Monotonicity of Non deterministic Graph Searching

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Introduction

Non deterministic graph searching

[Fomin, Fraigniaud, Nisse, 2005]
Parametrized variant that unifies visible and invisible graph searching.

Monotone non deterministic graph searching : interpretation in terms of graph decomposition

Does recontamination help ?

Our result : Non deterministic graph searching is monotone.

Introduction

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interpretation in terms of graph decomposition

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Our result : Non deterministic graph searching is monotone.

Non deterministic Graph Searching

An omniscient arbitrary fast invisible fugitive runs at the vertices of the graph.
The searchers cannot see the fugitive, however :
An Oracle permanently knows the position of the fugitive.

The searchers can perform a query to the oracle :

Answer of the oracle :

The connected component of the contaminated part of the graph, where the fugitive is currently standing.

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Non deterministic Search Strategy

three basic operations :

- Place a searcher ;
- Remove a searcher ;
- Perform a query to the oracle.

The searchers aim at catching the fugitive.

The fugitive is caught when it occupies the same vertex as a searcher and it has no way to escape.
An edge is cleared when both its ends are occupied.

q-limited non deterministic search number

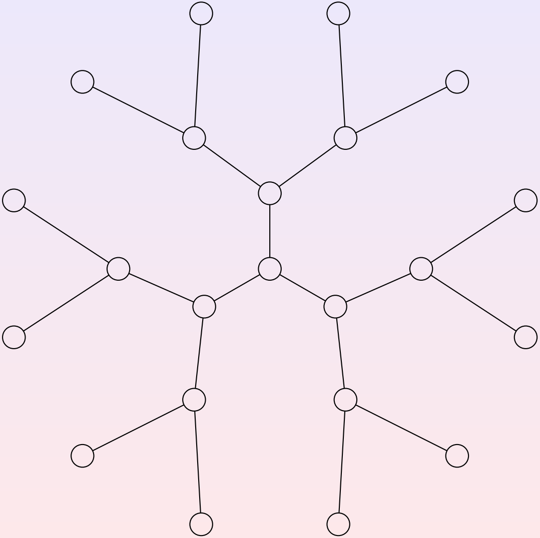
Tradeoff number of searchers / number of query steps

q-limited (non deterministic) search number, $s_q(G)$

- $\mathbf{s}_0(G) = \mathbf{pw}(G) + 1$, node search number of G ;
- $\mathbf{s}_\infty(G) = \mathbf{tw}(G) + 1$, visible search number of G.

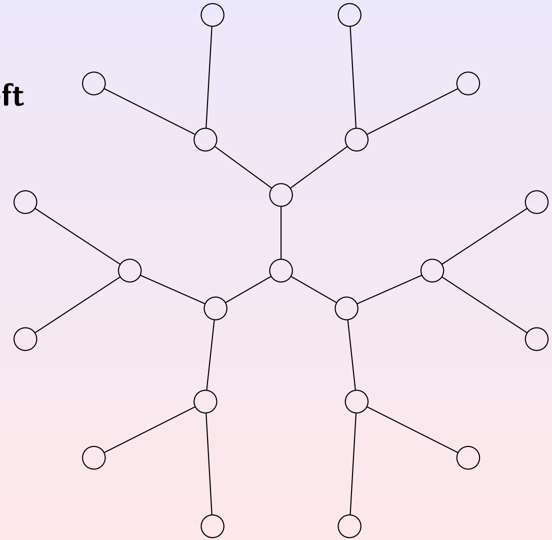
Exemple with q=2 :

$s_0(T)=3$



Exemple with q=2 :

2 queries left



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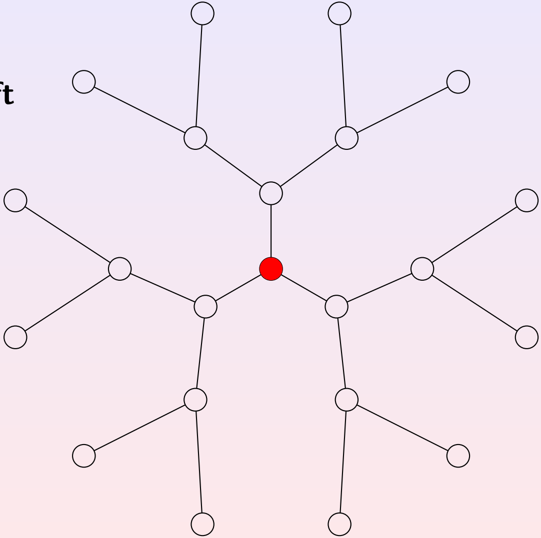

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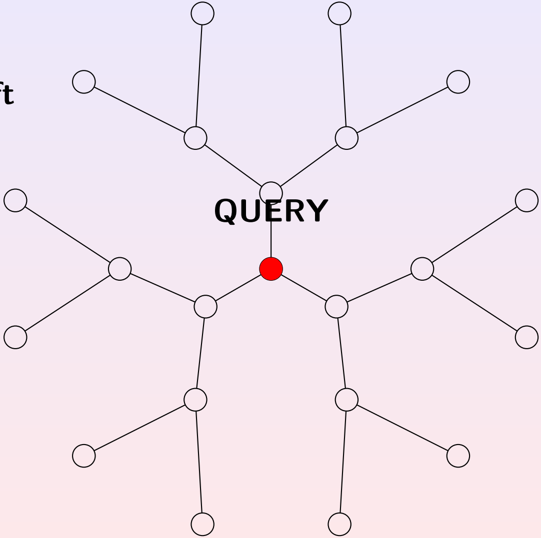
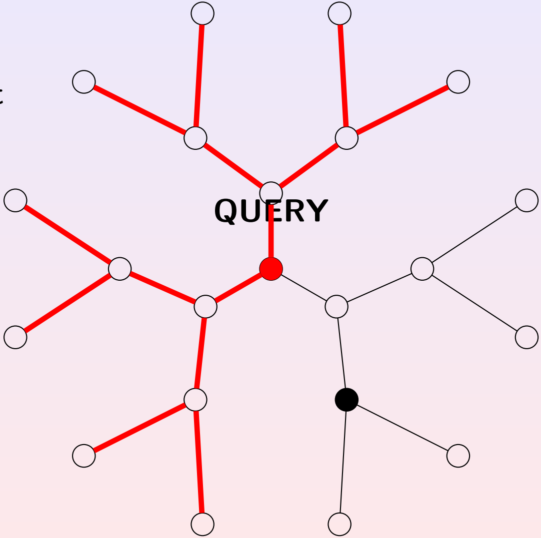


Exemple with q=2 :

1 query left



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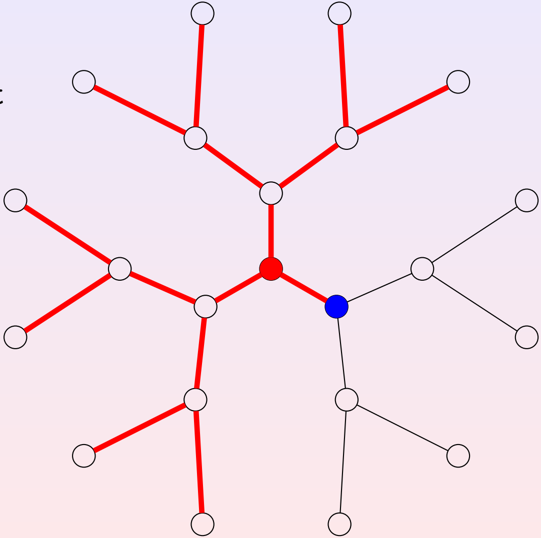
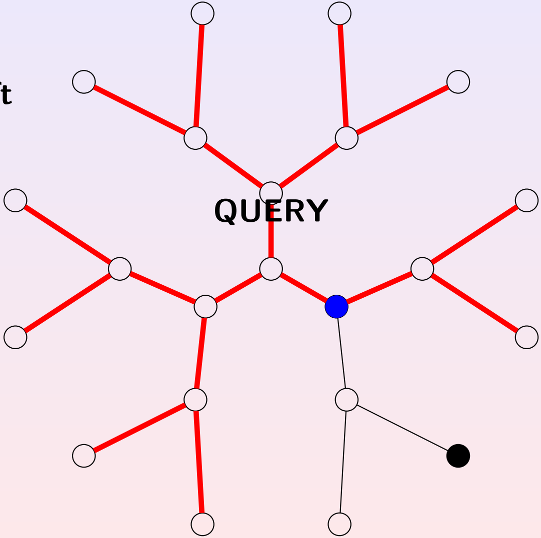


Exemple with q=2 :

no query left



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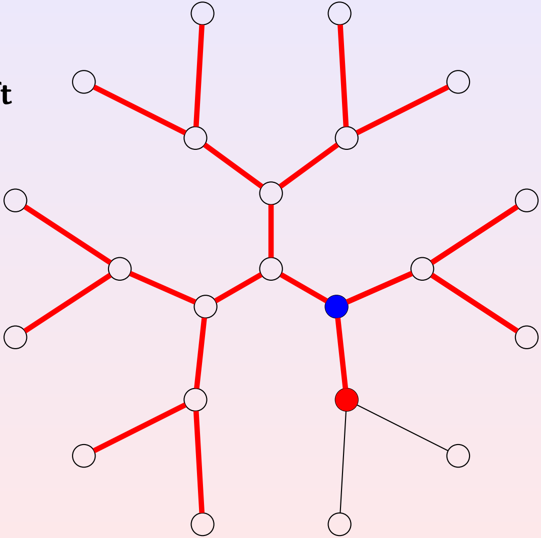

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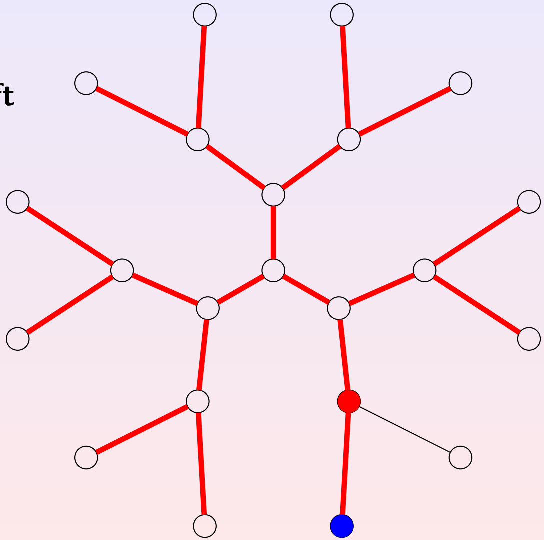

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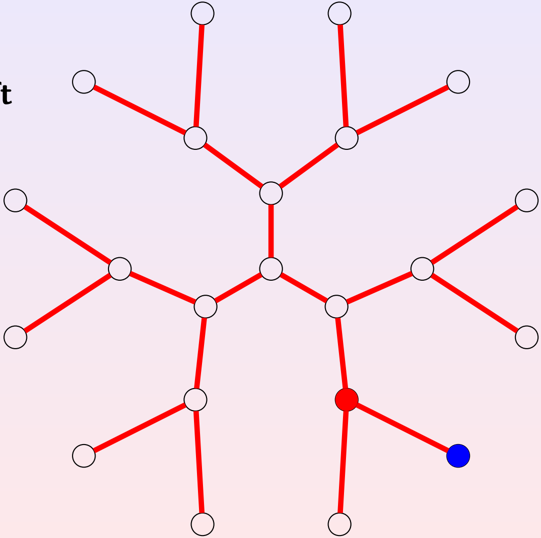
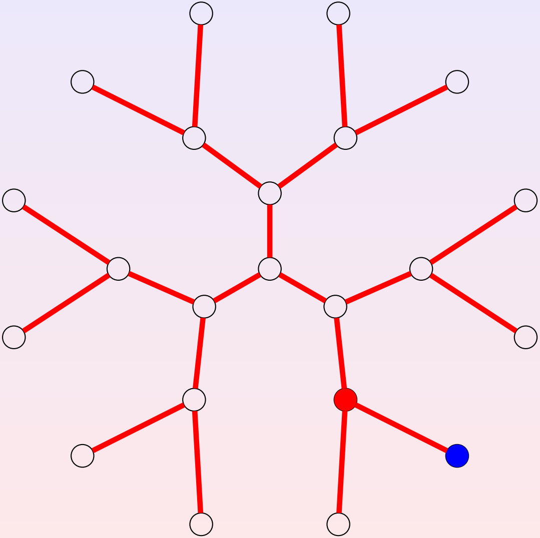


Exemple with q=2 :

$s_2(T)=2$



Some terminology

q-branched tree

- rooted tree ;
- branching node (at least two children) ;
- every path from the root to a leaf contains at most q branching nodes.

Interpretation in terms of Graph Decompositions

(T, X) : q-branched tree decomposition

(T, X) a tree-decomposition with T a q-branched tree.

q-branched treewidth, $tw_q(G)$, minimum width among any q-branched tree-decomposition of G.

- path decomposition $=$ 0-branched tree decomposition
  $\mathbf{pw}(G) = \mathbf{tw}_0(G)$;
- tree decomposition $=$ ∞-branched tree decomposition
  $\mathbf{tw}(G) = \mathbf{tw}_\infty(G)$;

Interpretation in terms of Graph Decompositions

The branched decompositions correspond to monotone search strategies for non deterministic graph searching.
$\mathbf{ms}_q(G)$: q-limited monotone search number

Theorem[Fomin, Fraigniaud, Nisse, 2005] :

1. For any $q \geq 0$, for any graph G, $\mathbf{ms}_q(G) = \mathbf{tw}_q(G) + 1$;
2. Computing $tw_q(G)$ is NP-complete for any q ;
3. Exact exponential algorithm that, for any graph G and any $q \geq 0$, computes $\mathbf{tw}_q(G)$ and an optimal decomposition.

Does recontamination help for any $q \geq 0$?

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Does recontamination help for any $q \geq 0$?

Related work

Recontamination does not help to catch an invisible fugitive.

Case of an invisible fugitive : $\mathbf{s}_0(G) = \mathbf{ms}_0(G)$

- **Bienstock and Seymour**, J.of Alg., 1991
  Monotonicity in graph searching.
- **LaPaugh**, J.of ACM, 1993
  Recontamination does not help to search a graph.

Constructive proof by Bienstock and Seymour :
Local optimisation that transforms a search strategy into a monotone one without increasing the number of searchers.

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Case of a visible fugitive : $\mathbf{s}_\infty(G) = \mathbf{ms}_\infty(G)$

- **Seymour and Thomas**, J. of Comb. Th., 1993.
  Graph searching and a min-max theorem for tree-width

scheme of the proof :

there is no search strategy using k searchers.

$\Rightarrow$ there is no monotone search strategy using k searchers

$\Rightarrow$ there exists an escape strategy for the fugitive

$\Rightarrow$ there exists a general escape strategy for the fugitive

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Our result

Recontamination never helps in non deterministic graph searching.

For any $q \geq 0$ and any graph G, $\mathbf{s}_q(G) = \mathbf{ms}_q(G)$

Remarks :

- Constructive proof that unifies the existing proofs ;
- Deciding $\mathbf{s}_q(G) \leq k$ is in NP ;
- The algorithm of Fomin *et al.* actually computes $\mathbf{s}_q(G)$.

Sketch of the proof

new structure inspired by tree-labelling [Robertson and Seymour, Graph Minor X] : Search-tree

Let G be a connected graph, $q \geq 0$ and $k \geq 1$. The following are equivalent

1. there exists a non deterministic search strategy using $\leq k$ searchers and at most q queries ;
2. there is a q-branched search-tree with width $\leq k$;
3. there is a monotone q-branched search-tree with width $\leq k$;
4. there exists a non deterministic monotone search strategy using $\leq k$ searchers and at most q queries ;

Some terminology

Border of subsets of edges $E_1, \cdots, E_p \subseteq E(G)$:

$\delta(E_1, E_2)$= set of vertices incident to an edge of E_1 and an edge of E_2.

$\delta(E_1) = \delta(E_1, E(G) \setminus E_1)$.

$\delta(E_1, \cdots, E_p) = \bigcup_{i \neq j} \delta(E_i, E_j)$.

Definition of search-tree

(T, α, β, r) a search-tree of a graph G ;

- T : a tree rooted in $r \in V(T)$;
- α : incidence of $T \longrightarrow$ subset of $E(G)$;
  $v \in V(T)$, e incident to $v \longrightarrow \alpha(v, e) \subseteq E(G)$.
- β : $V(T) \longrightarrow$ subset of $E(G)$;
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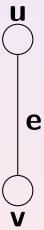
Definition of search-tree

(T, α, β, r) must satisfy two properties :

(P1) for any edge $e = \{u, v\}$ of T, $\alpha(u, e) \cap \alpha(v, e) = \emptyset$;

set of contaminated edges before one step $\alpha(u, e)$

cleared edges that remains clear after one step $\alpha(v, e)$



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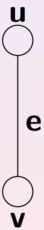

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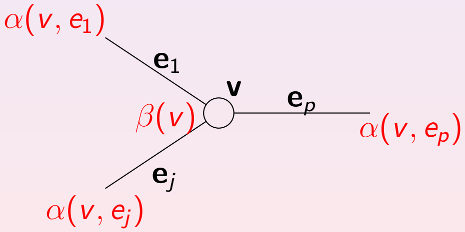
Definition of search-tree

(T, α, β, r) must satisfy two properties :

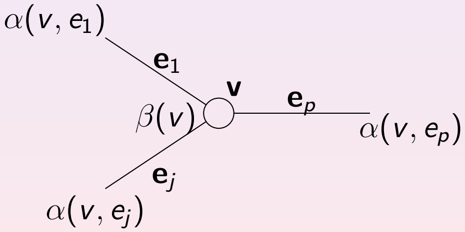
(P2) for any node v of T incident to $e_1, \ldots, e_p$,
$\{\beta(v), \alpha(v, e_1,), \ldots, \alpha(v, e_p)\}$ is a partition of E



Definition of search-tree

(T, α, β, r) is q-branched if T is q-branched.

width(T)$= \max_{v \in V(T)} |\delta(\alpha(v, e_1,), \ldots, \alpha(v, e_p)) \bigcup V[\beta(v)]|$;



From the strategy to the search-tree

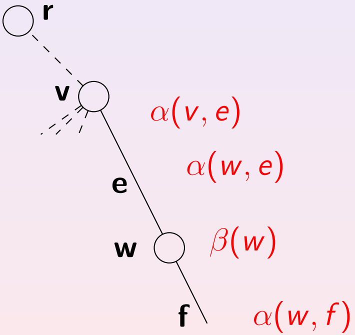
We construct the search-tree (T, α, β, r) recursively ;

Each edge **e** corresponds to a step of the strategy ;

Each branching node corresponds to a query step.

From the strategy to the search-tree

If the considered step i is a *Placing searcher* or *Removing searcher* step :



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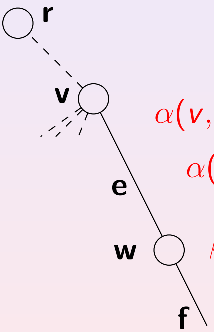

$\alpha(v, e)$, **contaminated edges before step i**

$\alpha(w, e)$

$\beta(w)$

$\alpha(w, f)$

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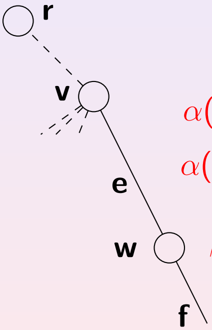

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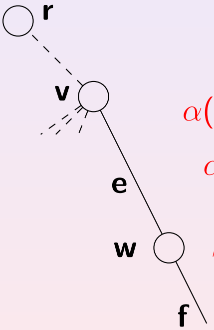

$\alpha(v, e)$, contaminated edges before step i

$\alpha(w, e)$, edges cleared before step i
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$\beta(w)$, **edges cleared at step i**
thanks to placement of new searchers

$\alpha(w, f)$

From the strategy to the search-tree

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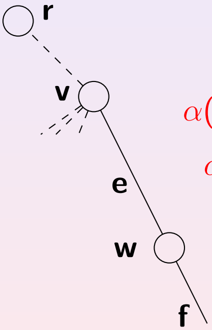

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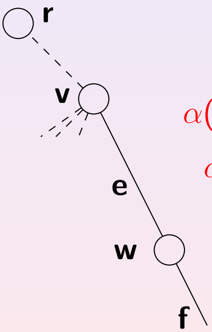

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P1 and **P2** are satisfied.

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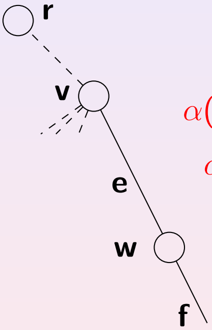

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If initial search strategy allows recontamination at this step, the corresponding edge e is not monotone.

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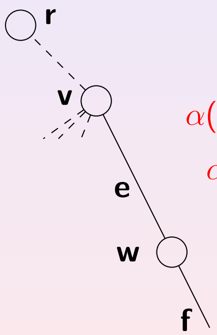

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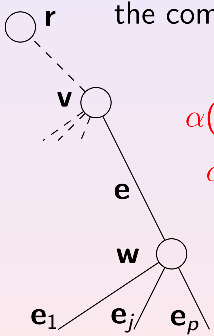
$\alpha(w, f)$, contaminated edges after step i

$$\delta(\alpha(w, e), \alpha(w, f)) \bigcup V[\beta(w)] \leq \#(\text{searchers}).$$

From the strategy to the search-tree

If we consider a *Performing a query* step, with $C_1, \cdots, C_p$ the components where the fugitive can stand



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$\alpha(w, e_j) = E(C_j)$, **contaminated edges after step** i **according to the answer of the oracle**

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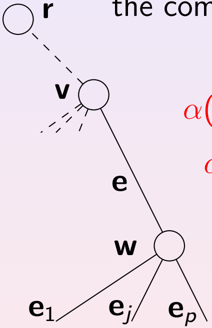

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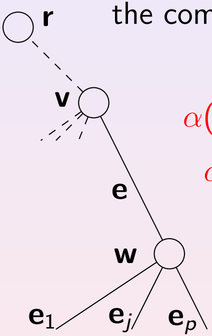

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From a search-tree to a monotone search-tree

A search-tree is monotone if all its edges are monotone.

By local optimizations, we build a monotone search-tree from a search tree (T, α, β, r).

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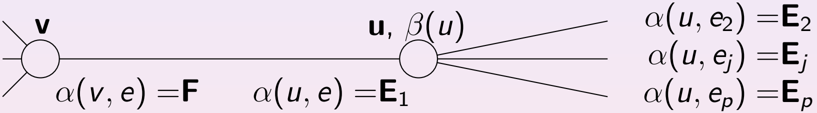
Local Optimisation

Let $e = \{v, u\}$ a non monotone edge (i.e., $E_1 \cup F \neq E$)



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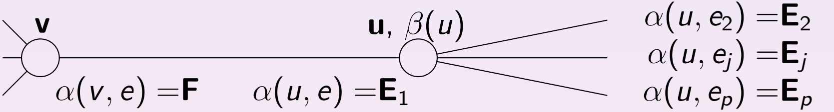

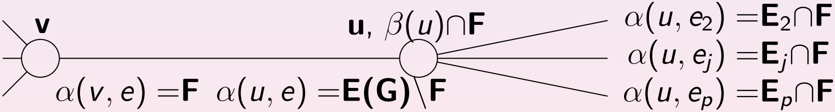

We define a weight function that strictly decreases by this optimisation.

From a search-tree to a monotone search-tree

2 weight functions

- w(T)= $\sum_{v \in V(T)} |\delta\big(\alpha(v, e_1)..\alpha(v, e_\ell)\big) \bigcup V[\beta(v)]|$
- bd(T)= $\sum n^{-\mathbf{dist}(r,e)}$ the sum being taken over the non monotone edges

From monotone search-tree to monotone strategy

(T, α, β, r) a q-branched monotone search-tree with width $\leq k$

Then, $(T, (X_v)_{v \in V(T)})$ with

$$X_v = \delta(\alpha(v, e_1)..\alpha(v, e_\ell)) \bigcup V[\beta(v)]$$

is a q-branched tree-decomposition with width $\leq k - 1$

Using the Theorem of Fomin *et al.*, we get the result.

Open problems

About monotony

Does recontamination help for catching a visible fugitive that runs in a directed graph ?

About non deterministic graph searching

Linear algorithm in case of trees ?
Linear algorithm in case of the class of graph with $\mathbf{tw}_q$ bounded ?

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