A Kuratowski theorem for general surfaces

Graph minors VIII, Robertson and Seymour, JCTB 90

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Talk mainly based on *Graphs on Surfaces* [Mohar,Thomassen]

A Kuratowski theorem for general surfaces

Minor of G: subgraph of H got from G by edge-contractions.

$\mathcal{F}(S)$: set of graphs embeddable in a surface S (minor closed)

ex: S_0 the sphere, $\mathcal{F}(S_0)$: set of planar graphs

$\mathcal{O}(S)$: set of minimal obstructions of $\mathcal{F}(S)$.

$G \in \mathcal{F}(S)$ iff no graph in $\mathcal{O}(S)$ is a minor of G

Kuratowski's Theorem

A graph is planar iff it does not contain K_5 or $K_{3,3}$ as a minor.

Corollary: $\mathcal{O}(S_0)$ is finite.

Generalization to any surface [Graph Minor VIII, 90]

For any (orientable or not) surface S, $\mathcal{O}(S)$ is finite.

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For any (orientable or not) surface S, $\mathcal{O}(S)$ is finite.

"Application"

Theorem [Graph Minor XIII, 95]

Let H be a fixed graph. There is a $O(n^3)$ algorithm deciding whether a n-node graph G admits H as minor.

Corollary

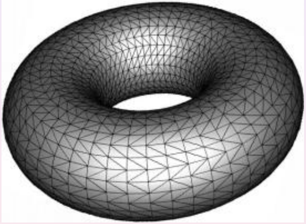
For any surface S, there is a polynomial-time algorithm deciding whether a graph $G \in \mathcal{F}(S)$.

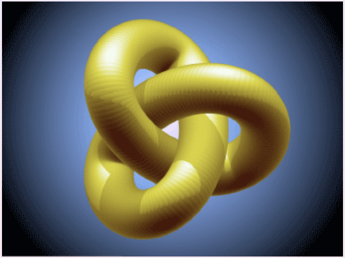
Limitations

- time-complexity: huge constant depending on $|H|$
- #obstructions: projective plan=103 [Ar81], torus ≥ 3178
- explicit obstruction set (constructive algo. [FL89])

Surfaces

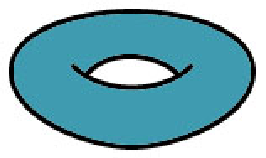
- Surface: connected compact 2-manifold.

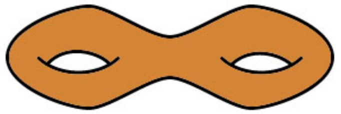


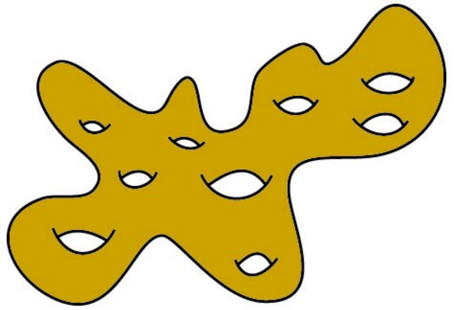


* Thanks to Ignasi for this slide and the next 4 slides

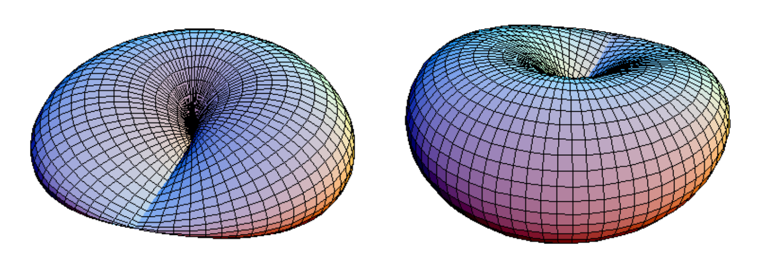
Handles







Cross-caps

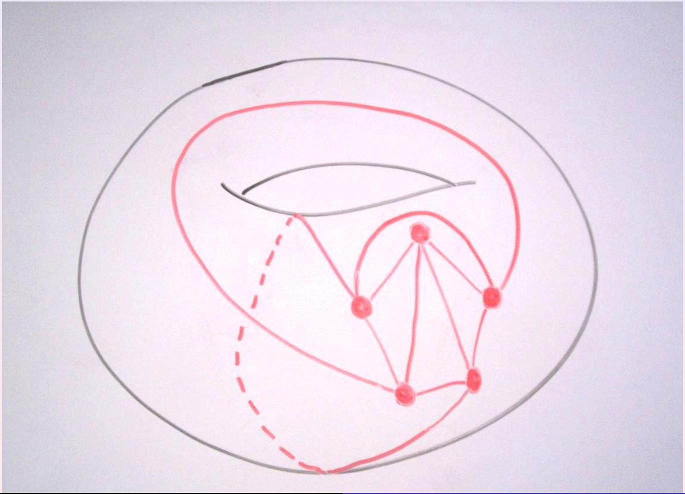


Genus of a surface

- The surface classification Theorem: any compact, connected and without boundary surface can be obtained from the sphere $\mathcal{S}^2$ by adding handles and cross-caps.
- Orientable surfaces: obtained by adding $g \geq 0$ *handles* to the sphere S_0, obtaining the g-torus S_g with Euler genus $eg(S_g) = 2g$.
- Non-orientable surfaces: obtained by adding $h > 0$ *cross-caps* to the sphere S_0, obtaining a non-orientable surface $\mathbb{P}_h$ with Euler genus $eg(\mathbb{P}_h) = h$.

Graphs on surfaces

- An embedding of a graph G on a surface Σ is a drawing of G on Σ without edge crossings.



Graphs on surfaces

- An embedding of a graph G on a surface Σ is a drawing of G on Σ without edge crossings.
- An embedding defines vertices, edges, and faces.

Euler Formula: $|V| - |E| + |F| = 2 - eg$

- The Euler genus of a graph G, $eg(G)$, is the least Euler genus of the surfaces in which G can be embedded.

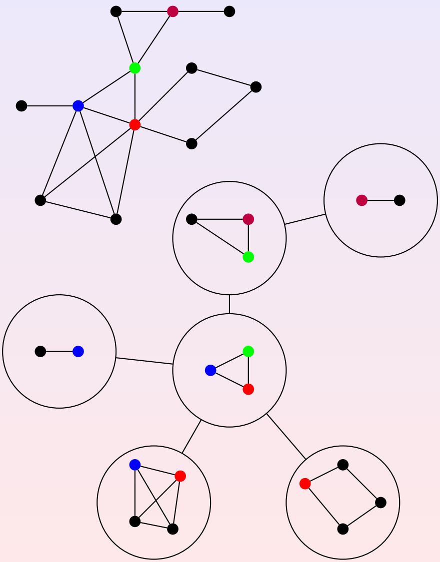
Some usefull relations

G' connected subgraph of G and Π embedding of G:
$genus(G', \Pi) \leq genus(G, \Pi)$

v a cut-vertex of $G = G_1 \cup G_2$ with $G_1 \cap G_2 = \{v\}$ and G_2 non planar. Then, $genus(G) > genus(G_1)$.

G_1, G_2 disjoint connected graphs and xy edge of G_2. Let G obtained from $G_1 \cup G_2$ by deleting xy and adding an edge from x to G_1 and from y to G_1.
If G_2 non planar, then, $genus(G) > genus(G_1)$.

Tree Decomposition of a graph G



a tree T and bags $(X_t)_{t \in V(T)}$

- every **vertex** of G is at least in one bag;
- both ends of an **edge** of G are at least in one bag;
- Given a vertex of G, all bags that contain it, form a **subtree**.

Width = Size of larger Bag -1

Treewidth
tw(G), minimum width among any tree decomposition

Any bag is a separator

A Kuratowski theorem for orientable surfaces

We focus on orientable surfaces.
genus(G): minimum genus of an orientable embedding of G.

$\mathcal{F}_g$: the set of graphs with genus $\leq g$ (minor closed)

ex: $\mathcal{F}_0$: set of planar graphs

$\mathcal{O}_g$: the set of minimal obstructions of $\mathcal{F}_g$.

$G \in \mathcal{F}_g$ iff no graph in $\mathcal{O}_g$ is a minor of G

Theorem [Graph Minor VIII, 90]

For any $g \geq 0$, $\mathcal{O}_g$ is finite.

Finitness of $\mathcal{O}_g$: Sketch of proof of [T97] (1/3)

If the treewidth of the graphs in $\mathcal{O}_g$ is bounded $\Rightarrow$ $\mathcal{O}_g$ is finite.

Bounded treewidth graphs are WQO [RS90]

$\{G_1, G_2, \cdots\}$ infinite set of bounded treewidth graphs.
Then, $\exists i, j$ such that G_i is a minor of G_j.

Assume $\mathcal{O}_g$ is an infinite set of bounded tw graphs. Then, $\exists H, G \in \mathcal{O}_g$ such that H is a minor of G. A contradiction.

A weaker but sufficient resut [M01]

S surface of euler-genus g. $\exists N > 0$ s.t., any $H \in \mathcal{O}(S)$ with treewidth $< w$ has at most N vertices.

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So, we aim at proving that the treewidth of the graphs in $\mathcal{O}_g$ is bounded.

How to characterize a graph with high treewidth?

If $tw(G) < k$, then G does not contain a $k * k$ grid as a minor

A kind of converse holds

Grid exclusion Theorem [RS86, DJGT99]

If $tw(G) > r^{4m^2(r+2)}$, then G contains either K_m or the $r * r$-grid as a minor.

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Finitness of $\mathcal{O}_g$: Sketch of proof of [T97] (3/3)

So, if $G \in \mathcal{O}_g$ has no "big" grid as a minor, it has bounded tw.

No $G \in \mathcal{O}_g$ has a "big" grid as a minor [T97]

Let G be 2-connected, s.t. $genus(G \setminus e) < genus(G) = g$, $\forall e \in E(G)$. Then G contains no subdivision of $J_{\lceil 1100 g^{3/2} \rceil}$

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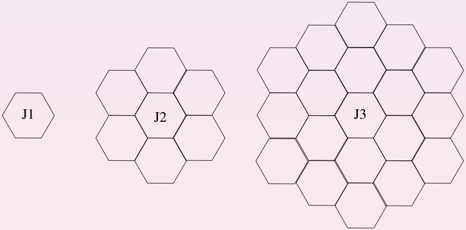


Note that J_k is a subgraph of a $4k * 2k$ grid.

Finitness of $\mathcal{O}_g$: Sketch of proof of [T97]

1) S surface of euler-genus g. $\exists N > 0$ s.t., any $H \in \mathcal{O}(S)$ with treewidth $< w$ has at most N vertices. [M01]

2) If $tw(G) > r^{4m^2(r+2)}$, then G contains either K_m or the $r * r$-grid as a minor. [RS86, DJGT99]

3) Let G be 2-connected, s.t. $genus(G \setminus e) < genus(G) = g$, $\forall e \in E(G)$. Then G contains no subdivision of $J_{\lceil 1100 g^{3/2} \rceil}$ [T97]

2) + 3) $\Rightarrow$ 4) Graphs in $\mathcal{O}_g$ have bounded treewidth

1) + 4) $\Rightarrow$ For any $g \geq 0$, $\mathcal{O}_g$ is finite.

Minimal obstructions of bounded treewidth

Theorem 1 [Mohar 01]

Let S be a surface of euler-genus g. $\exists N > 0$ s.t., any $H \in \mathcal{O}(S)$ with treewidth $< w$ has at most N vertices.

Proof Th. 1 (bounded size of bounded tw obstr.)

Let S be any surface with euler genus g.
Assume $H \in \mathcal{O}(S)$ is arbitrary large with $tw(H) \leq w$.
Let T be a tree-decomposition of H with width $\leq w$.

First step. T has bounded degree.
Thus, T contains an arbitrary large path P.

2nd step. Using P, $G \in \mathcal{F}(S)$ major of H can be built

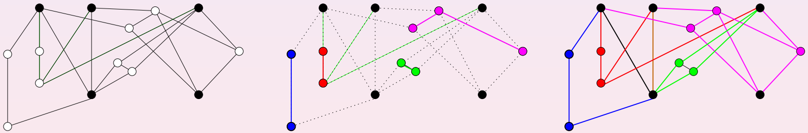
A contradiction.

Proof Th. 1, first step

X a subgraph of a graph G s.t. $V(X)$ is a separator of G.

X-bridge

- either an edge in $E(G) \setminus E(X)$, or
- a connected component of $G \setminus X$ together with all edges (and their enpoint) with one end in $V(G)$ and the other in $V(X)$.



X is the set of black vertices. There are 6 X-bridges (right).

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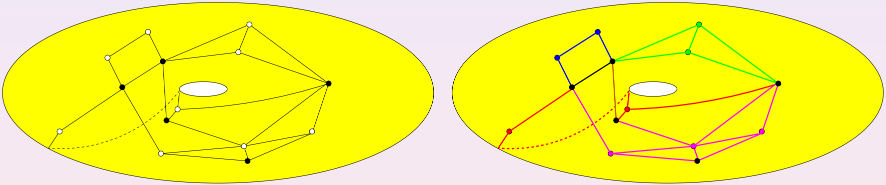
Property

$(T, (X_t)_{t \in V(T)})$ tree-decomposition of G and $t_0 \in V(T)$.
The degree of t_0 in T is less than the number of X_{t_0}-bridges

Proof Th. 1, count the X-bridges

G Π-embedded in S of Euler genus $eg = 2 - 2g$ (orientable). $X \subseteq V(G)$.



Note that by Euler Formula: $V - E + F = 2 - eg \Rightarrow 14 - 23 + F = 0$, i.e., $F = 9$.

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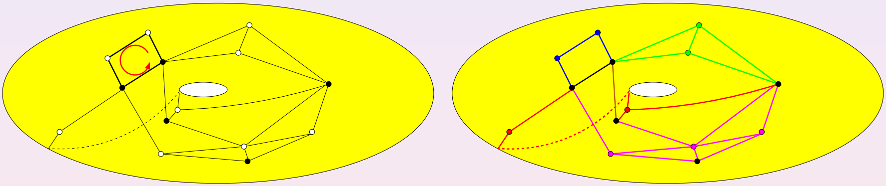

For any Π-facial walk W, add edges between consecutive vertices in W that are incident to edges in W from different X-bridges.

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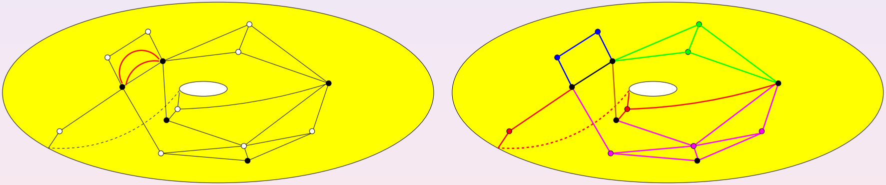

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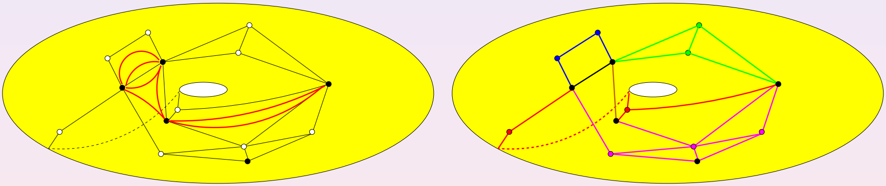

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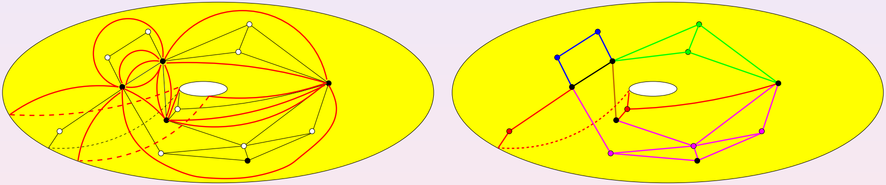

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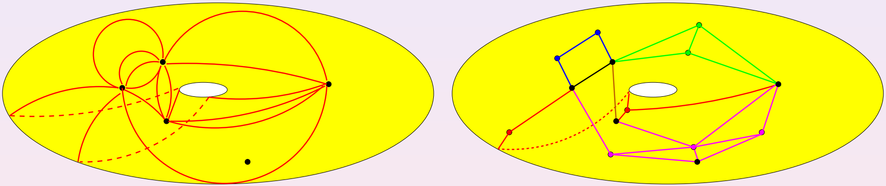
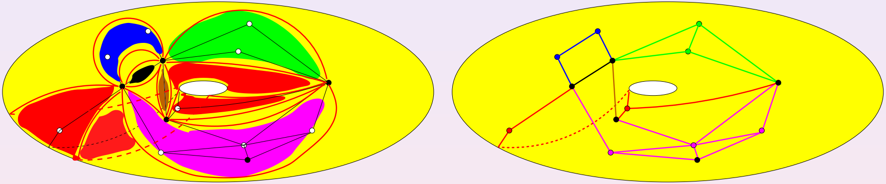


For any Π-facial walk W, add edges between consecutive consecutive vertices in W that are incident to edges in W from different X-bridges. X^* the induced graph.

Proof Th. 1, count the X-bridges

G Π-embedded in S of Euler genus $eg = 2 - 2g$ (orientable). $X \subseteq V(G)$.



Lem. Any X-bridge contains in a face of X^*.
Each face of X^* is either included in a face of G or contains a X-bridge
Each edge of X^* is incident to exactly one face contaning a X-bridge.
Lem. If no $x \in X$ is a cutvertex and $\forall x, y \in X$, a x, y-bridge is planar only if it is an edge: by Euler Formula, $\#$ X-bridge $\leq f(g, |X|)$.

Proof Th. 1: Summerize the first step

Let S be any surface with euler genus g.
Assume $H \in \mathcal{O}(S)$ is arbitrary large with $tw(H) \leq w$.
Let T be a tree-decomposition of H with width $\leq w$.

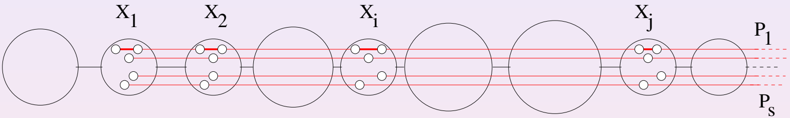
$t_0 \in V(T)$. **degree(t_0) in T**$\leq \#$ **$\mathbf{X}_{t_0}$-bridges**

no $x \in X_{t_0}$ is a cutvertex and $\forall x, y \in X_{t_0}$, a x, y-bridge is planar only if it is an edge because $H \in \mathcal{O}(S)$
By previous lemma: $\#$ **$\mathbf{X}_{t_0}$-bridge** $\leq$ **$\mathbf{f(g,}|\mathbf{X}_{t_0}|)$**.

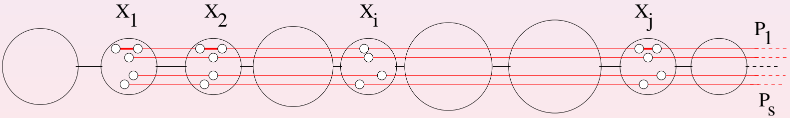
$\Rightarrow$ T has bounded degree: T contains an arbitrary large path.

Proof Th. 1: 2nd step, what about this long path?

Menger Theorem+pigeonhole princ.: $\exists (X_i)_{i \geq 1}$ large familly of bags s.t. $|X_i| = s \leq w$, s disjoint paths between the X_i, and 1 edge $\in P_1 \cap X_i$, $\forall i$.



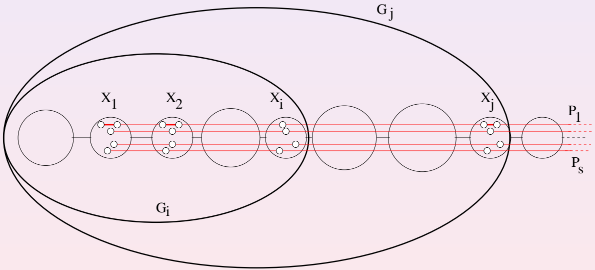
$H \in \mathcal{O}(S) \Rightarrow G^i$ got by contr. of the edge in $P_1 \cap X_i$ embeddable in S



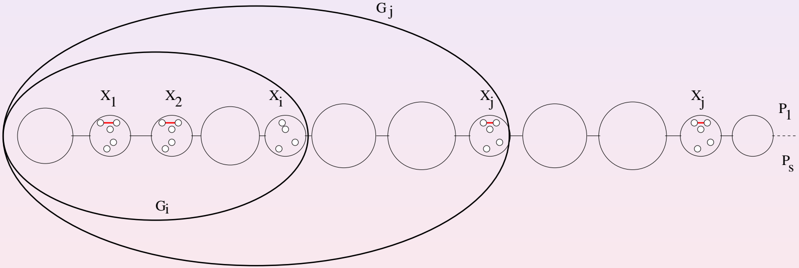
Proof Th. 1: 2nd step, what about this long path?

Th.: two surfaces with same genus are isomorphic

g, $|X_i|$ bounded $\forall i$, $\exists i, j$ such that X_j strongly isomorphic with X_i (with same embedding) and G_j can be embedded in the same surface as G_i



Hence, G (below) is embeddable in S. But H minor of G cannot ???
A Contradiction.



Finitness of $\mathcal{O}_g$: Sketch of proof of [T97]

1) S surface of euler-genus g. $\exists N > 0$ s.t., any $H \in \mathcal{O}(S)$ with treewidth $< w$ has at most N vertices. [M01] OK

2) If $tw(G) > r^{4m^2(r+2)}$, then G contains either K_m or the $r * r$-grid as a minor. [RS86, DJGT99]

3) Let G be 2-connected, s.t. $genus(G \setminus e) < genus(G) = g$, $\forall e \in E(G)$. Then G contains no subdivision of $J_{\lceil 1100 g^{3/2} \rceil}$ [T97]

2) + 3) $\Rightarrow$ 4) Graphs in $\mathcal{O}_g$ have bounded treewidth
1) + 4) $\Rightarrow$ For any $g \geq 0$, $\mathcal{O}_g$ is finite.

Minimal obstructions do not contain "big" grids

Theorem 2 [Thomassen 97]

Let G be 2-connected, s.t. $genus(G \setminus e) < genus(G) = g$, $\forall e \in E(G)$. Then G contains no subdivision of $J_{\lceil 1100 g^{3/2} \rceil}$.

Assume $G \in \mathcal{O}_g$ contains a subdivision of $J_{\lceil 1100 g^{3/2} \rceil}$.

1. Find a "big" and "good" planar subgraph H
2. Show that for any embedding Π of G, "small" parts of H have genus 0.
   That is Π induces a planar embedding of these parts.

Remove edge e of a "small" part, $G \setminus e$ embeddable in S_g
The corresponding embedding of the small part is planar
Extend it into an embeding of G into S_g. A contradiction.

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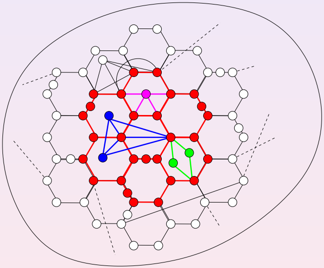
Proof of Th. 2: find good subgraph

A subdivision H of J_k is good in G if the union of H and those H-bridges with an attachment not in the outer face of H is planar.

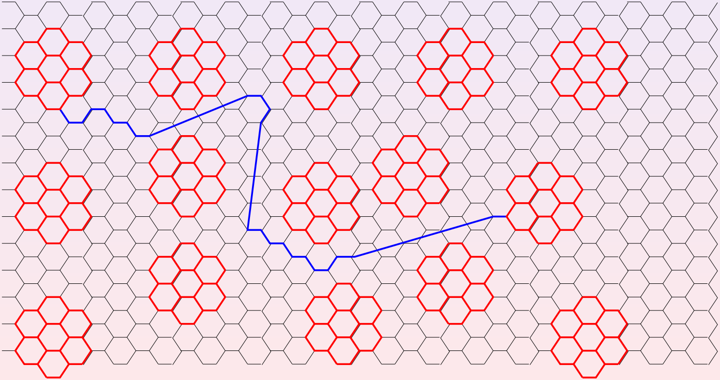


G of genus g with a subdivision H' of J_m as a subgraph.
If $m > 100k\sqrt{g}$, H' contains a good (in G) subdivision of J_k.

Proof of Th. 2: find good subgraph

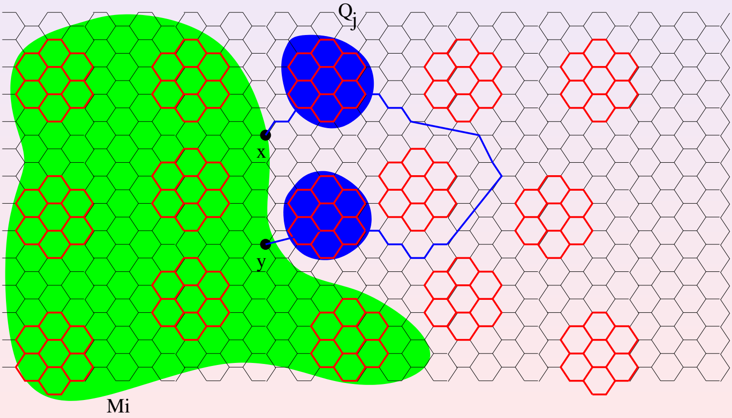
$(Q_j)_{j \leq 2g+2}$ pairwise disjoint sudivisions of J_k.
$\forall i, j$, there is a path between Q_i and Q_j avoiding the others.

Assume all are not good



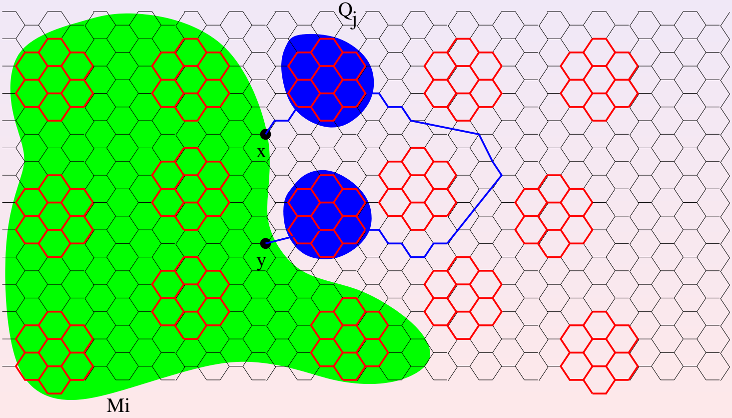
Proof of Th. 2: find good subgraph

We build $(M_i)_{i \leq g+1}$ with $M_1 = Q_1$, and
M_i intersects at most $2i - 1$ graphs Q_j, and
$genus(M_i) \geq i - 1$



Proof of Th. 2: find good subgraph

M_{i+1} got from M_i by adding Q_j and corresponding bridges
Since Q_j and its bridges are not planar, the genus increases until a subgraph of G with genus $G+1$. A contradiction.



Minimal obstructions do not contain "big" grids

Assume $G \in \mathcal{O}_g$ contains a subdivision of $J_{\lceil 1100 g^{3/2} \rceil}$.

G of genus g with a subdivision H' of J_m as a subgraph.
If $m > 100k\sqrt{g}$, H' contains a good (in G) subdivision of J_k.

G of genus g with a good subdivision H of J_k as a subgraph.
If $k \geq 4g + 6$, then any embedding of genus g induces a planar embedding of J_{k-4g-4}.

Remove edge e of a "small" part, $G \setminus e$ embeddable in S_g
The corresponding embedding of the small part is planar
Extend it into an embeding of G into S_g. A contradiction.

Finitness of $\mathcal{O}_g$: Sketch of proof of [T97]

1) S surface of euler-genus g. $\exists N > 0$ s.t., any $H \in \mathcal{O}(S)$ with treewidth $< w$ has at most N vertices. [M01] OK

2) If $tw(G) > r^{4m^2(r+2)}$, then G contains either K_m or the $r * r$-grid as a minor. [RS86, DJGT99]

3) Let G be 2-connected, s.t. $genus(G \setminus e) < genus(G) = g$, $\forall e \in E(G)$. Then G contains no subdivision of $J_{\lceil 1100 g^{3/2} \rceil}$ [T97] OK

2) + 3) $\Rightarrow$ 4) Graphs in $\mathcal{O}_g$ have bounded treewidth

1) + 4) $\Rightarrow$ For any $g \geq 0$, $\mathcal{O}_g$ is finite.

Big treewidth graphs contain big grids

Theorem 3 [RS86, DJGT99]

2) If $tw(G) > r^{4m^2(r+2)}$, then G contains either K_m or the $r * r$-grid as a minor.

Big treewidth graphs contain big grids

- If many paths with "good properties" G has a big grid

$d \geq r^{2r+2}$. Let G contains a set $\mathcal{H}$ of $r^2 - 1$ disjoint paths, and a set $\mathcal{V} = \{V_1, \cdots, V_d\}$ of d disjoint paths such that each $V \in \mathcal{V}$ intersects all $H \in \mathcal{H}$, and that any $H \in \mathcal{H}$ consists of d disjoint segments such that V_i meets H only in its i^{th} segment. Then G has a $r * r$ grid as minor.

- If G has big treewidth, it contains a big mesh.

If G contains no k-mesh of order h, then $tw(G) \leq h + k - 1$.

- If big mesh, G has many paths with "good properties".

How to build a big grid: Intuition

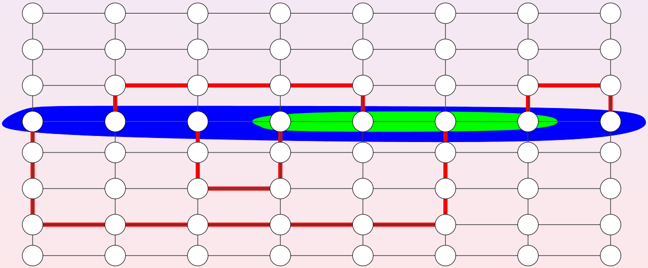
$d \geq r^{2r+2}$. Let G contains a set $\mathcal{H}$ of $r^2 - 1$ disjoint paths, and a set $\mathcal{V} = \{V_1, \cdots, V_d\}$ of d disjoint paths such that each $V \in \mathcal{V}$ intersects all $H \in \mathcal{H}$, and that any $H \in \mathcal{H}$ consists of d disjoint segments such that V_i meets H only in its i^{th} segment. Then G has a $r * r$ grid as minor.

Because of the number of "vertical" paths (in$\mathcal{V}$) , sufficient such paths can be found that intersect r horizontal paths (in $\mathcal{H}$) in the "same order".

Structure in big treewidth graph

Externally k-connected set

$X \subseteq V(G)$ with $|X| \geq k$ and for any $Y, Z \subset X$, $|Y| = |Z|$, there are $|Y|$ disjoint Y-Z paths without internal vertices or edges in X.

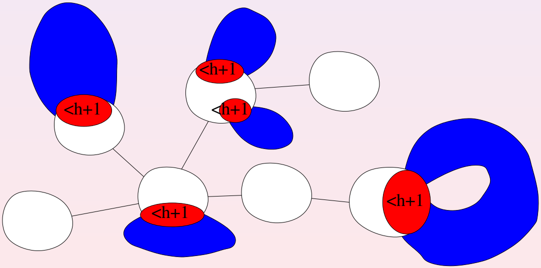


Structure in big treewidth graph

If $h \geq k$ and G contains no externally k-connected set with h vertices, then $tw(G) < h + k - 1$

$U \subseteq V(G)$ maximal such that $G[U]$ has a tree-decomposition $\mathcal{D}$ of width $< h + k - 1$ and $\forall$ component C of $G \setminus U$, $|N(C) \cap U| \leq h$ and $N(C) \cap U$ lies into a bag of $\mathcal{D}$.

Assume $U \neq V(G)$



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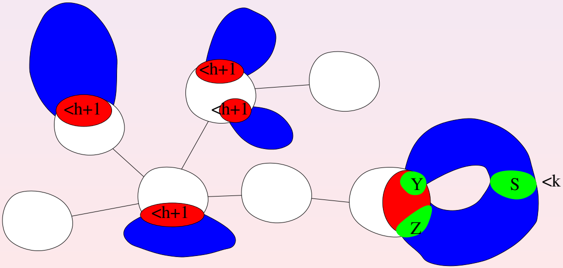
Let C a component of $G \setminus U$ and $X = N(C) \cap U$. X not externally k-conneted, thus

by Menger th., let S be a Y-Z separator in C, with $Y, Z \subset X$ and $|S| < k$.



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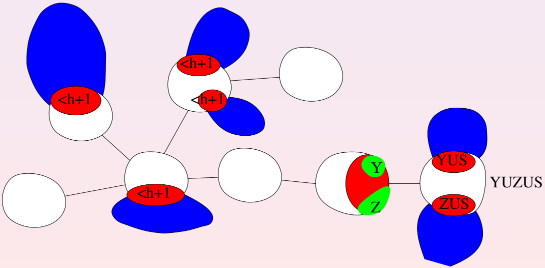
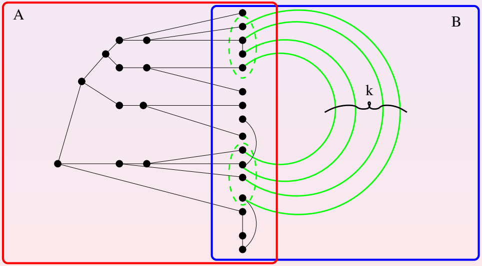
U can be extended, contradicting its maximality.



A better structure in big treewidth graph

A separation (A, B) is a k-**mesh** if

- all edges of $G[V(A \cap B)]$ lie in A,
- A contains a tree T with maximum degree 3
- all vertices of $A \cap B$ lie in T with degree ≤ 2, and some has degree 1
- $V(A \cap B)$ is externally k-connected in B



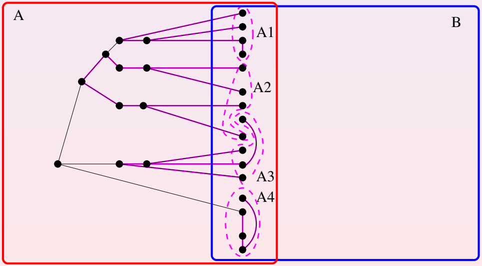
If G contains no k-mesh of order $h \geq k$, $tw(G) < h + k - 1$

Proof of Th. 3

Theorem 3 [RS86, DJGT99]

2) If $tw(G) > r^{4m^2(r+2)}$, then G contains either K_m or the $r * r$-grid as a minor.

Let $c = r^{4(r+2)}$ and $k = c^{m(m-1)}$. $\exists$ a k-mesh of order $m(2k-1)+k-1$.
There are m disjoint subtrees each containing $\geq k$ vertices of $A \cap B$.



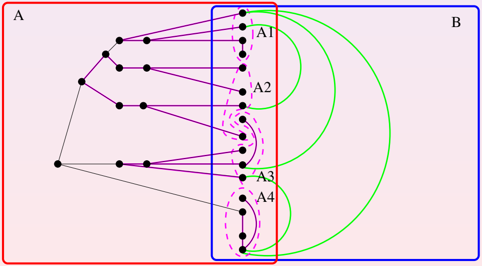
Proof of Th. 3

Theorem 3 [RS86, DJGT99]

2) If $tw(G) > r^{4m^2(r+2)}$, then G contains either K_m or the $r * r$-grid as a minor.

Intempt to find vertex disjoint paths between A_i and A_j for all i, j

If not, exhibit many paths with good properties to build a $r * r$ grid



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References

- *Graphs on Surfaces*, Mohar and Thomassen, Johns Hopkins Univ. Press, 2001
- *Graph Theory*, Diestel, Graduate texts in Maths. 2005

- *A Kuratowski theorem for the projective plan*, Archdeacon, JGT 81
- *Highly connected sets abd the excluded grid theorem*, Diestel, Jensen, Gorbunov and Thomassen, JCTB 99
- *An Analogue of the Myhill-Nerode Theorem and its Use in Computing Finite-Basis Charaterizations*, Fellow and Langston, FOCS 89
- *Sur le probleme des courbes gauches en Topologie*, Kuratowski, Fund. Math. 30
- *Graph minors and graphs on surfaces*, Mohar, Survey in Combinatorics 01
- *Graph Minors IV, Tree-width and well-quasi-ordering*, Robertson and Seymour, JCTB 90
- *Graph Minors V,Excluding a Planar Graph*, ——, JCTB 86
- *Graph Minors VIII, A Kuratowski Theorem for General Surfaces*, ——, JCTB 90
- *Graph Minors XIII, the Disjoint Paths Problem*, ——, JCTB 95
- *A simpler Proof of the Excluded Minor Theorem for Higher Surfaces*, Thomassen, JCTB 97