Dimension Métrique des Graphes Orientés

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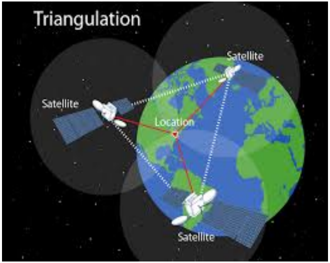
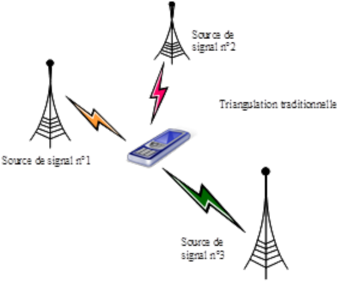
AlgoTel, Saint Laurent de la Cabrerisse, 6 juin 2019

Metric Dimension of graphs





Precisely locate using few information
How to generalize to graph metric ?

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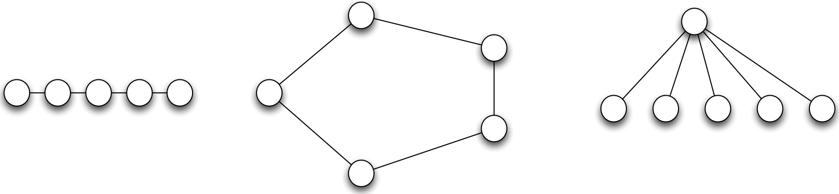
A target is **hidden** at some (**unknown**) vertex t of a graph $G = (V, E)$
Probing a vertex $v \in V(G) \Rightarrow$ the distance $dist_G(t, v)$ between v and t.

Resolving set : set of vertices to probe s.t. the target is uniquely located

Set $R = \{v_1, \cdots, v_i\} \subseteq V$ s.t. $(dist_G(v, v_j))_{j \leq i}$ pairwise distinct $\forall v \in V$.



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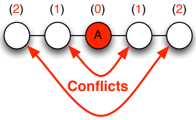

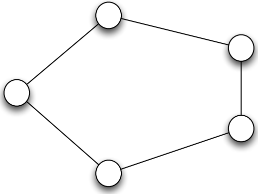

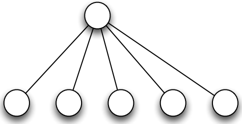

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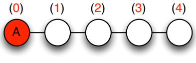

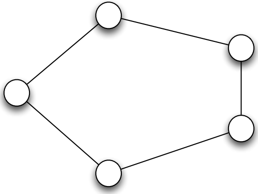

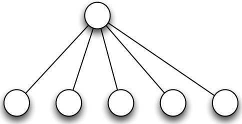

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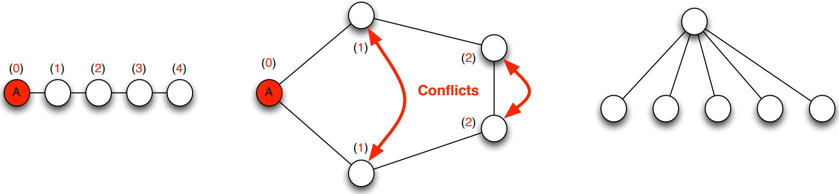

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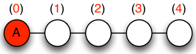

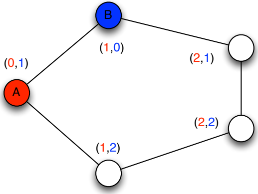

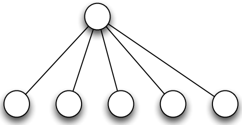

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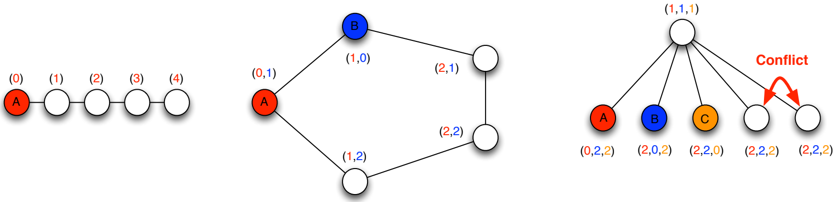

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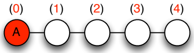

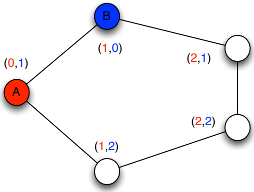

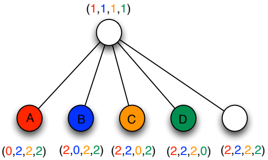

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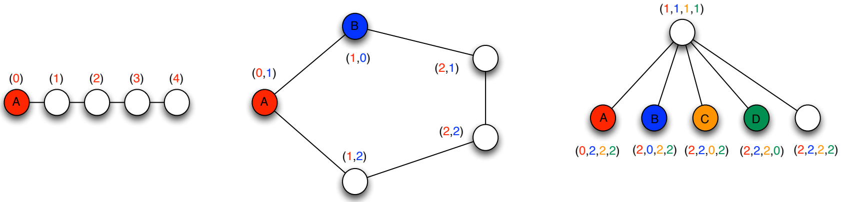

Metric Dimension $MD(G)$: min. size of a resolving set in G. $(MD(G) \leq |V|)$

example : for any tree T, $MD(T) = \#leaves - \#$"branching nodes"

Computing $MD(G)$ [Harary, Melter 76, Slater 75] is NP-c in planar graphs [Díaz *et al.* 17], W[2]-hard [Hartung,Nichterlein 13], FPT in tree-length [Belmonte *et al.* 17]...

Sequential Metric Dimension

Fast localization of an Immobile target : *[Bensmail et al. 2018]*

Minimize number of turns when probing $k \leq MD(G)$ per turn ?

- NP-complete, for any fixed $k \geq 1$, in general graphs ;
- In trees : NP-complete (k part of the input), $+1$-approximation.

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Localization of a moving target in a graph G : *[Seager 13]*

$\zeta(G)$: min. number of probes per turn to locate the target moving in G ?

- $\zeta(T) \leq 2$ for any tree T. Characterization of trees with $\zeta(T) = 2$ *[Seager 14]*
- Deciding if $\zeta(G) \leq k$ is NP-hard ; *[Bosek et al. 18]*
- $\zeta(G) \leq pw(G)$ for any graph G, but $\zeta(G)$ unbounded in the class of graphs with treewidth ≤ 2 ; *[Bosek et al. 18]*
- $\zeta(G) \leq 2$ for any outerplanar graph G. *[Bonato,Kinnersley 18]*

Metric Dimension in Oriented Graphs

Orientation of G **:** each edge $\{u, v\}$ becomes exactly one arc among uv or vu.

Probing a vertex $v \in V(G) \Rightarrow$ the distance $dist_G(v, t)$ FROM v TO t.

Resolving set : set of vertices to probe s.t. the target is uniquely located

Set $R = \{v_1, \cdots, v_i\} \subseteq V$ s.t. $(dist_D(v_i, v))_{j \leq i}$ pairwise distinct $\forall v \in V$.

Remark : *A priori*, $dist_D(v, t)$ may be ∞.

$\rightarrow$ ONLY strongly connected orientations.

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$MD(D)$: min. size of a resolving set in a **strong** oriented graph D.

Few related work

- upper bounds *[Chartrand et al 00]*
- NP-complete in strong oriented graphs *[Rajan et al. 14]*
- complete graphs *[Lozano 13]*, Cayley digraphs *[Fehr et al. 06]*, de Bruijn and Kautz *[Rajan et al. 14]*

Worst/Best Oriented Metric Dimension (WOMD/BOMD)

Given a class $\mathcal{G}$ of undirected n-node graphs,

- $WOMD(\mathcal{G}) = \sup_{D \text{ strong orientation of } G \in \mathcal{G}} \frac{MD(D)}{n}$
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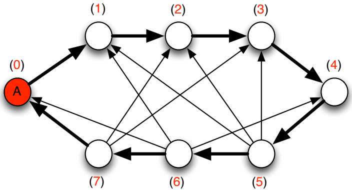
Very few previous work

[Chartrand et al. 01]

- tournaments : $WOMD(K_n) = 1/2$
- $\mathcal{H}_n$, class of Hamiltonian n-node graphs : $BOMD(\mathcal{H}_n) = 1/n$.

Every Hamiltonian graph has an orientation D with $MD(D) = 1$.



Our contributions

Focus on Worst Orientations (WOMD) for various graph classes.

$\mathcal{G}_\Delta$: class of graphs with maximum degree $\leq \Delta$.

- $\frac{2}{5} \leq WOMD(\mathcal{G}_3) \leq \frac{1}{2}$
- $\frac{1}{2} \leq WOMD(\mathcal{G}_4) \leq \frac{6}{7}$
- $\lim_{\Delta \to \infty} WOMD(\mathcal{G}_\Delta) = 1$

Grids **: class of cartesian grids.**

- $\frac{1}{2} \leq WOMD(Grids) \leq \frac{2}{3}$

$WOMD^*$ defined as $WOMD$ but over Eulerian orientations (in-degree=out-degree).

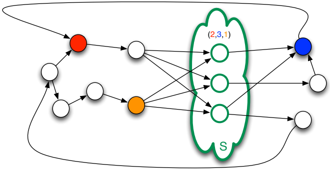
Tori **: class of cartesian tori.**

- $WOMD^*(Tori) = \frac{1}{2}$

Easy lemmas but very useful

Lower bound

S set of vertices with exactly same in-neighbors
$\Rightarrow$ Every resolving set contains $\geq |S| - 1$ vertices in S.



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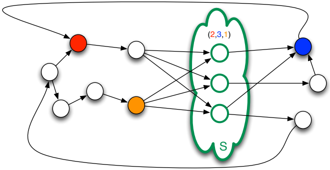

Upper bound : $D = (V, A)$ be any strong oriented graph

G_{aux} undirected graph with vertex-set V

$\{u, v\} \in E(G_{aux}) \Leftrightarrow N_D^-(u) \cap N_D^-(v) \neq \emptyset$ (intersecting in-neighborhoods).

Lemma : Every (non-empty) vertex cover of G_{aux} is a resolving set for D

Application : If G_{aux} has max. degree Δ', then $\chi(G_{aux}) \leq \Delta' + 1$ (chromatic number), so $\alpha(G_{aux}) \leq \frac{\Delta'}{\Delta'+1} n$, and so $MD(D) \leq \frac{\Delta'}{\Delta'+1} n$ for any strong orientation D of G.

Lower Bounds for $\mathcal{G}_\Delta$

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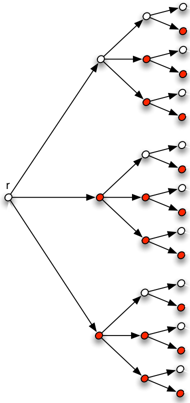

Maximum degree $\Delta = d + 1 \geq 3$

- "Complete" d-ary tree depth k
  (Force a "large" resolving set)

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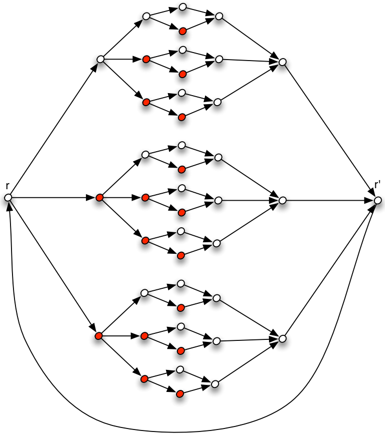

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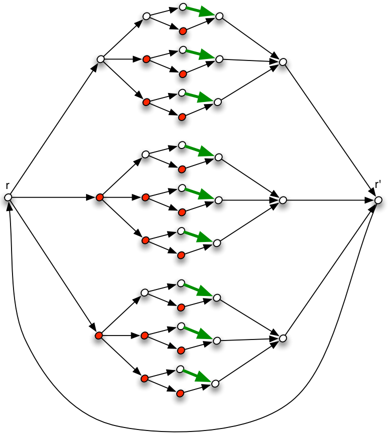

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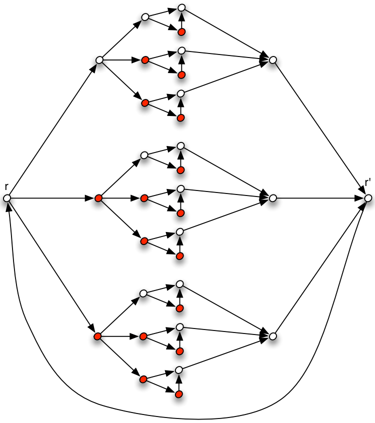

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Do the maths :

$$\lim_{k \to \infty} \frac{MD}{|V|} \geq \frac{2}{5} \text{ for } \Delta = 3$$
$$\lim_{k \to \infty} \frac{MD}{|V|} \geq \frac{1}{2} \text{ for } \Delta = 4$$
$$\lim_{\Delta \to \infty} \frac{MD}{|V|} = 1$$

Lower Bounds for Grids and Tori

Lower bound : S set of vertices with exactly same in-neighbors
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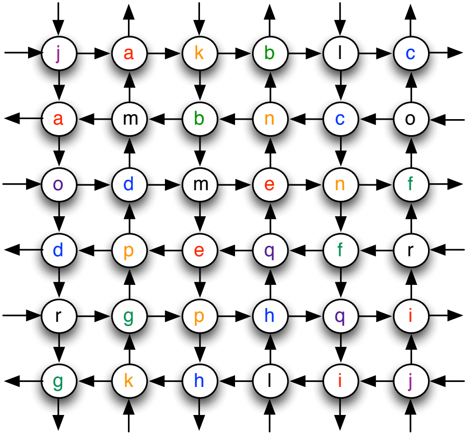
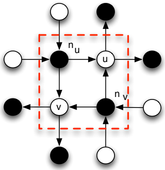
Lemma : For $\mathcal{G} \in \{Grids, Tori\}$, $WOMD(\mathcal{G}) \geq \frac{1}{2}$



Upper Bound for Tori : ad-hoc proof

Thm : Eulerian orientation $\vec{T}$ of the torus $(d^+ = d^- = 2) \Rightarrow MD(\vec{T}) \leq n/2$

Start with a MIS X (in **black**), local modifications till resolving set of same size.



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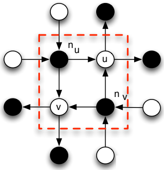

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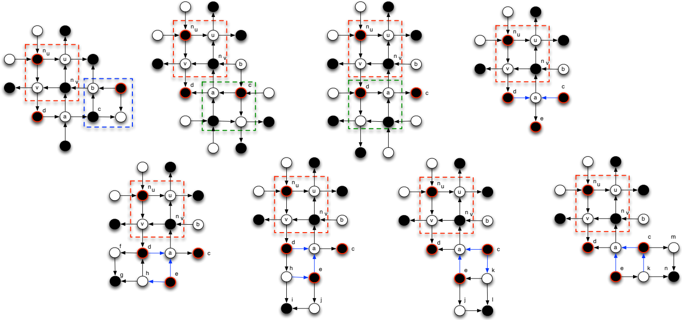
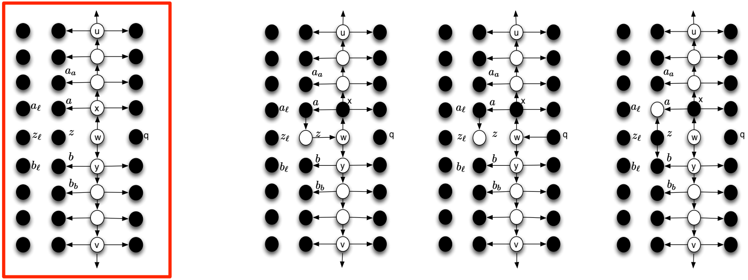
Replace n_v by u in X, sequentially in all "bad squares", makes X a resolving set (proof by case analysis).



Upper Bound for Grid : another ad-hoc proof :(

Thm : Any orientation $\vec{G}$ of a grid $\Rightarrow MD(\vec{G}) \leq 2n/3$

Start with a set X (in **black**), local modifications till resolving set of same size.



Again, proof by case analysis...

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This is a preliminary work.

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