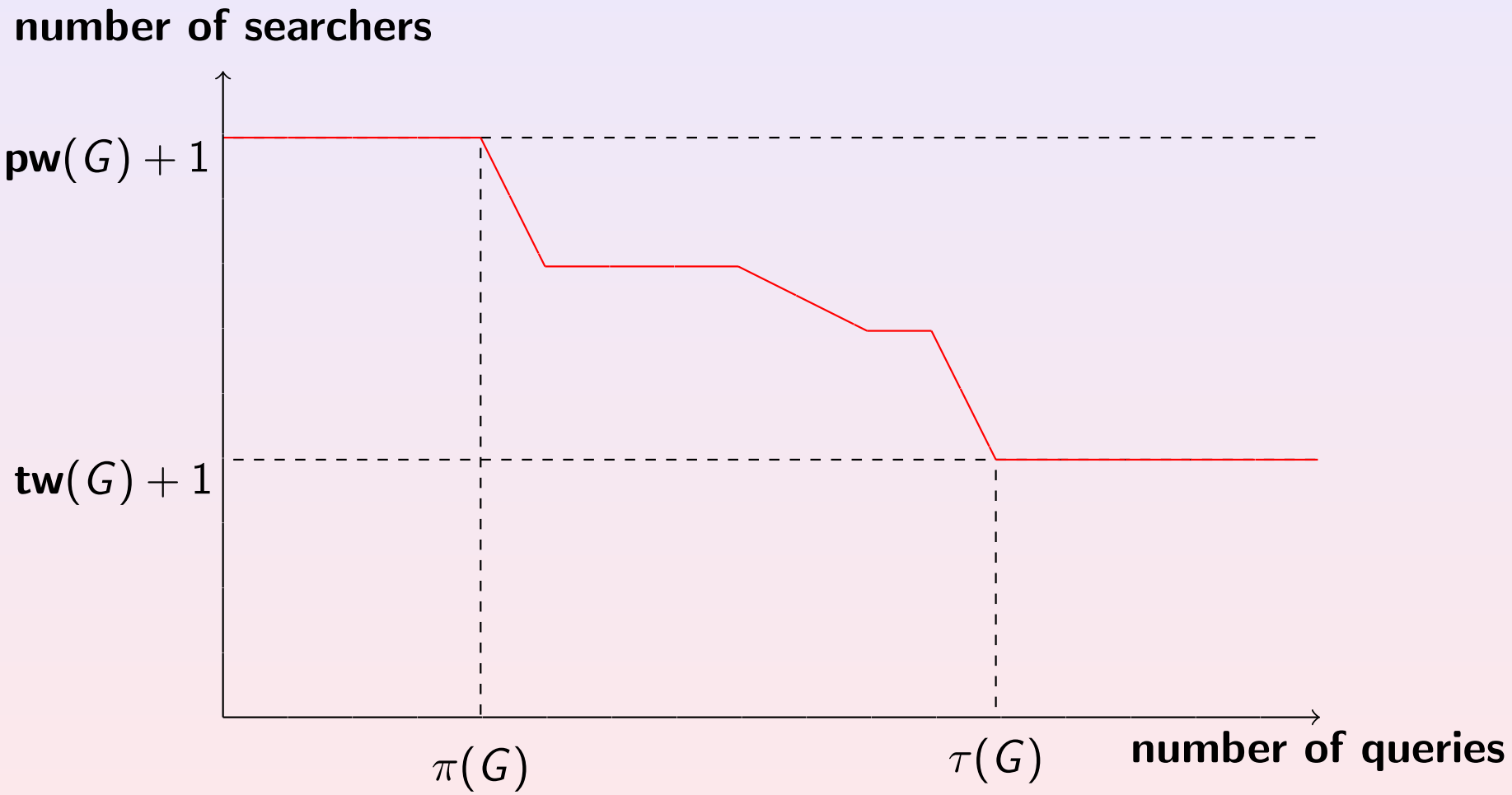


Controlled Amount of Nondeterminism



Branched Tree Decomposition

q-branched treewidth, $tw_q(G)$.

- rooted tree decomposition ;
- branching node (at least two children) ;
- every path from the root to a leaf contains at most q branching nodes.

- path decomposition $=$ 0-branched tree decomposition $\mathbf{pw}(G) = \mathbf{tw}_0(G)$;
- tree decomposition $=$ ∞-branched tree decomposition $\mathbf{tw}(G) = \mathbf{tw}_\infty(G)$;

Main Results (1)

Branched Treewidth vs. Limited Graph Searching

Theorem 1 : For any $q \geq 0$, for any graph G, $\mathbf{s}_q(G) = \mathbf{tw}_q(G) + 1$.

NP-Completness

Computing whether $\mathbf{tw}_q(G) \leq k$ is NP-complete for any q.

Main Results (2)

Exact Exponential Algorithm

Theorem 2 : There exists an algorithm that, for any graph G and any $q \geq 0$, computes $\mathbf{tw}_q(G)$ and an optimal q-branched tree decomposition of G.

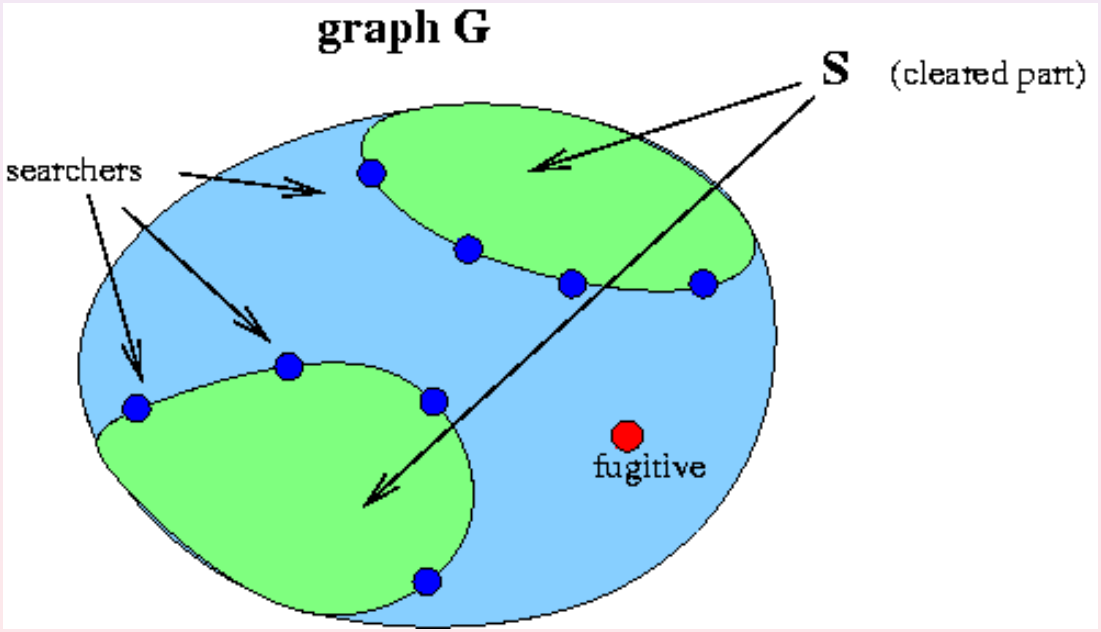
Bounding the Nondeterminism

Theorem 3 : For any $q \geq 1$, for any graph G, $\mathbf{tw}_{q-1}(G) \leq 2\ \mathbf{tw}_q(G)$.

Exact Exponential Algorithm

Configuration digraph H

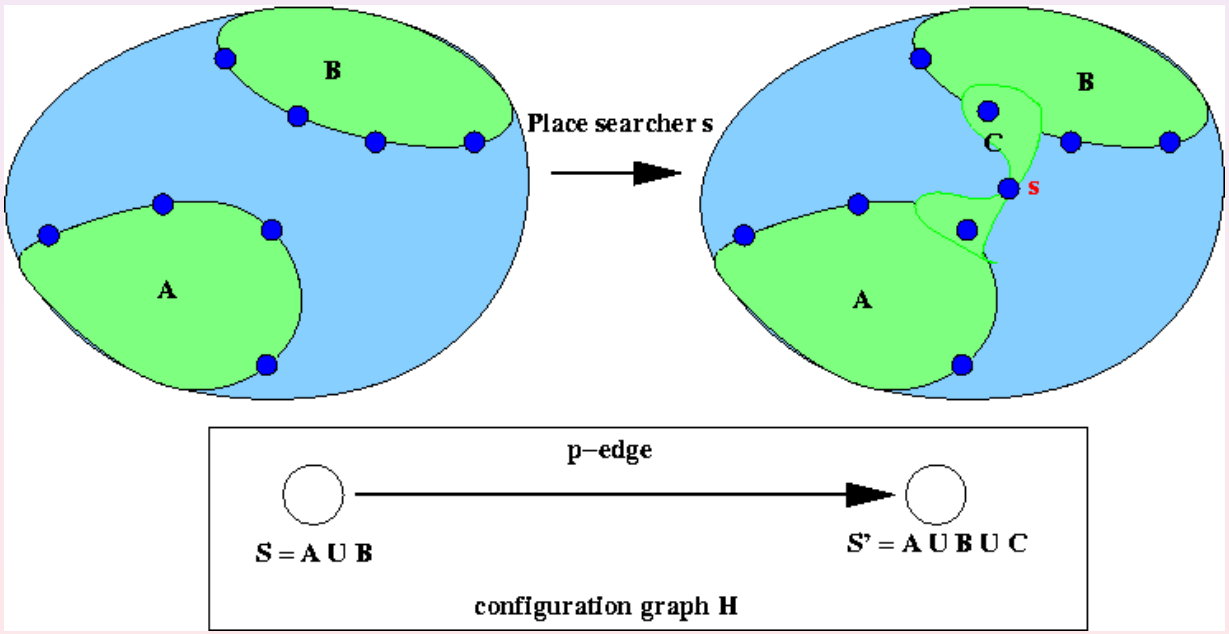
- a **vertex** S of H is a *set of clear vertices* of G *reachable* by a search program using k searchers. $V(H) = \{S \subseteq V(G)\ s.t.\ |\delta(S)| \leq k\}$;



Exact Exponential Algorithm

Configuration digraph H

- a **directed edge** $\{S, S'\}$ of H is a *search step* that allows to reach S' from S.
  place edges and *query* edges ;



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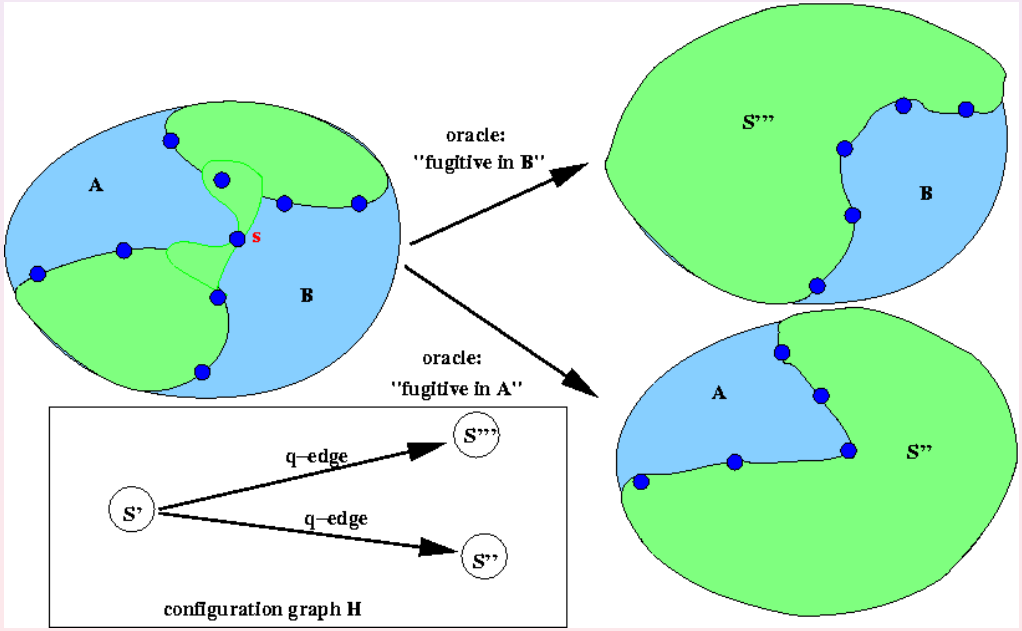
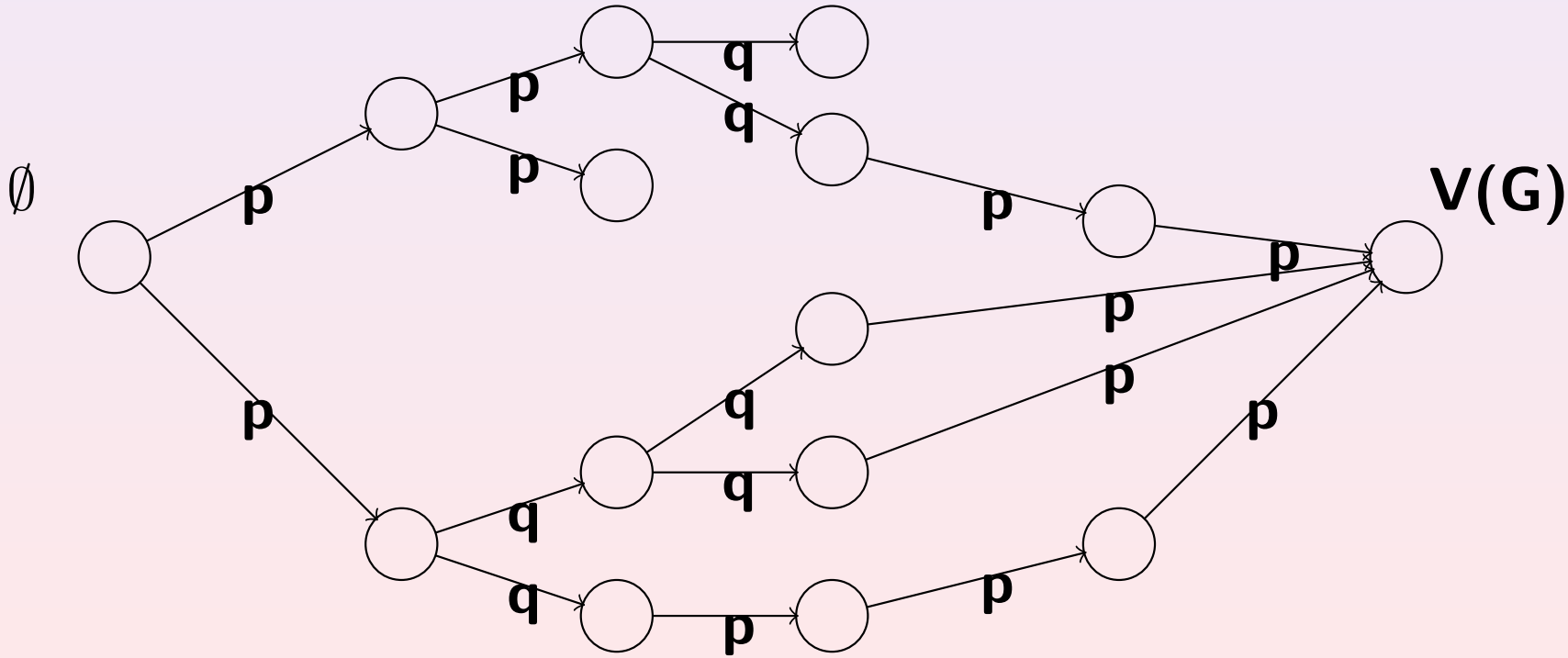


Exact Exponential Algorithm

Principle of the Algorithm

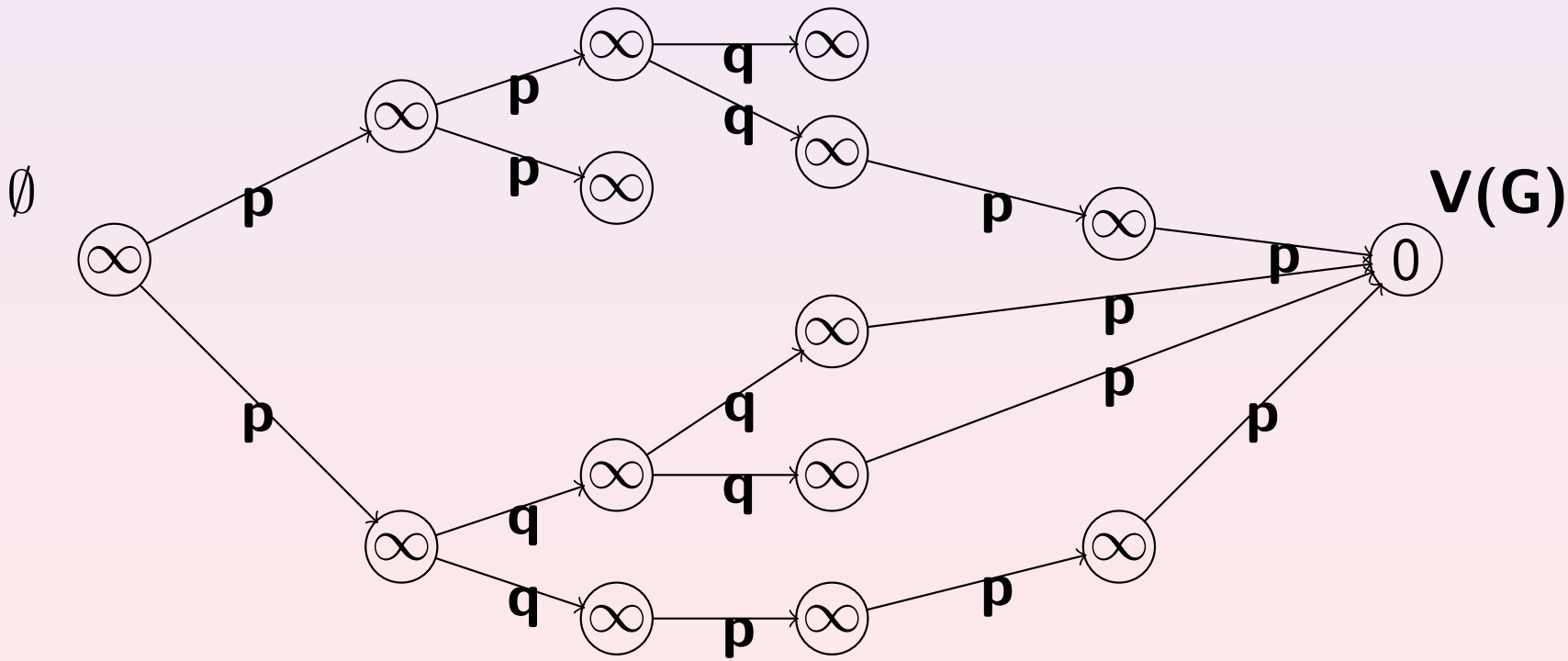
- To find a path in H from $S = \emptyset$ to $S = V(G)$.
  Correspondence with a search strategy.
- Labelling of vertices of the configuration graph.



Exact Exponential Algorithm

Principle of the Algorithm

- *label*(S)= number of queries required to reach $V(G)$;
- Propagation of the labelling from *label*$(V(G)) = 0$;
  label$(\emptyset)$ is the minimum q such that $\mathbf{s}_q(G) = k$;



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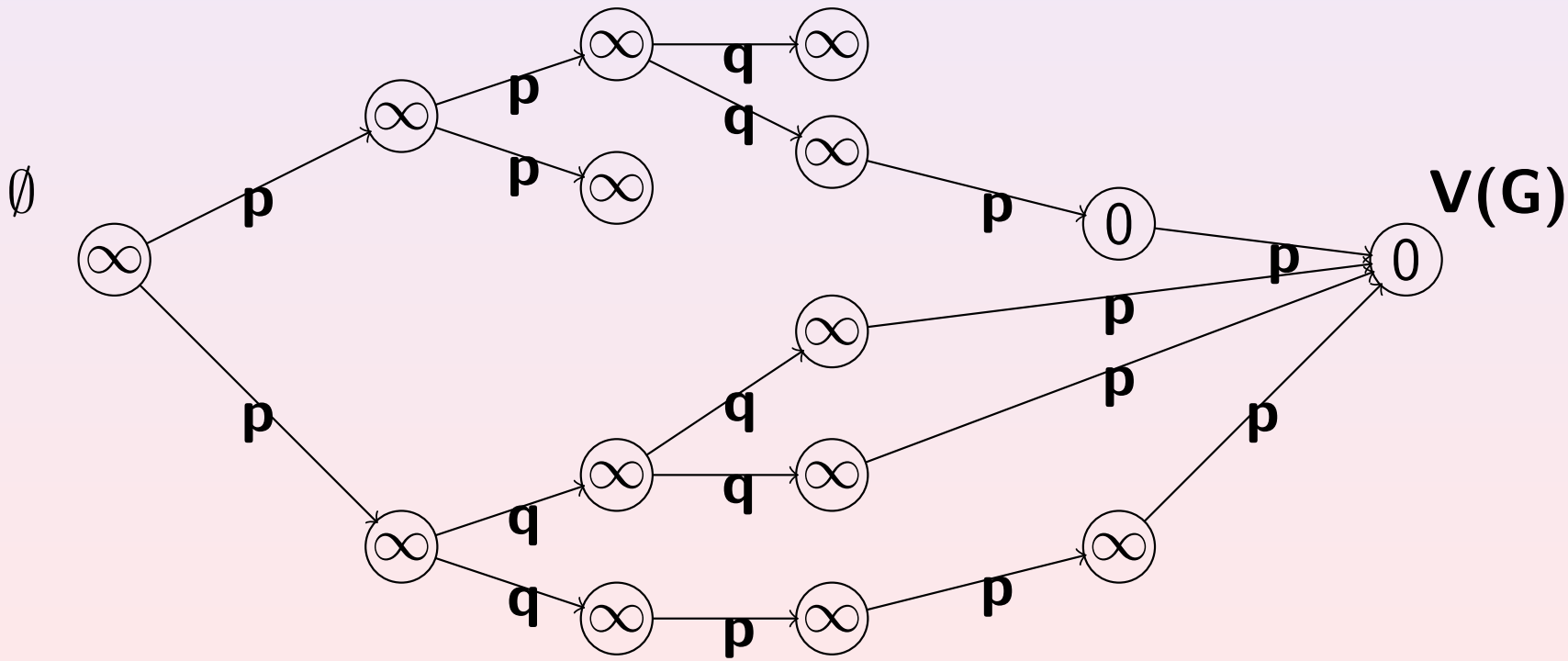

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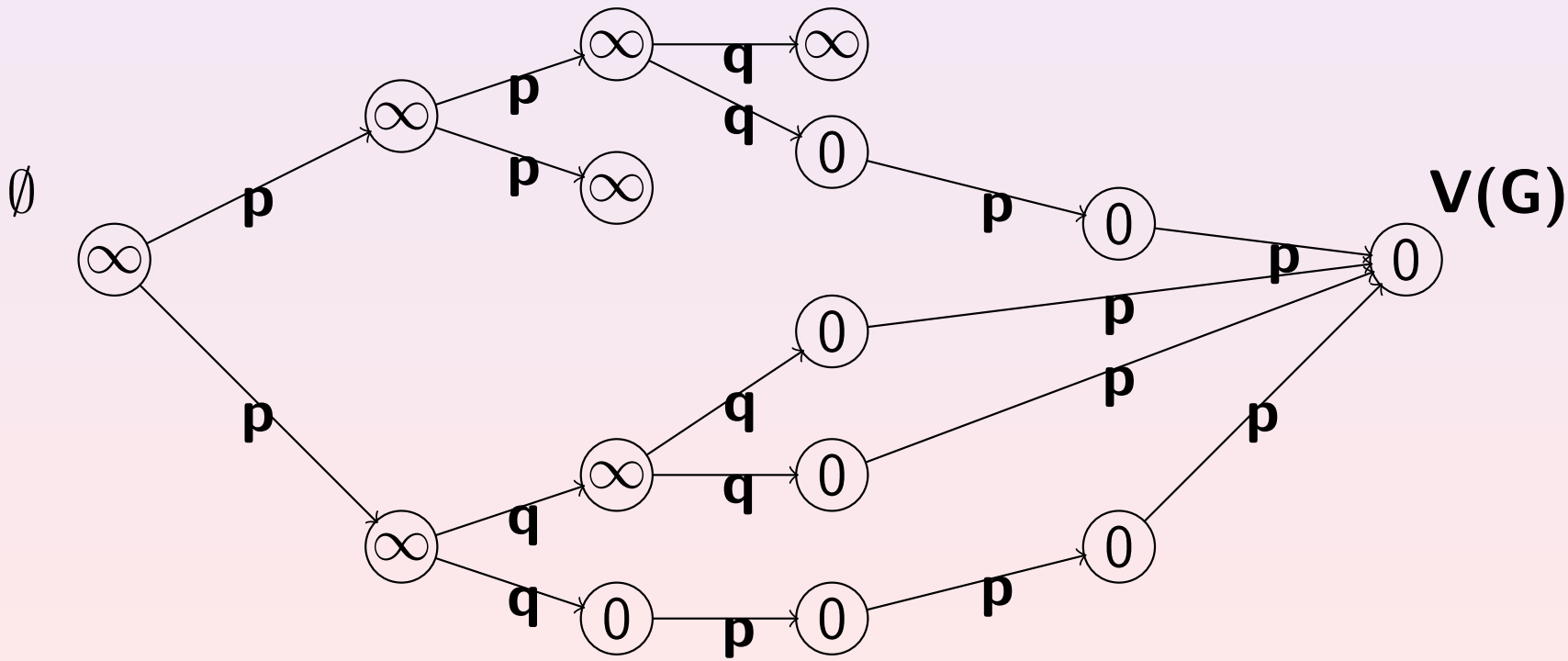

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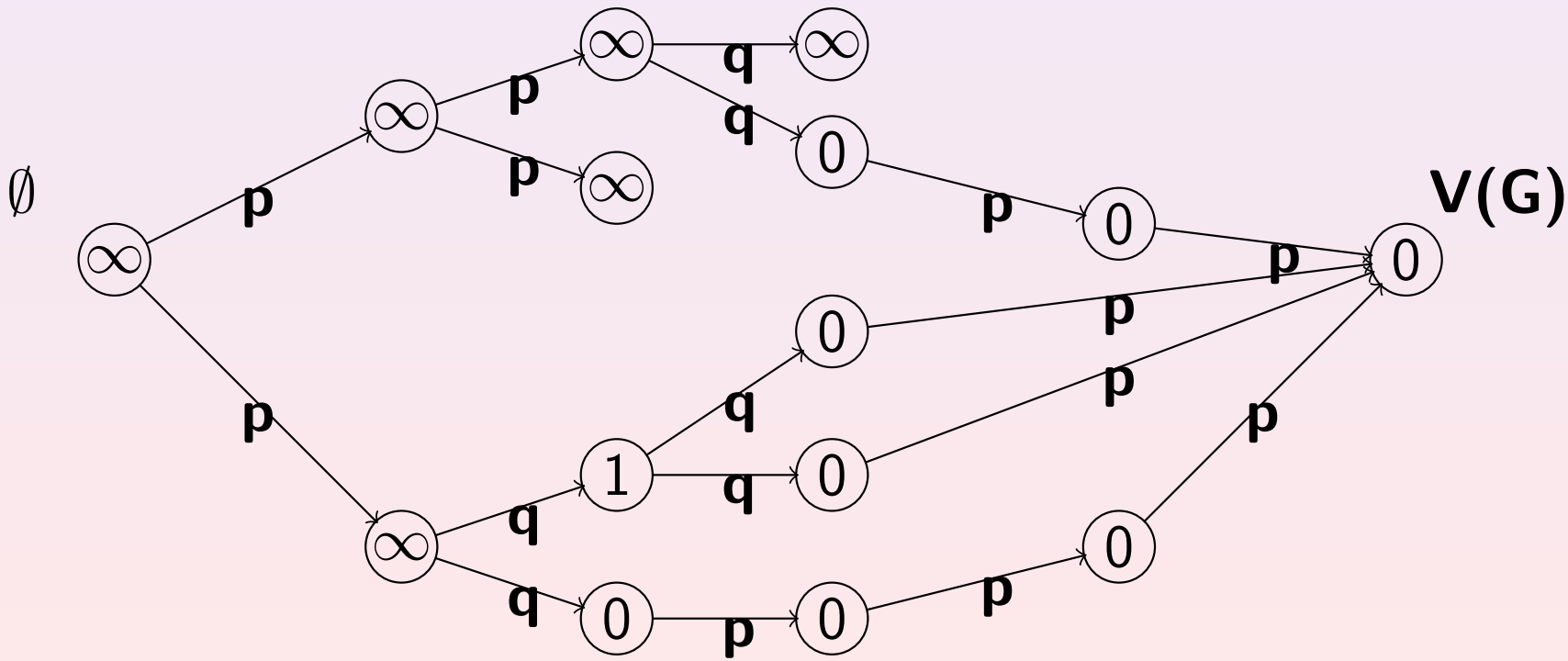

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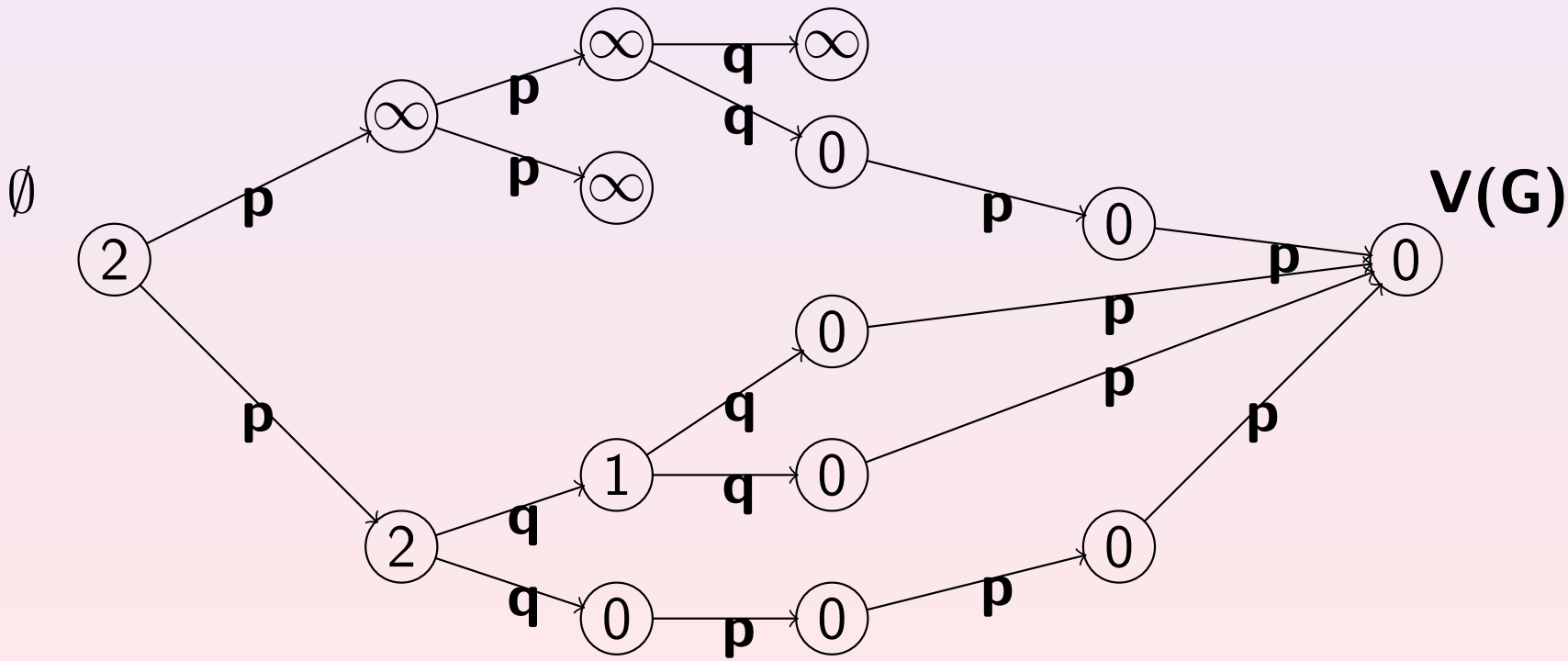

Conclusion and Further Work

New variant of search game

- nondeterministic search game ;
- unify pathwidth and treewidth.

Further Work

- design of an $O(c^n)$-time exact algorithm $(c < 2)$;
- **role of recontamination.**