

Connected Treewidth and Connected Graph Searching

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Graph Searching

Goal

In a contaminated network,

- an invisible omniscient arbitrary fast **fugitive** ;
- a team of **searchers** ;

We want to find a **strategy** that catch the fugitive **using the fewest searchers as possible.**

Motivations

- network security, speleological rescue...
- game related to well known graphs' parameters :
treewidth and **pathwidth** ;

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Search Strategy, Parson. [GTC,1978]

Sequence of three basic operations,...

- 1 Place a searcher at a vertex of the graph ;
- 2 Move a searcher along an edge of the graph ;
- 3 Remove a searcher from a vertex of the graph.

... that must result in catching the fugitive

The fugitive is caught when it meets a searcher at a vertex or in an edge of the graph.

We want to minimize the number of searchers.

Let $s(G)$ be the smallest number of searchers needed to catch a fugitive in a graph G .

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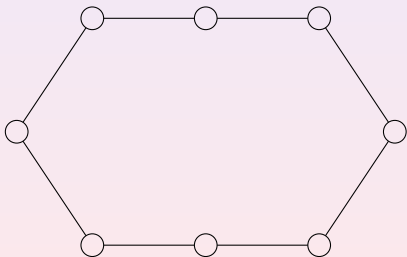
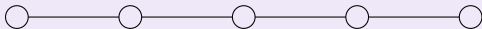
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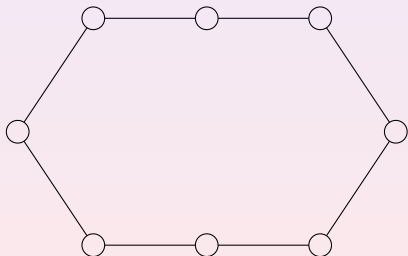
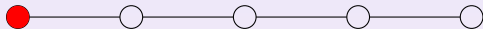
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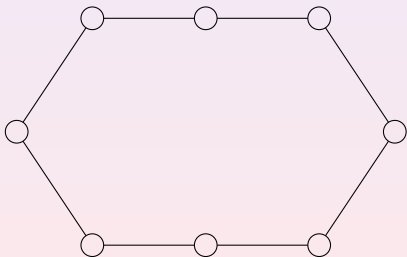
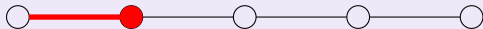
Simple Examples : Path and Ring



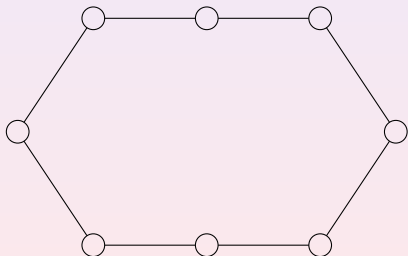
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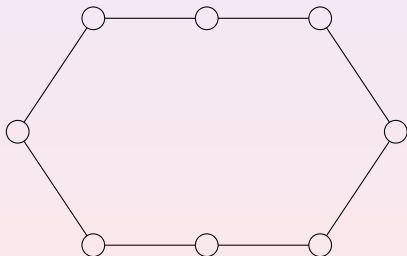
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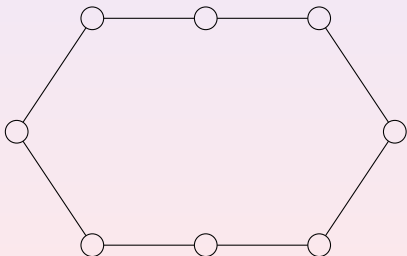
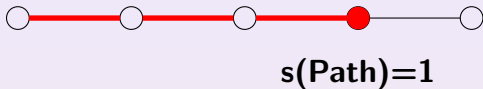
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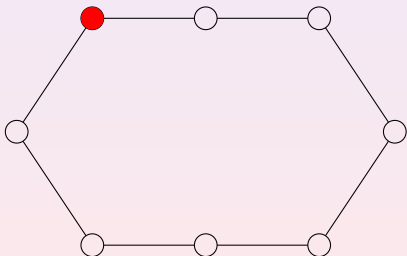
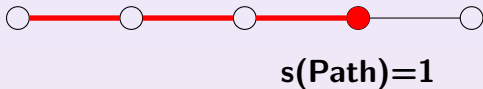
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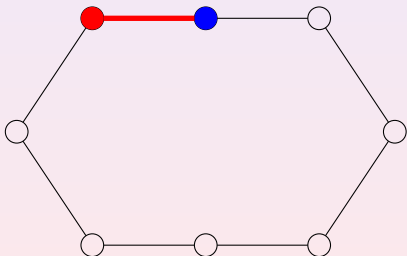
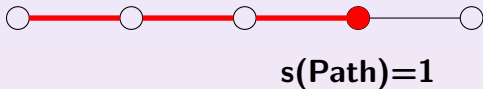
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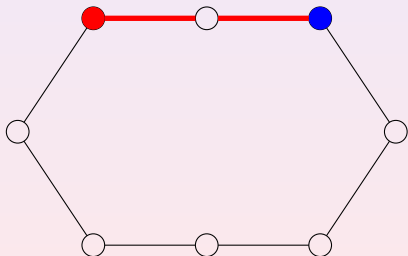
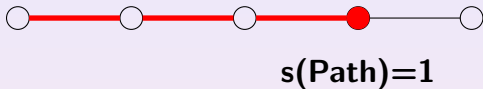
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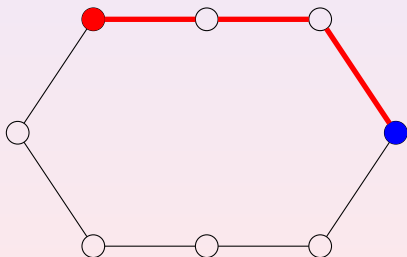
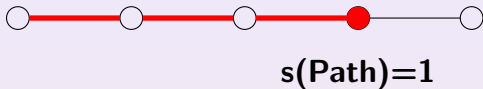
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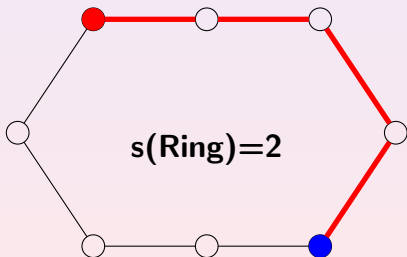
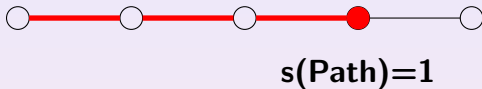
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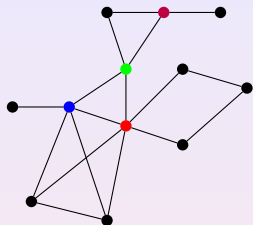
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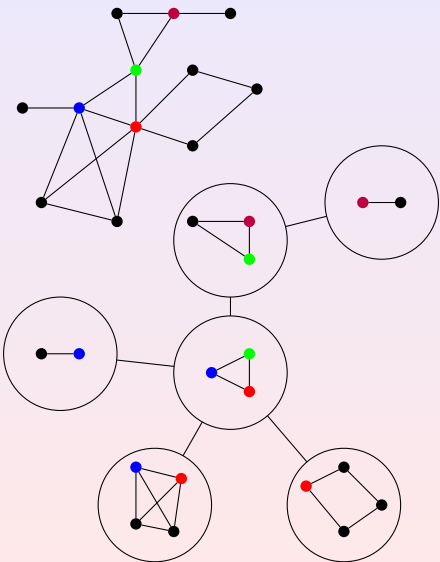


Tree and Path Decompositions

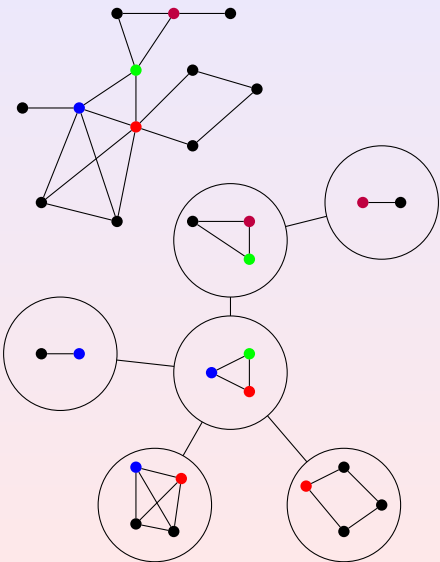


Tree and Path Decompositions

a tree T and bags $(X_t)_{t \in V(T)}$



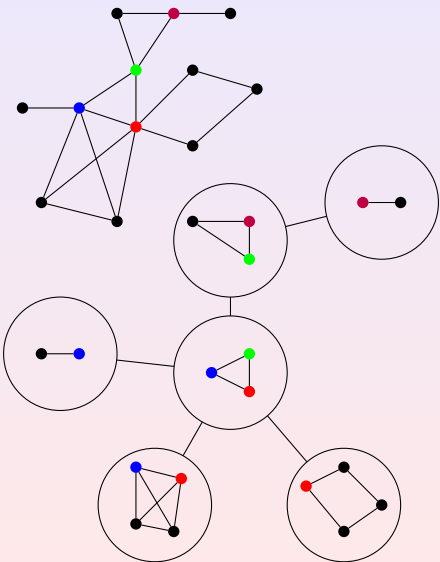
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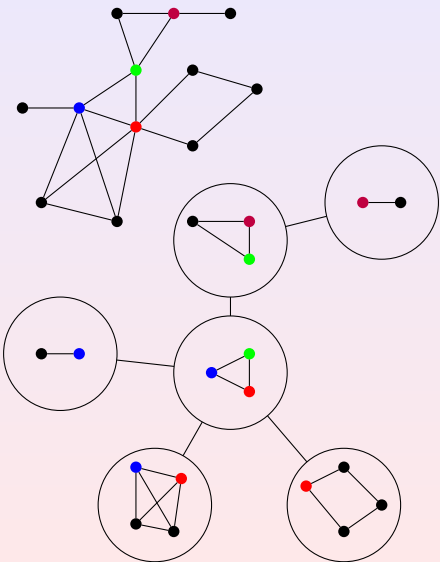
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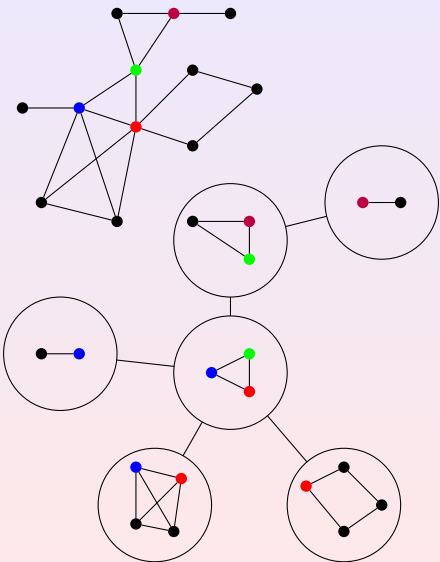


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- every **vertex** of G is at least in one bag ;
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- For any vertex of G , all bags that contain it, form a **subtree**.

Width = Size of largest Bag - 1

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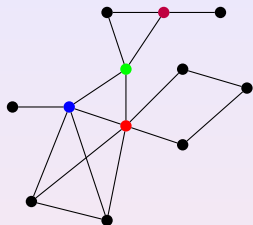
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treewidth of G

tw(G), minimum width among any tree-decomposition

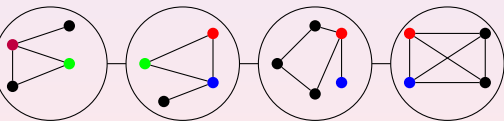
Tree and Path Decompositions



a path P and bags $(X_t)_{t \in V(P)}$

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- For any vertex of G , all bags that contain it, form a **subpath**.

Width = Size of largest Bag - 1



pathwidth of G

$\text{pw}(G)$, minimum width among any path-decomposition

Relationship between search number and pathwidth

Ellis, Sudborough and Turner. [Inf. Comput.,1994]

For any graph G , $vs(G) \leq s(G) \leq vs(G) + 2$

Kinnersley. [IPL.,1992]

For any graph G , $vs(G) = pw(G)$

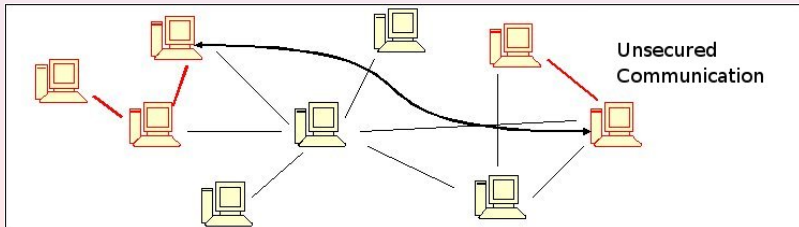
For any n -node graph G :

$pw(G) \leq s(G) \leq pw(G) + 2$

Connected Graph Searching

Limits of the Parson's model

- Searchers cannot move at will in a real network ;
- It would be better to let searchers be grouped.



Connected Graph Searching

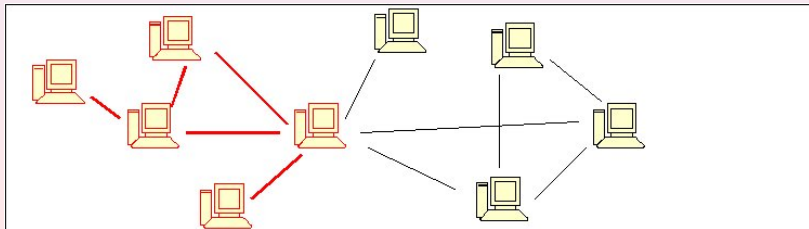
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Connected Search Strategy

At any step, the cleared part of the graph must induced a connected subgraph.

Let $cs(G)$ be the connected search number of the graph G .



Cost of connectedness : case of trees

Barrière, Flocchini, Fraigniaud and Santoro. [SPAA, 2002]

Linear Algorithm

Barrière, Fraigniaud, Santoro and Thilikos. [WG, 2003]

For any tree T , $s(T) \leq cs(T) \leq 2s(T) - 2$.

Moreover, these bounds are tight.

Cost of connectedness : case of arbitrary graphs

Seymour and Thomas. [Combinatorica, 1994]

Bond Carving

Fomin, Fraigniaud and Thilikos. [Technical report, 2004]

- Using a branch-decomposition, polynomial constructive algorithm that computes a connected search strategy.
- For any connected graph G ,
 $cs(G) \leq s(G) (2 + \log |E(G)|)$.

Connected Treewidth

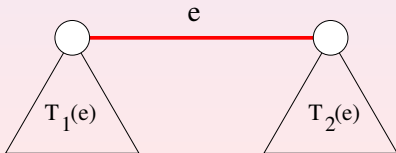
Connected e -cut of a tree-decomposition (T, X)

The edge e is said connected if both $G[T_1(e)]$ and $G[T_2(e)]$ induced connected subgraphs of G .

Connected tree-decomposition (T, X)

For any $e \in E(T)$, e is connected.

Connected treewidth, $\text{ctw}(G)$



Connected Treewidth

For any connected graph G , $ctw(G) = tw(G)$

Golumbic. Algorithmic graph theory and perfect graphs

A “clique tree” of a minimal triangulation H of a connected graph G is an optimal tree-decomposition of G .

Parra and Scheffler. [DAM 1997]

A “clique tree” of a minimal triangulation H of a connected graph G is a connected tree-decomposition of G .

Results (1)

Theorem 1 : new proof

For any connected graph G , $ctw(G) = tw(G)$

Constructive proof

Given a tree-decomposition of width $\leq k$ of a connected graph G with n vertices, our algorithm computes a connected tree-decomposition of width $\leq k$ of G , in time $O(n.k^3)$.

Results (2)

Theorem 2

For any connected graph G , $\mathbf{cs}(G) \leq \mathbf{s}(G) (1 + \log_2 |V(G)|)$.

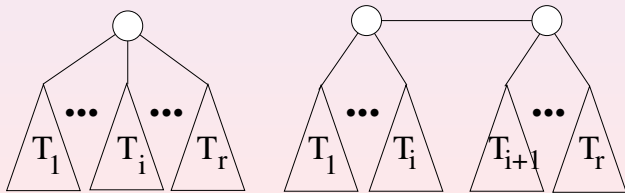
Constructive proof

Given a tree-decomposition of a graph G , our algorithm computes a connected search strategy for G , using at most $\mathbf{tw}(G) \log |V(G)|$ searchers, in polynomial time.

Squetch of the proof of Theorem 2

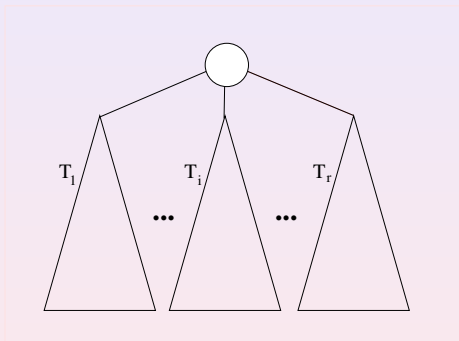
For any connected n -node graph G , $\text{cs}(G) \leq \text{s}(G) (1 + \log_2 n)$

- proof by induction on n
- **Robertson and Seymour.** Graph Minors II. Algorithmic Aspects of Tree-Width. J. of Alg 7, 1986.
 - For any tree-decomposition (T, X) of a n -node graph G , there are one (or two adjacent vertices) of T such that :
 - for any $1 \leq j \leq r$, $|G[T_j]| \leq n/2$



Squetch of the proof of Theorem 2

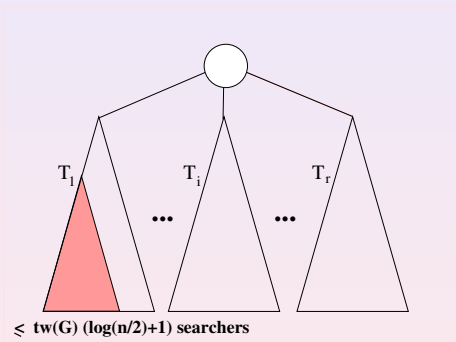
Starting from a connected tree-decomposition of G



For any $1 \leq i \leq r$, $G[T_i]$ is a connected subgraph with at most $n/2$ vertices.

Squetch of the proof of Theorem 2

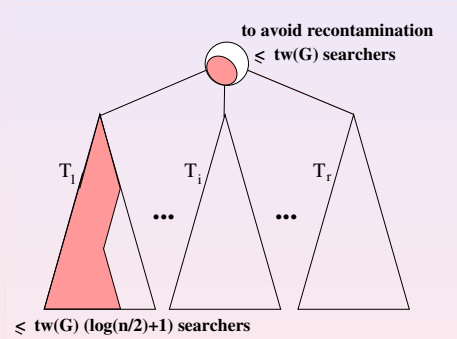
Starting from a connected tree-decomposition of G



There is a connected search strategy for $G[T_1]$, using at most $\text{tw}(G)(\log(n/2) + 1)$ searchers.

Squetch of the proof of Theorem 2

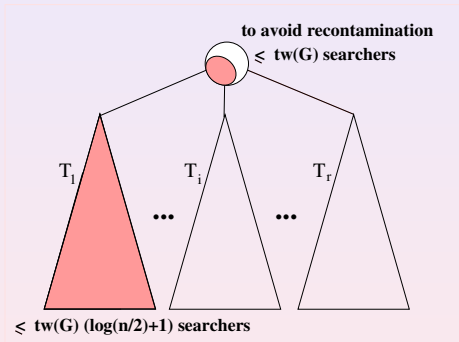
Starting from a connected tree-decomposition of G



At most $\text{tw}(G)$ searchers are required to protect $G[T_1]$ from recontamination from the remaining part of G .

Squetch of the proof of Theorem 2

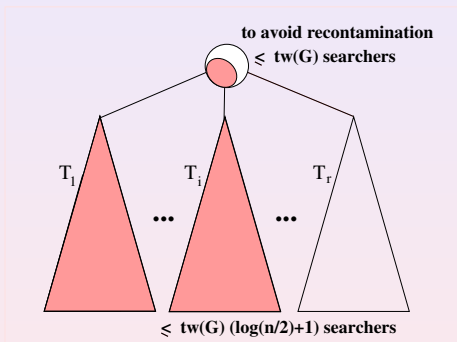
Starting from a connected tree-decomposition of G



Then we can terminate the clearing of $G[T_1]$.

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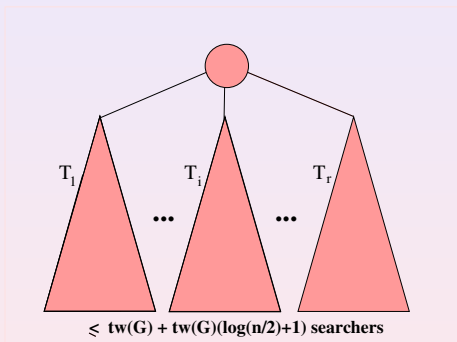
Starting from a connected tree-decomposition of G



Then we can use our $\text{tw}(G)(\log(n/2) + 1)$ searchers to clear another subgraph $G[T_i]$, and so on...

Squetch of the proof of Theorem 2

Starting from a connected tree-decomposition of G



Connected search strategy using at most $tw(G)(\log n + 1)$ searchers. Thus, $cs(G) \leq s(G)(\log n + 1)$

Conclusion and Further Work

Cost of connectedness

- new upper bound of the ratio $\mathbf{cs}(G)/\mathbf{s}(G)$
- constructive algorithm

Open problems

- What is the optimal bound ?
In trees : $\mathbf{cs}(T)/\mathbf{s}(T) \leq 2$ and this bound is tight
[Barrière et al.].
If the fugitive is visible : $\mathbf{cs}(G)/\mathbf{s}(G) \leq \log n$ and this
bound is tight.
- Is the problem of computing $\mathbf{cs}(G)$ NP-complete ?
It is known to be NP-hard.