Localization in graphs and sequential metric dimension

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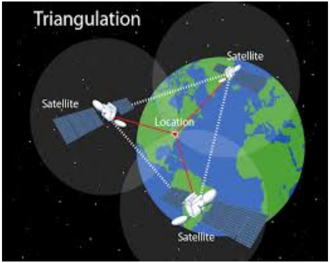
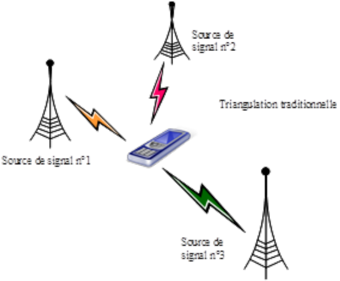
Xidian University, Xi'an, September 5th, 2019

Outline

1. Metric dimension
2. Sequential localization of an **immobile** target
3. Metric dimension in oriented graphs

Metric Dimension of graphs





Precisely locate using few information

Fix any three points A, B and C in the plane. For any point v, $(dist(A, v), dist(B, v), dist(C, v))$ is sufficient to locate v ! !

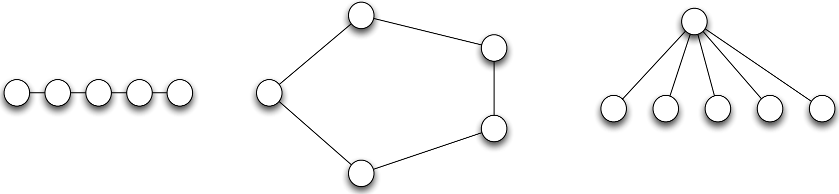
How to generalize to graph metric ?

Metric Dimension of graphs

A target is **hidden** at some (**unknown**) vertex t of a graph $G = (V, E)$
Probing a vertex $v \in V(G) \Rightarrow$ the distance $dist_G(t, v)$ between v and t.

Resolving set : set of vertices to probe s.t. the target is uniquely located

Set $R = \{v_1, \cdots, v_i\} \subseteq V$ s.t. $(dist_G(v, v_i))_{j \leq i}$ pairwise distinct $\forall v \in V$.



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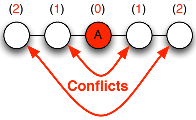

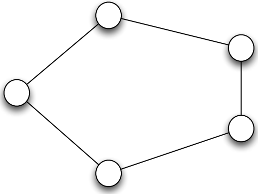

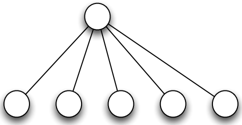

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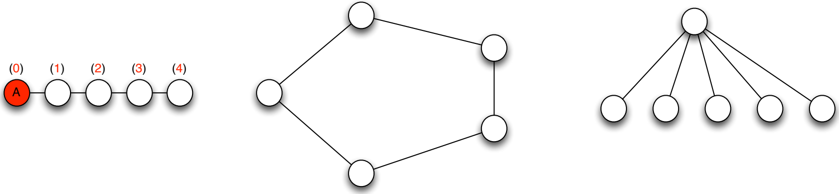

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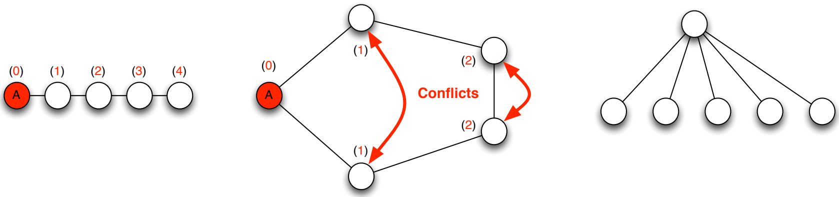

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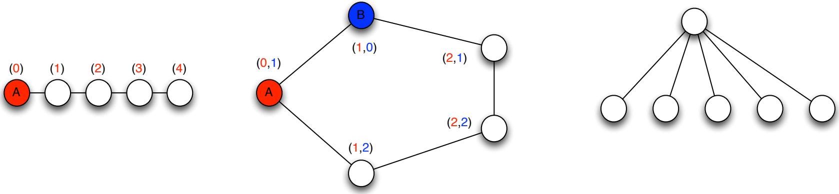

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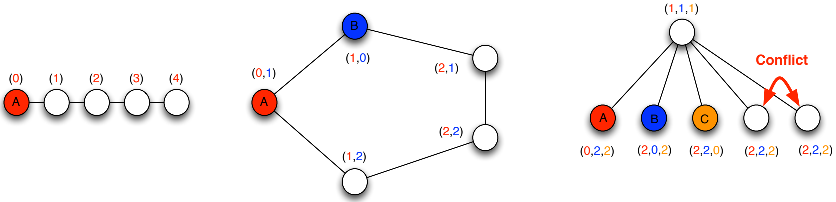

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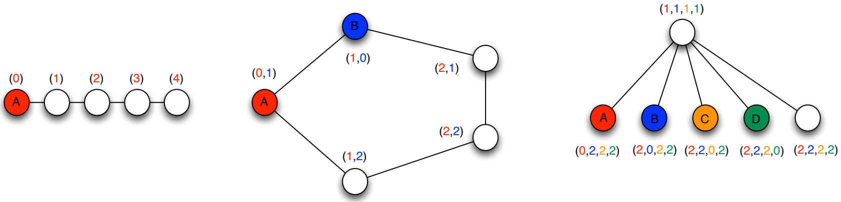

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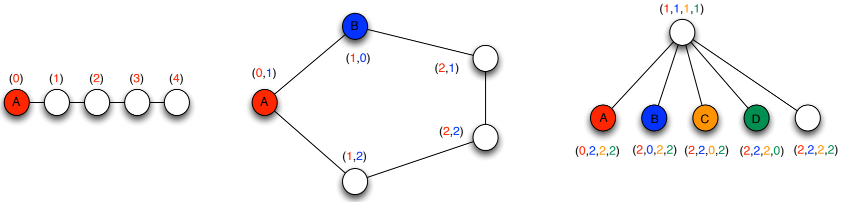

Metric Dimension $MD(G)$: min. size of a resolving set in G. $(MD(G) \leq |V|)$

example : for any tree T, $MD(T) = \#leaves - \#$"branching nodes"

Computing $MD(G)$ [Harary, Melter 76, Slater 75] is NP-c in planar graphs [Díaz *et al.* 17], W[2]-hard [Hartung,Nichterlein 13], FPT in tree-length [Belmonte *et al.* 17]...

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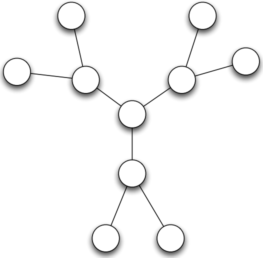
Sequential Metric Dimension

Sequentiel variant : Seager (2013) : Probe only ONE vertex per turn.



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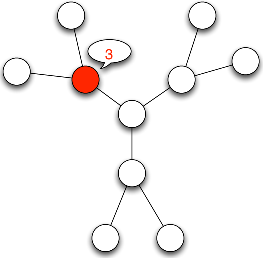

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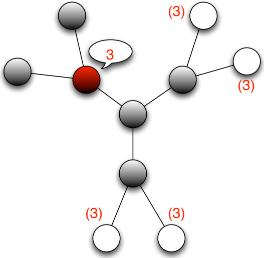

Each turn brings some new information

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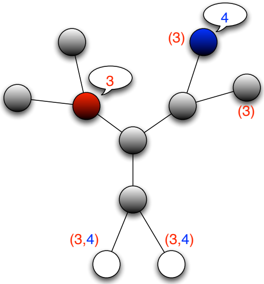

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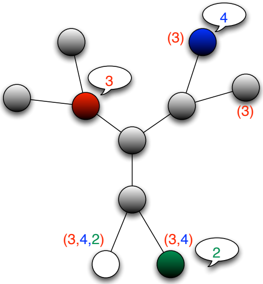

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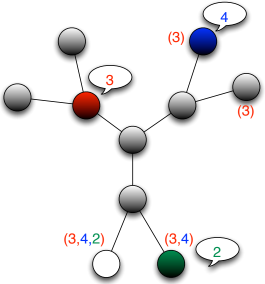

Target found in $< n$ turns in any n-node graph :

Probe each vertex (but one) one by one

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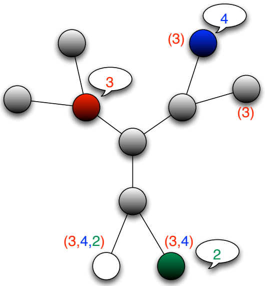

Target found in $< n$ turns in any n-node graph :

Probe each vertex (but one) one by one

Goal : Minimize # of turns to locate an **immobile** target hidden in G.

Sequential Metric Dimension & Game of Guess Who ?

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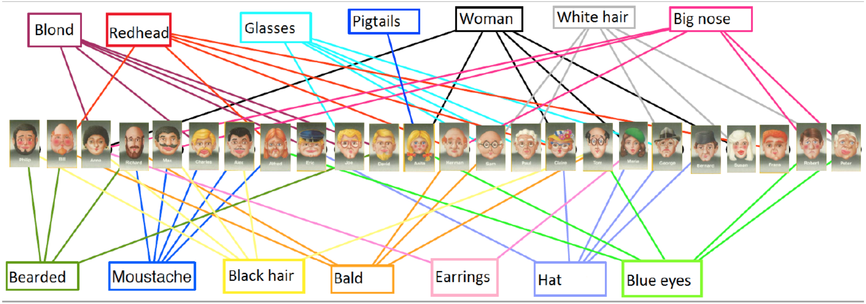

Note : One universal vertex is not depicted on the figure

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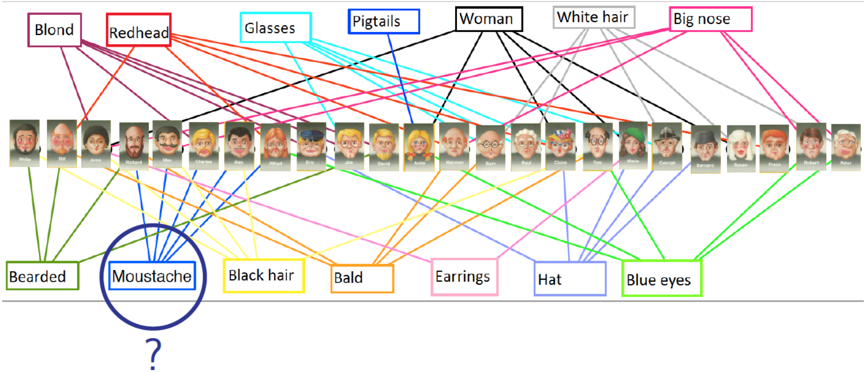

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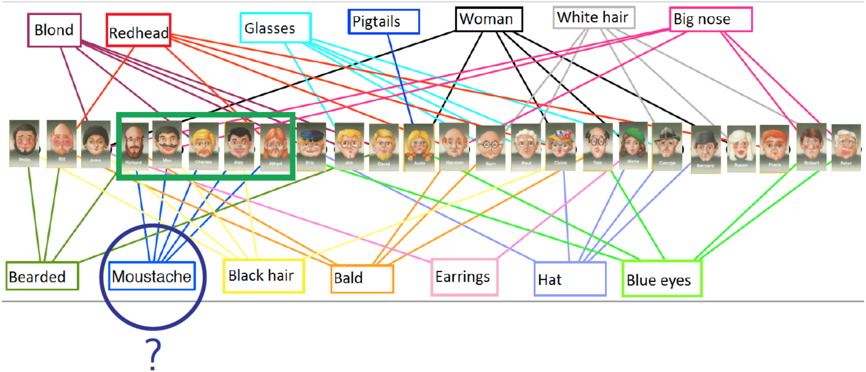

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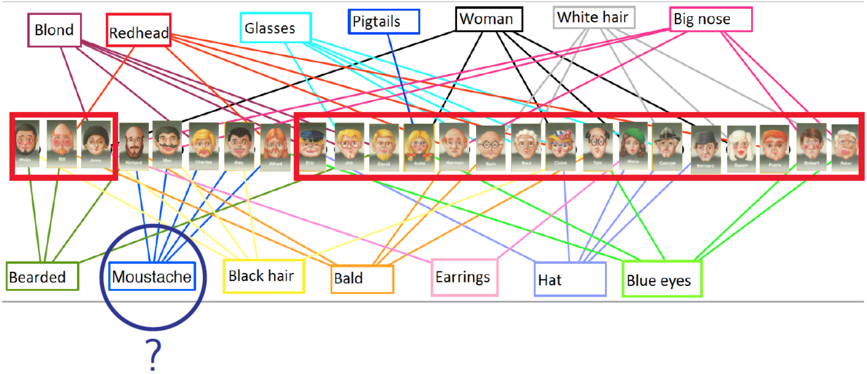

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What if more than one vertex can be probed per turn ?

Sequential Metric Dimension of G

Given k, ℓ, G, is it possible to locate the **immobile** target in G in at most ℓ turns by probing at most $k \geq 1$ vertices each turn.

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Remark : for any $G, k \geq 1$, $\lambda_k(G) \leq \lceil \frac{MD(G)}{k} \rceil$

(at each turn, probe k vertices of an optimal resolving set)

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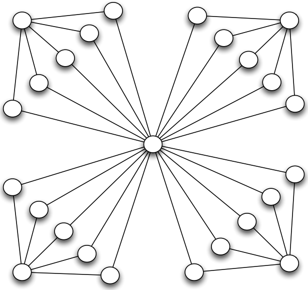

Metric Dimension $MD(G) = ?$

Sequential Metric Dimension

[Bensmail, Mazauric, Mc Inerney, N., Pérennes, 2018]

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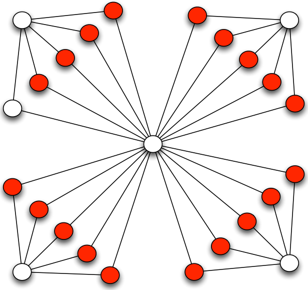

Metric Dimension $MD(G) = 19$

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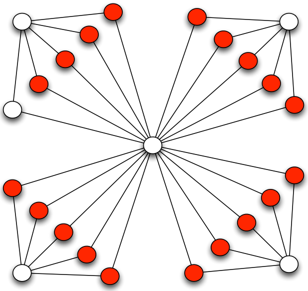

Metric Dimension $MD(G) = 19$

$\lambda_4(G) \leq \lceil \frac{19}{4} \rceil = 5.$

But...

Sequential Metric Dimension [Bensmail, Mazauric, Mc Inerney, N., Pérennes, 2018]

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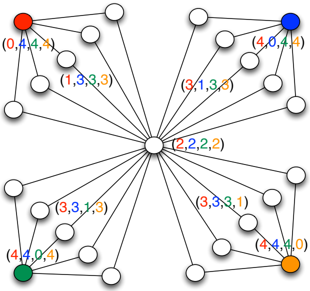

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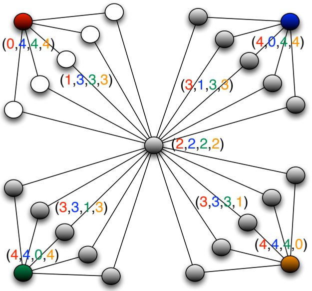

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In one turn, only five locations remain possible

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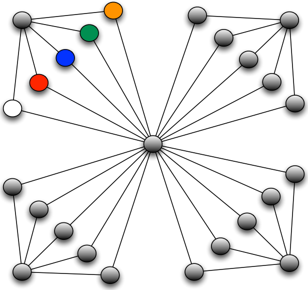

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But...
$\lambda_4(G) = 2 < \lceil \frac{19}{4} \rceil$.

[Bensmail, Mazauric, Mc Inerney, N., Pérennes, 2018]

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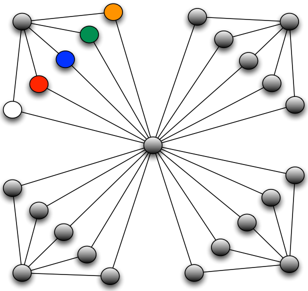

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$\lambda_4(G) = 2 < \lceil \frac{19}{4} \rceil$.

Lemma : for any $k \geq 1$

$\lambda_k(G)$ may be arbitrary smaller than $\lceil \frac{MD(G)}{k} \rceil$

Sequential Metric Dimension [Bensmail, Mazauric, Mc Inerney, N., Pérennes, 2018]

$\lambda_k(G)$: min. # turns to locate an immobile target, probing k vertices per turn.

Our contribution in general graphs :

Computational complexity *[Bensmail et al. 2018]*

- Let $k \geq 1$ be a **fixed** integer.

  The problem that takes any graph G with **diameter** 2 and an integer $\ell \geq 1$ as inputs and decides if $\lambda_k(G) \leq \ell$ is NP-complete.

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Polynomial-time algorithm *[Bensmail et al. 2018]*

- Let $k, \ell \geq 1$ be two **fixed** integers. The problem of deciding if $\lambda_k(G) \leq \ell$, for any n-node graph G, can be solved in time $n^{O(k\ell)}$.

Sequential Metric Dimension
[Bensmail, Mazauric, Mc Inerney, N., Pérennes, 2018]

$\lambda_k(G)$: min. # turns to locate an immobile target, probing k vertices per turn.

Our contribution in TREES :

Computational complexity
[Bensmail et al. 2018]

- The problem that takes any tree T and two integers $k, \ell \geq 1$ as inputs and decides if $\lambda_k(T) \leq \ell$ is NP-complete.

Polynomial-time algorithms
[Bensmail et al. 2018]

- There exists a polynomial-time $+1$-approximation to compute $\lambda_k(T)$.

  Precisely, there exists an algorithm that computes, in time $O(n \log n)$, a localization strategy using ℓ turns, probing k vertices per turn in any n-node tree T, with $\lambda_k(T) \leq \ell \leq \lambda_k(T) + 1$.
- Let $k \geq 1$ be a **fixed** integer. The problem of deciding if $\lambda_k(T) \leq \ell$, for any n-node tree T and any $\ell \geq 1$, can be solved in time $O(n^{k+2} \log n)$.

Deciding $\lambda_k(T)$ NP-hard in trees [Bensmail, Mazauric, Mc Inerney, N., Pérennes, 2018]

Sketch : Reduction from `Hitting Set` : Given $k \geq 1$, a set $\mathcal{B} = \{b_1, \cdots, b_n\}$ of elements and a set of subsets $\mathcal{S} = \{S_1, \cdots, S_m\} \subseteq 2^{\mathcal{B}}$ with $|S_i| = \sigma$ for any $1 \leq i \leq m$, is there a set $F \subseteq \mathcal{B}$ with $F \cap S_i$ for all $i \leq m$ and $|F| \leq k$?

Deciding $\lambda_k(T)$ NP-hard in trees [Bensmail, Mazauric, Mc Inerney, N., Pérennes, 2018]

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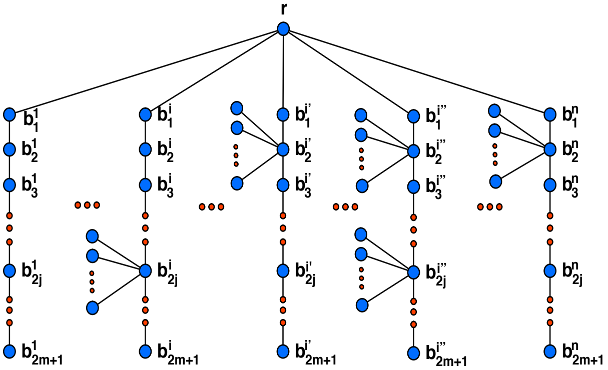
Tree T_γ (γ depends on n, k) s.t. $\lambda_k(T_\gamma) \leq 1 + \lceil \frac{\sigma-1}{k} \rceil + \lceil \frac{\gamma-1}{k} \rceil$ iff such a set F exists



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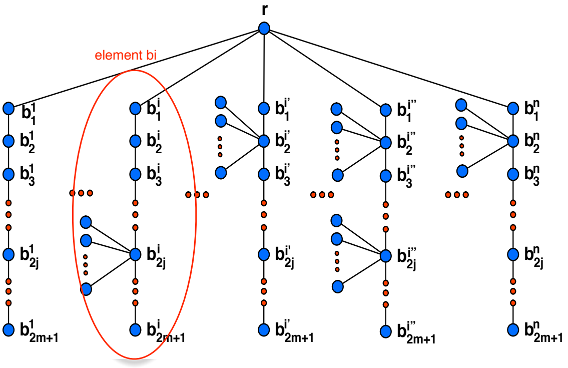

i^{th} branch $\sim$ element b_i

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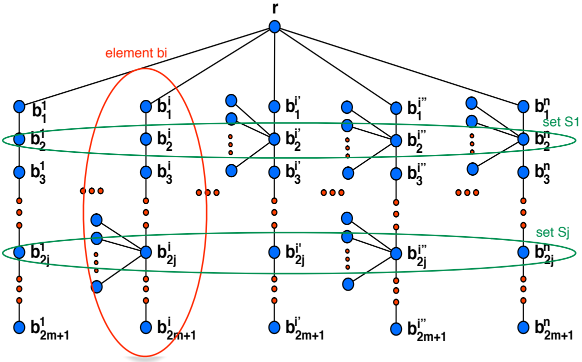


i^{th} branch $\sim$ element b_i

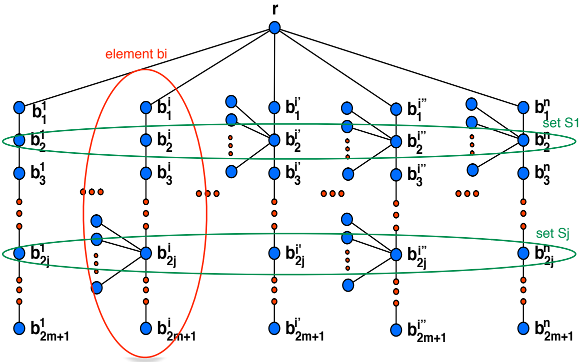
"level" $2j \sim$ set S_j

$b_i \in S_j \Leftrightarrow b^i_{2j}$ center of star (γ leaves)

e.g., $b_i, b_{i''} \in S_j$, $b_1, b_{i'}, b_n \notin S_j$

Deciding $\lambda_k(T)$ NP-hard in trees [Bensmail, Mazauric, Mc Inerney, N., Pérennes, 2018]

Sketch : Reduction from Hitting Set : Given $k \geq 1$, a set $\mathcal{B} = \{b_1, \cdots, b_n\}$ of elements and a set of subsets $\mathcal{S} = \{S_1, \cdots, S_m\} \subseteq 2^{\mathcal{B}}$ with $|S_i| = \sigma$ for any $1 \leq i \leq m$, is there a set $F \subseteq \mathcal{B}$ with $F \cap S_i$ for all $i \leq m$ and $|F| \leq k$?



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Recall that locating in "big" stars is "long"

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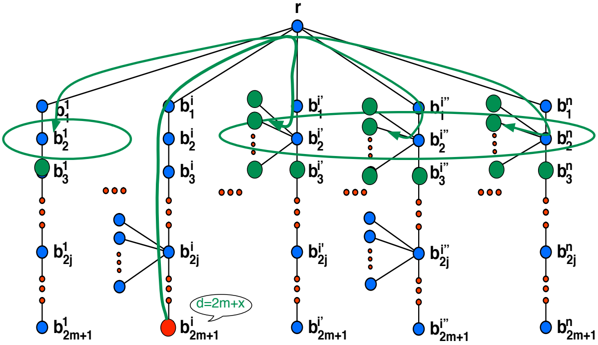

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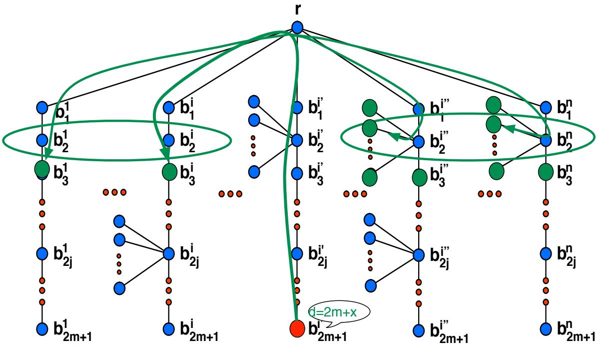

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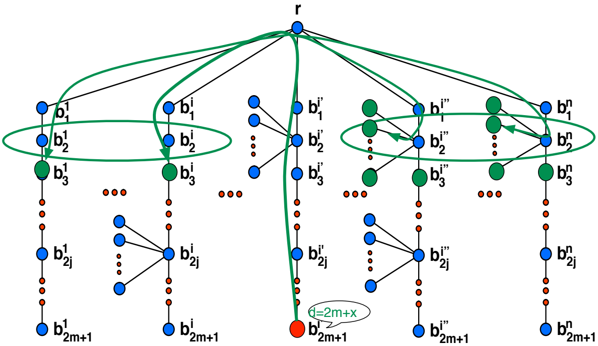

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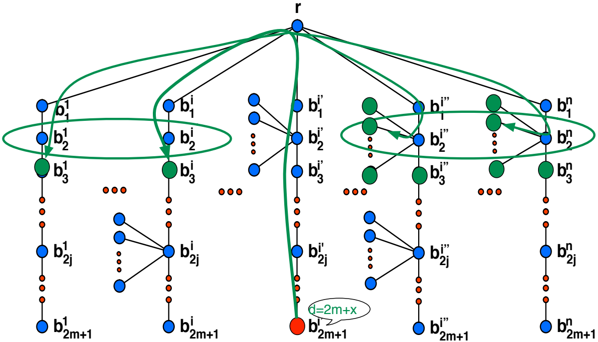

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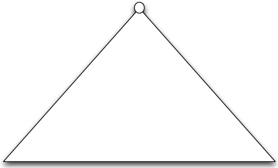
The difficulty only comes from the FIRST turn!

After the first turn, it becomes “easy”.

Indeed : probe any vertex



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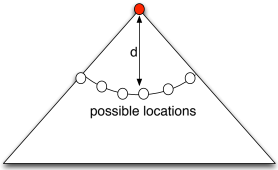

+1-approximation in trees

[Bensmail, Mazauric, Mc Inerney, N., Pérennes, 2018]

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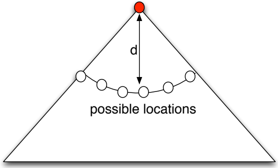
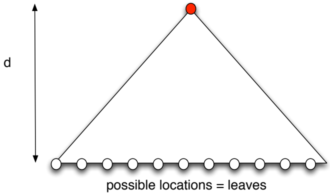
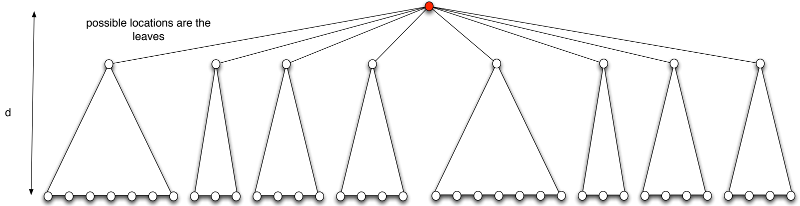


So, after this first turn, the instance becomes :
A rooted tree, with all leaves at same distance from the root, and the target on some leaf.



Theorem : In a rooted tree with all leaves at same distance from the root and the target at some leaf, an optimal strategy can be computed in time $O(n \log n)$

(1) Find in which subtree the target hide (2) recursively search in this subtree.



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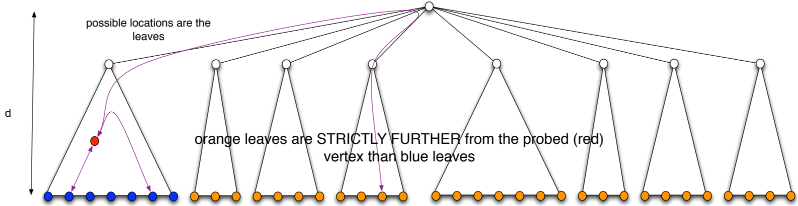

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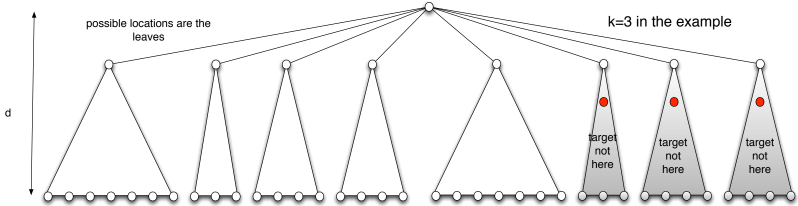

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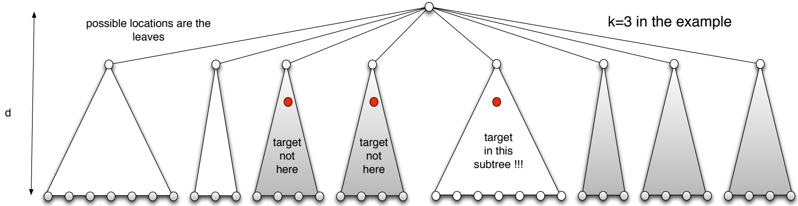

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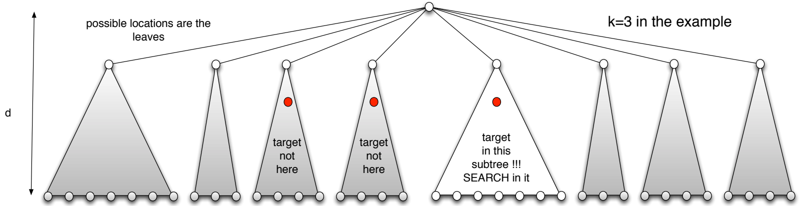

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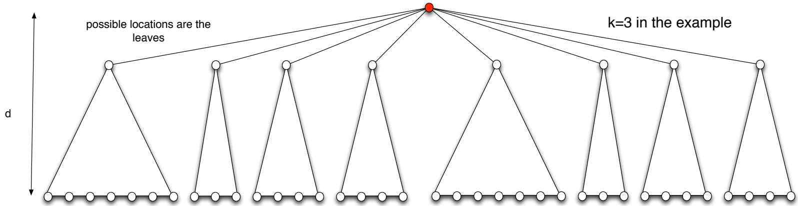

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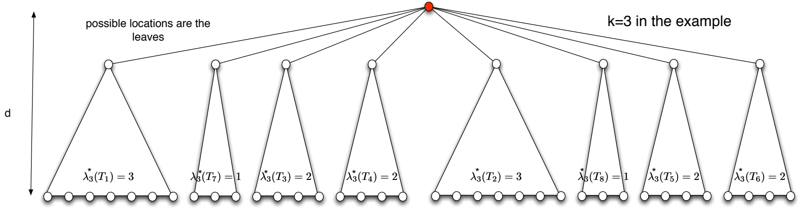

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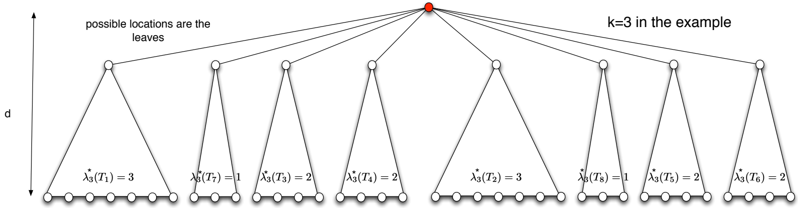

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Question : In which order to probe the subtrees ? $\Rightarrow$ by non-increasing order of their λ^*_k : probe first the subtrees that are long to search

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(1) Find in which subtree the target hide (2) recursively search in this subtree.

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Subtelty : what is the difference between these two graphs ?

possible locations are the leaves

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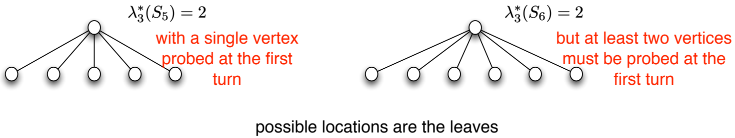

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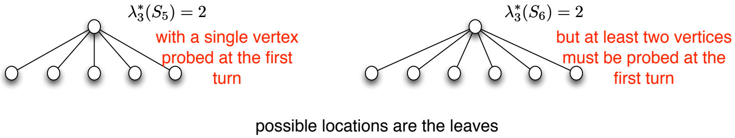

⇒ During Phase (1) : probing more than one vertex in some subtrees may "accelerate" the search in these subtrees.

$+1$-approximation in trees

[Bensmail, Mazauric, Mc Inerney, N., Pérennes, 2018]

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Optimal approach : Probe one or some vertices per subtree (in non-increasing order of their λ^*_k) until finding the "good" one.

The number of vertices to probe in each subtree in Phase (1) is decided by the (a bit) technical part of our Dynamic Programming algorithm.

Further work on sequential metric dimension

Open questions

- Can we compute $\lambda_k(T)$ in FPT time in trees (i.e., in time $f(k) * poly(n)$)?
- Complexity of deciding if $\lambda_k(G) \leq \ell$ in other graph classes (interval graphs, chordal graphs, bounded treewidth, etc.)

Other variant : when the target can move

At each turn, after k vertices have been probed, the target may move to a neighbor of its current position

- Already many results on this variant
- but also many open questions

...sorry, no time to go further on it

Outline

Metric Dimension in Oriented Graphs

Orientation of G : each edge $\{u, v\}$ becomes exactly one arc among uv or vu.

Probing a vertex $v \in V(G) \Rightarrow$ the distance $dist_G(v, t)$ FROM v TO t.

Resolving set : set of vertices to probe s.t. the target is uniquely located

Set $R = \{v_1, \cdots, v_i\} \subseteq V$ s.t. $(dist_D(v_i, v))_{j \leq i}$ pairwise distinct $\forall v \in V$.

Remark : *A priori*, $dist_D(v, t)$ may be ∞.

$\rightarrow$ ONLY strongly connected orientations.

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$MD(D)$: min. size of a resolving set in a **strong** oriented graph D.

Few related work

- upper bounds *[Chartrand et al 00]*
- NP-complete in strong oriented graphs *[Rajan et al. 14]*
- complete graphs *[Lozano 13]*, Cayley digraphs *[Fehr et al. 06]*, de Bruijn and Kautz *[Rajan et al. 14]*

Worst/Best Oriented Metric Dimension (WOMD/BOMD)

Given a class $\mathcal{G}$ of undirected n-node graphs,

- $WOMD(\mathcal{G}) = \sup_{D \text{ strong orientation of } G \in \mathcal{G}} \frac{MD(D)}{n}$
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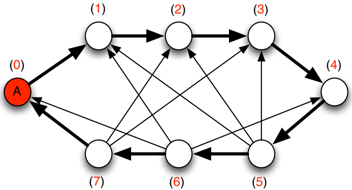
Very few previous work

[Chartrand et al. 01]

- tournaments : $WOMD(K_n) = 1/2$
- $\mathcal{H}_n$, class of Hamiltonian n-node graphs : $BOMD(\mathcal{H}_n) = 1/n$.

Every Hamiltonian graph has an orientation D with $MD(D) = 1$.



Our contributions

Focus on Worst Orientations (WOMD) for various graph classes.

$\mathcal{G}_\Delta$: class of graphs with maximum degree $\leq \Delta$.

- $\frac{2}{5} \leq WOMD(\mathcal{G}_3) \leq \frac{1}{2}$
- $\frac{1}{2} \leq WOMD(\mathcal{G}_4) \leq \frac{6}{7}$
- $\lim_{\Delta \to \infty} WOMD(\mathcal{G}_\Delta) = 1$

***Grids* : class of cartesian grids.**

- $\frac{1}{2} \leq WOMD(Grids) \leq \frac{2}{3}$

$WOMD^*$ defined as $WOMD$ but over Eulerian orientations (in-degree=out-degree).

***Tori* : class of cartesian tori.**

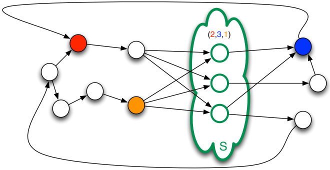
- $WOMD^*(Tori) = \frac{1}{2}$

Easy lemmas but very useful

Lower bound

S set of vertices with exactly same in-neighbors

$\Rightarrow$ Every resolving set contains $\geq |S| - 1$ vertices in S.



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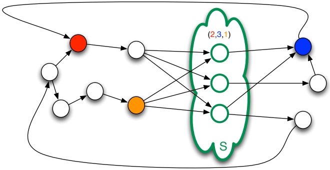

Upper bound : $D = (V, A)$ be any strong oriented graph

G_{aux} undirected graph with vertex-set V

$\{u, v\} \in E(G_{aux}) \Leftrightarrow N_D^-(u) \cap N_D^-(v) \neq \emptyset$ (intersecting in-neighborhoods).

Lemma : Every (non-empty) vertex cover of G_{aux} is a resolving set for D

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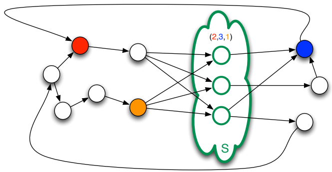

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Application : If G_{aux} has max. degree Δ', then $\chi(G_{aux}) \leq \Delta' + 1$ (chromatic number), so $\alpha(G_{aux}) \leq \frac{\Delta'}{\Delta'+1} n$, and so $MD(D) \leq \frac{\Delta'}{\Delta'+1} n$ for every strong orientation D of G.

Lower Bounds for $\mathcal{G}_\Delta$

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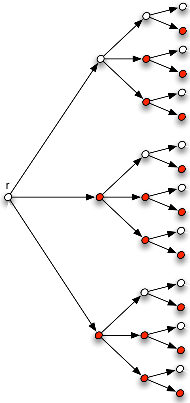

Maximum degree $\Delta = d + 1 \geq 3$

- "Complete" d-ary tree depth k (Force a "large" resolving set)

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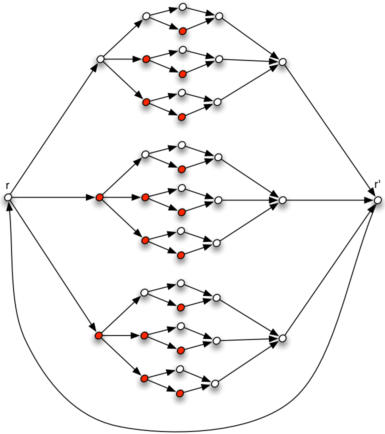

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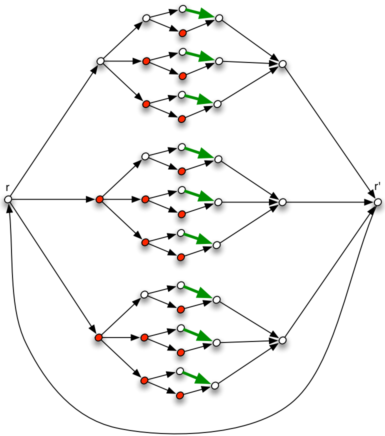

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- Contract green edges
  (reduce the size, preserving resolving set)

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Lower bound : S set of vertices with exactly same in-neighbors
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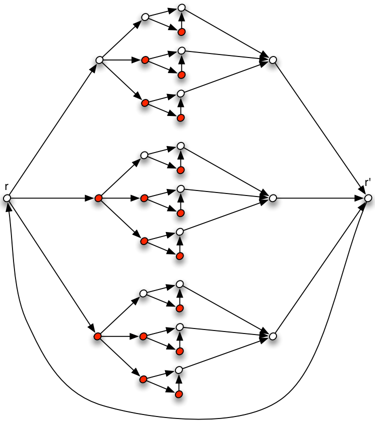


Maximum degree $\Delta = d + 1 \geq 3$

- "Complete" d-ary tree depth k (Force a "large" resolving set)
- Add a "reversed" complete d-ary tree depth $k - 1$

  (ensure strong connectedness)
- Contract green edges

  (reduce the size, preserving resolving set)

Do the maths :

$$\lim_{k \to \infty} \frac{MD}{|V|} \geq \frac{2}{5} \text{ for } \Delta = 3$$

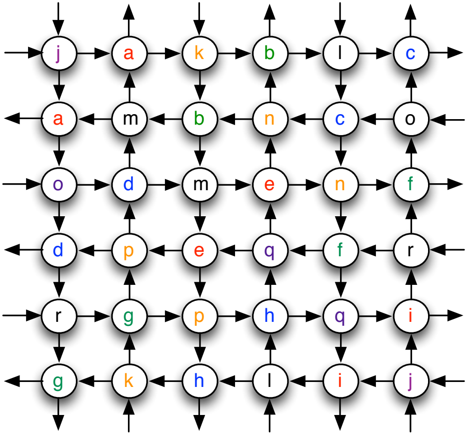
$$\lim_{k \to \infty} \frac{MD}{|V|} \geq \frac{1}{2} \text{ for } \Delta = 4$$

$$\lim_{\Delta \to \infty} \frac{MD}{|V|} = 1$$

Lower Bounds for Grids and Tori

Lower bound : S set of vertices with exactly same in-neighbors
$\Rightarrow$ Every resolving set contains $\geq |S| - 1$ vertices in S.

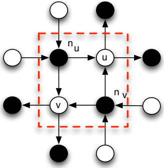
Lemma : For $\mathcal{G} \in \{Grids, Tori\}$, $WOMD(\mathcal{G}) \geq \frac{1}{2}$



Upper Bound for Tori : ad-hoc proof

Thm : Eulerian orientation $\vec{T}$ of the torus $(d^+ = d^- = 2) \Rightarrow MD(\vec{T}) \leq n/2$

Start with a MIS X (in **black**), local modifications till resolving set of same size.



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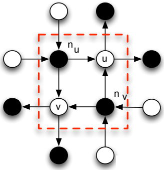

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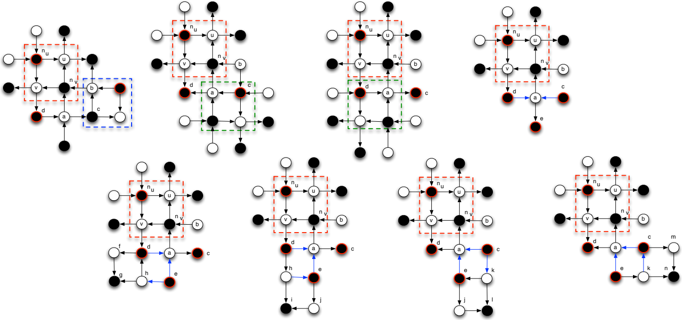
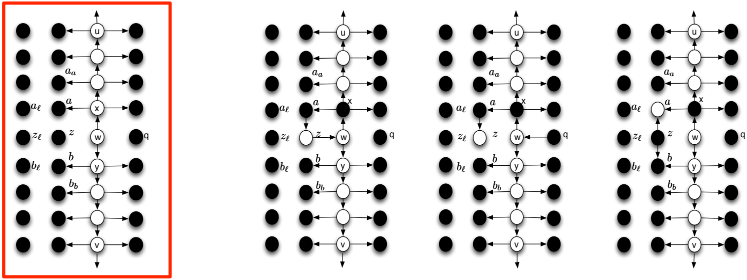
Replace n_v by u in X, sequentially in all "bad squares", makes X a resolving set (proof by case analysis).



Upper Bound for Grid : another ad-hoc proof :(

Thm : Any orientation $\vec{G}$ of a grid $\Rightarrow MD(\vec{G}) \leq 2n/3$

Start with a set X (in **black**), local modifications till resolving set of same size.



Again, proof by case analysis...

Further work on Directed Metric Dimension

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Thank you, 谢谢 , Merci !