Nondeterministic Graph Searching: From Pathwidth to Treewidth

Fedor V.Fomin[1] Pierre Fraigniaud[2] Nicolas Nisse[2]

Department of Informatics, University of Bergen,
PO Box 7800, 5020 Bergen, Norway.

CNRS, Lab. de Recherche en Informatique, Université Paris-Sud,
91405 Orsay, France.

Workshop on Graph Classes, Width Parameters and Optimisation, Prague, October 19, 2005

Graph Searching

Goal

In an undirected simple graph,

- omniscient and arbitrary fast fugitive ;
- a team of searchers ;

We want to find a strategy that catch the fugitive
using the fewest searchers as possible.

Motivation

- game related to treewidth and pathwidth ;
- we introduce a parametrized version of treewidth.

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Search Strategy, Parson. [GTC,1978]

Sequence of two basic operations,...

1. Place a searcher at a vertex of the graph ;
2. Remove a searcher from a vertex of the graph.

...that must result in catching the fugitive

The fugitive is caugth when it is at a vertex occupied by a searcher and it has no way to escape.

We want to minimize the number of searchers.

Let $s(G)$ be the smallest number of searchers needed to catch a fugitive in a graph G.

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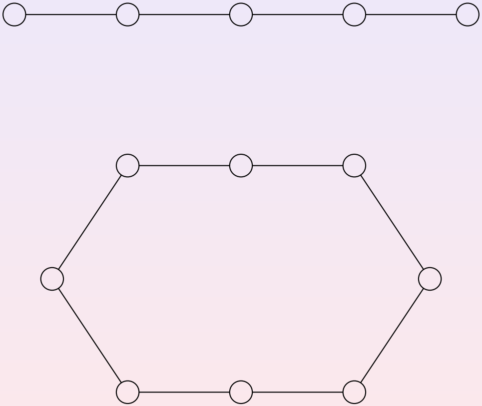
Simple Examples : Path and Ring



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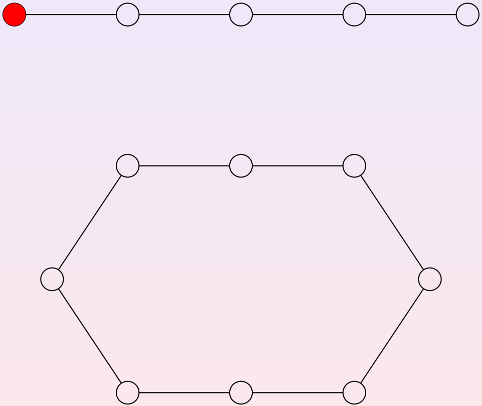

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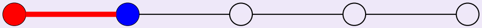

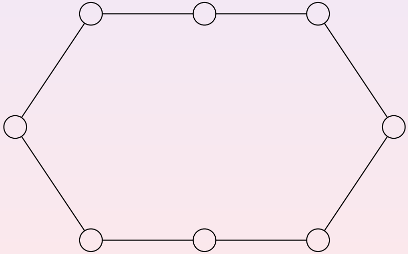

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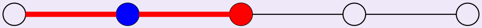

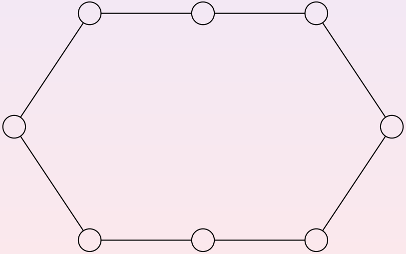

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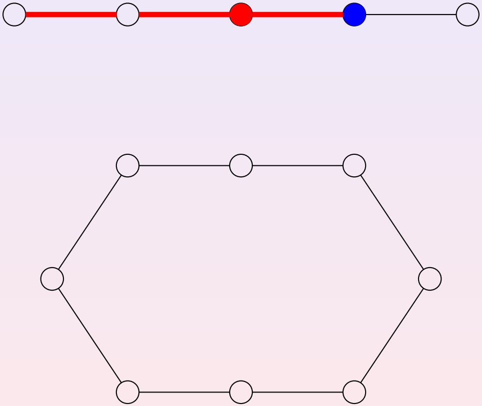
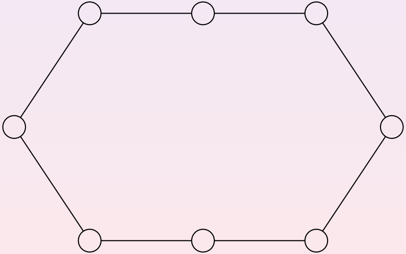


s(Path)=2



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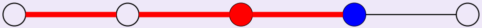

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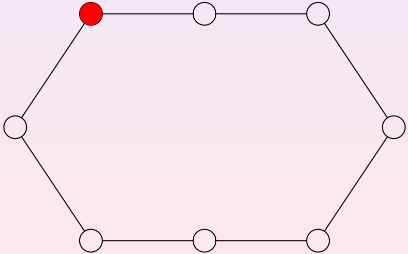

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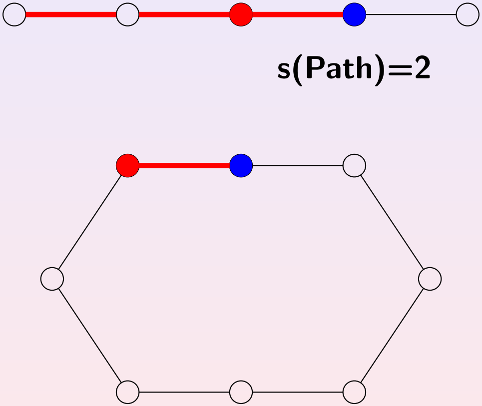

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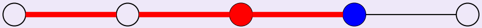

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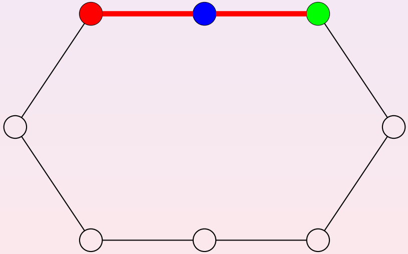

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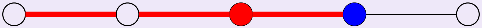

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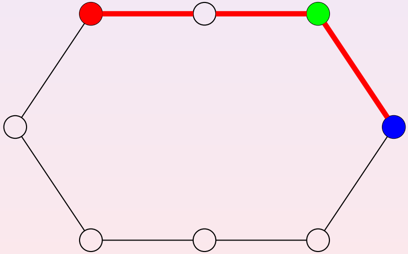

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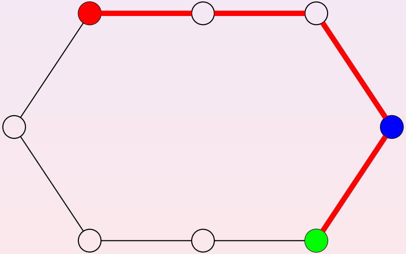

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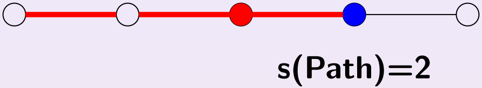
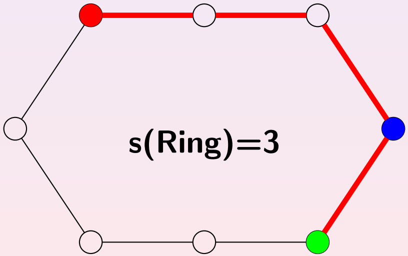
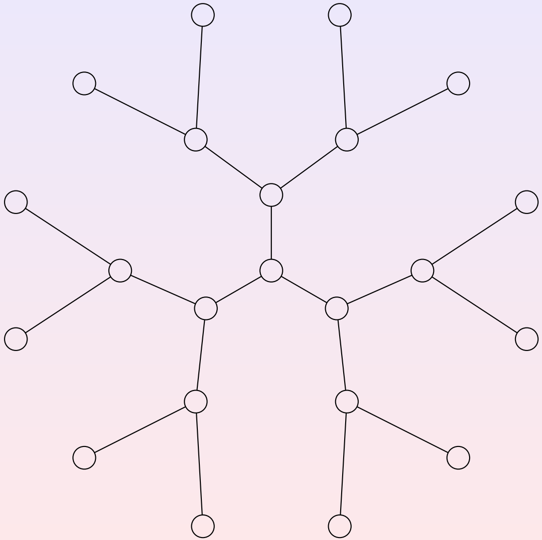




Graph searching in a tree



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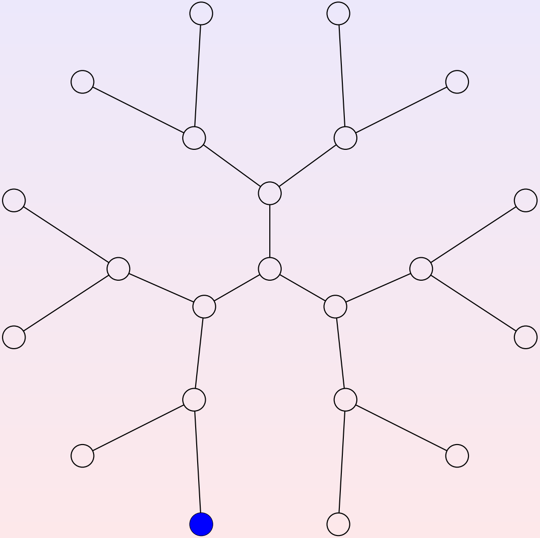

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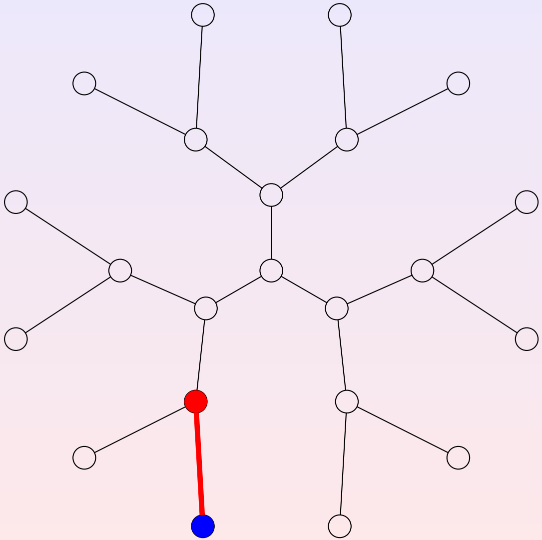

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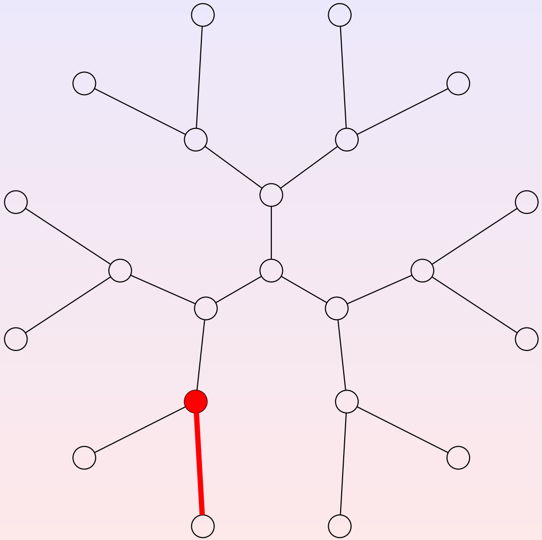

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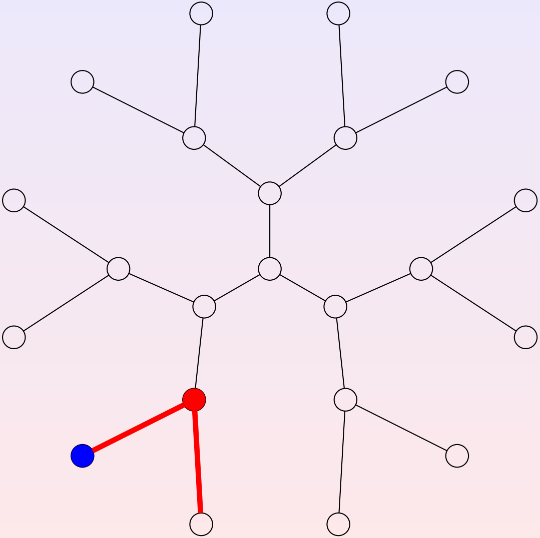

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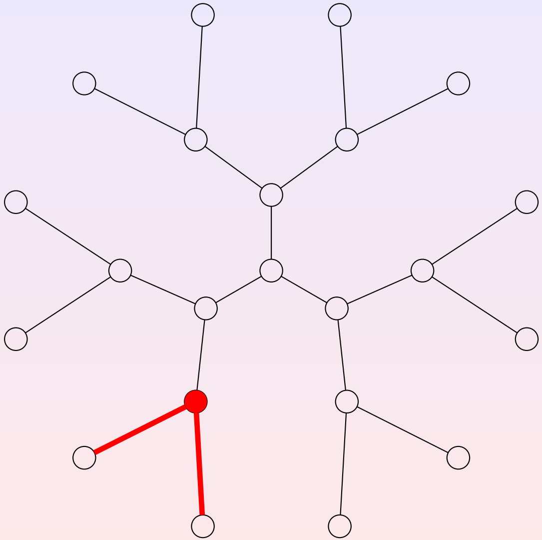

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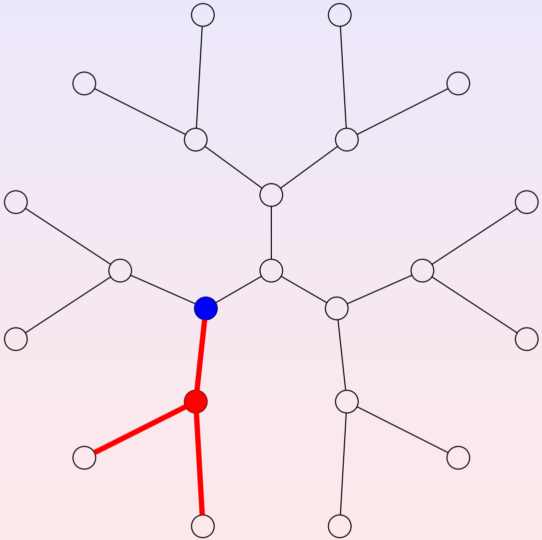

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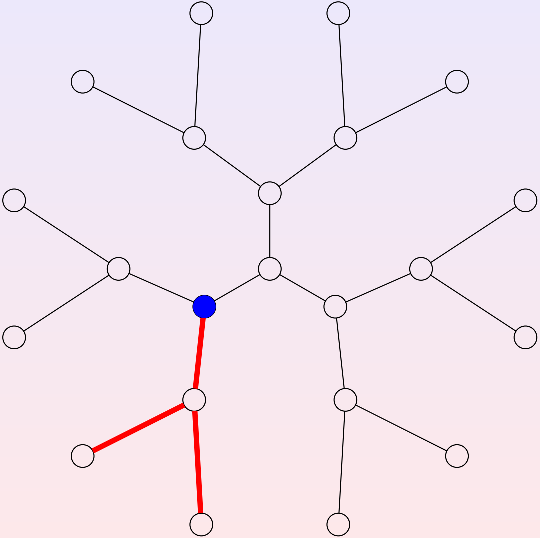

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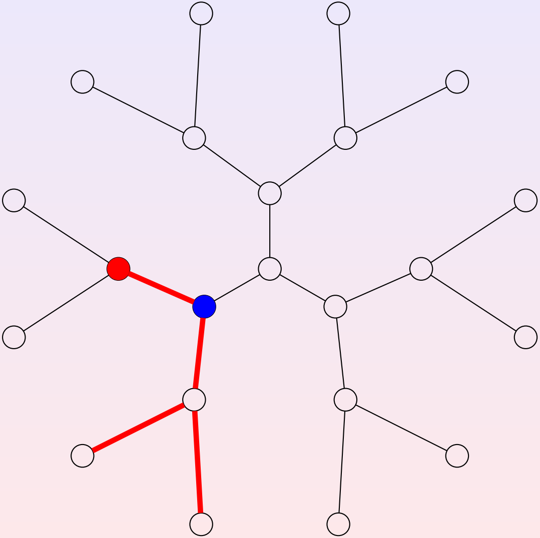

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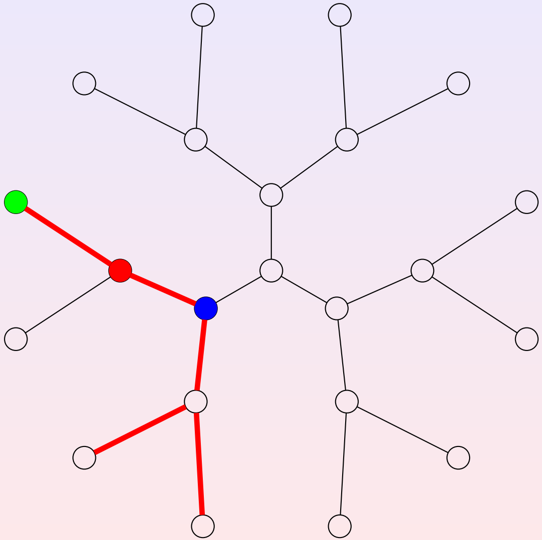

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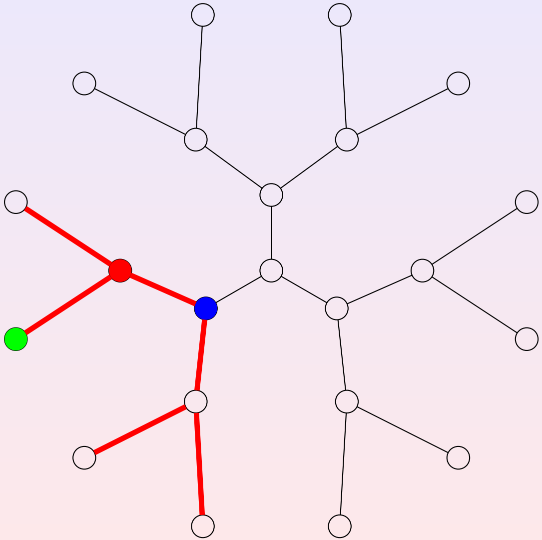

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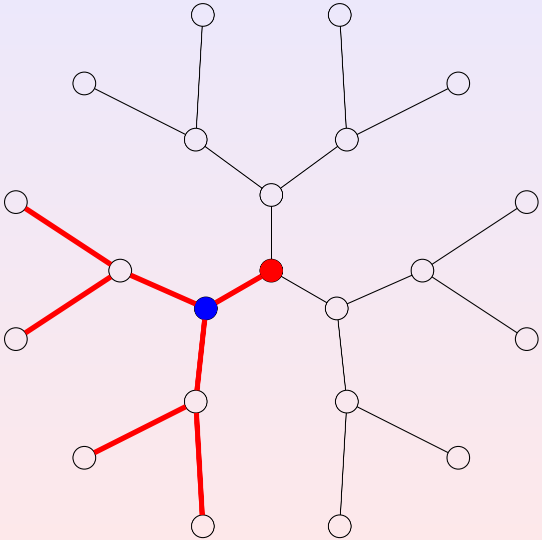

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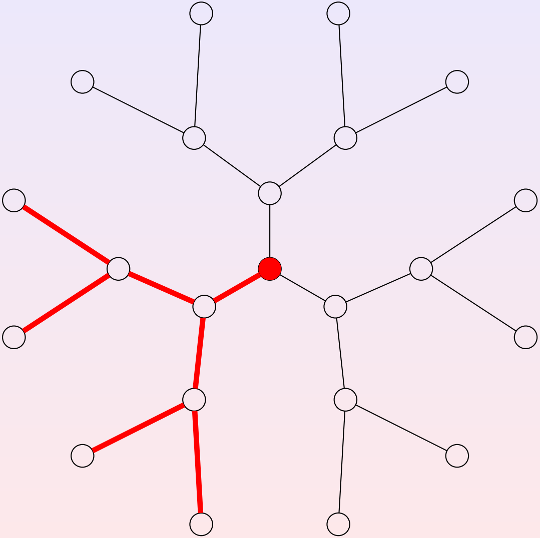

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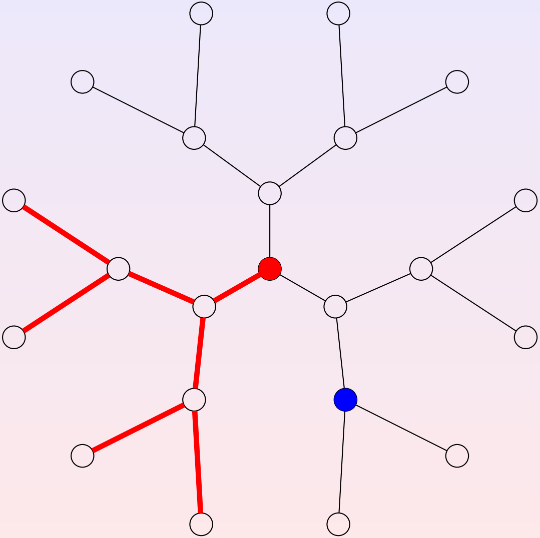

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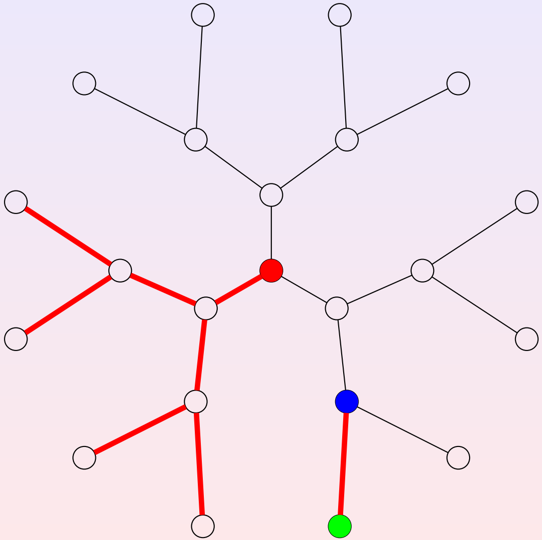

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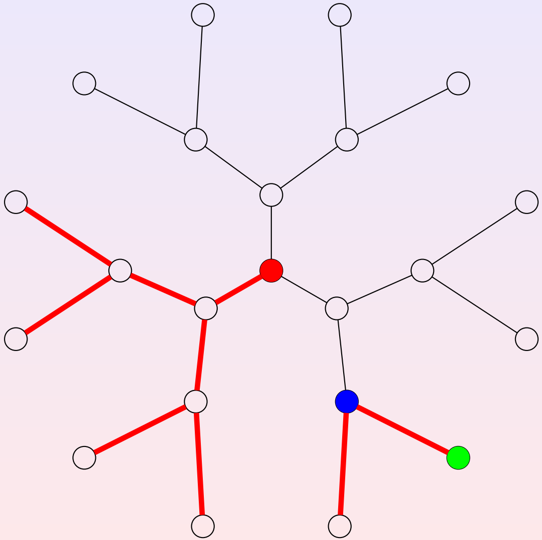

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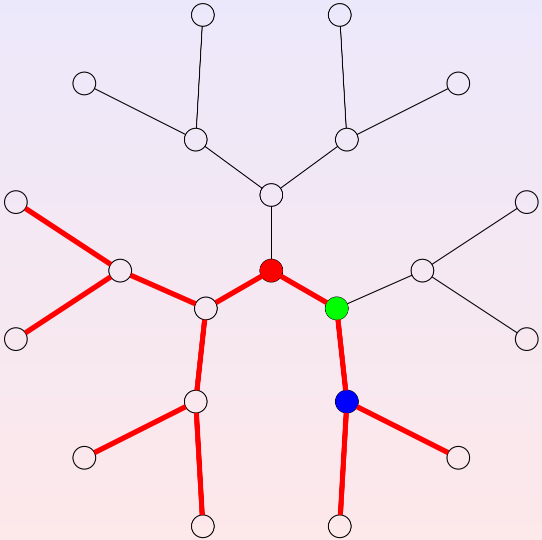

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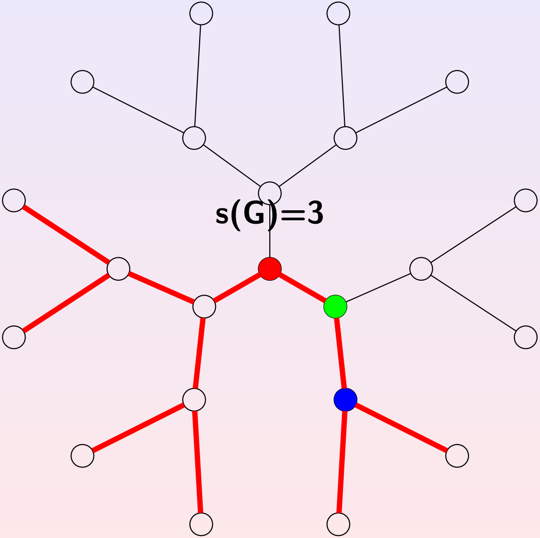
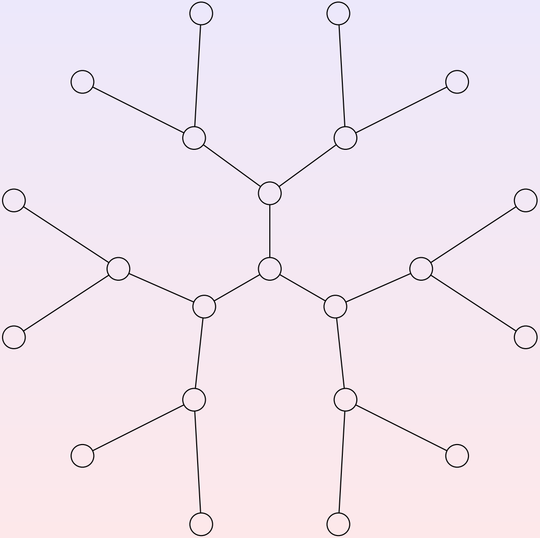


Visible graph searching in a tree



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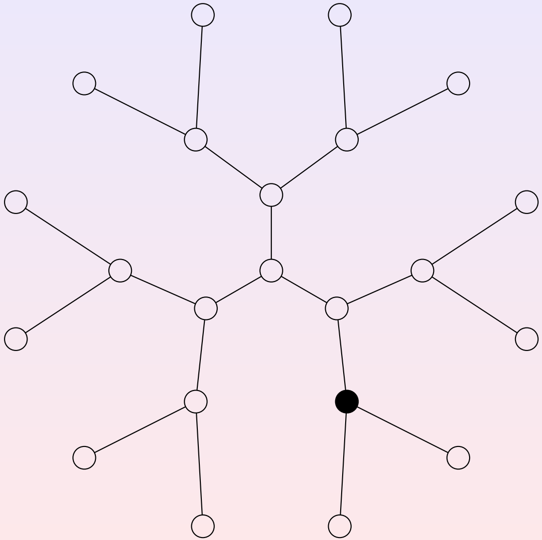

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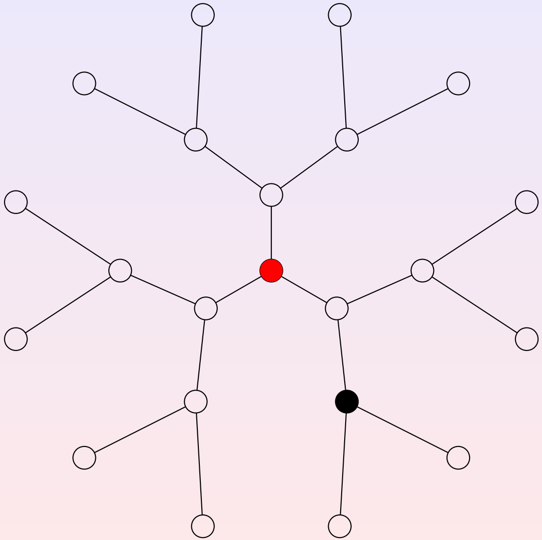

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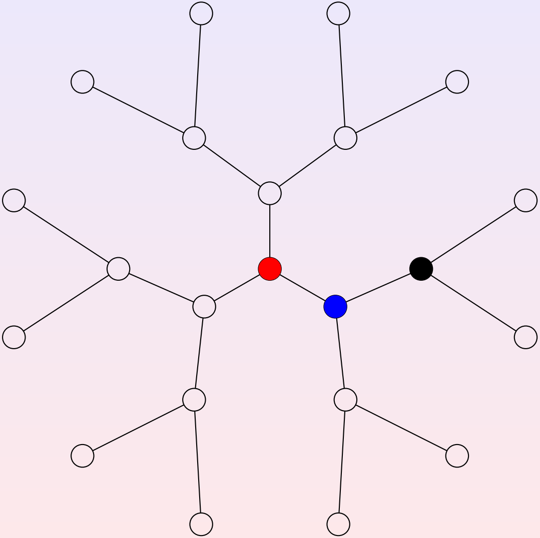

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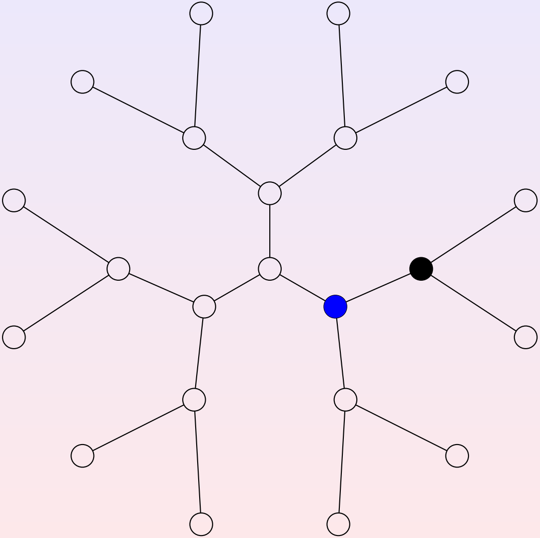

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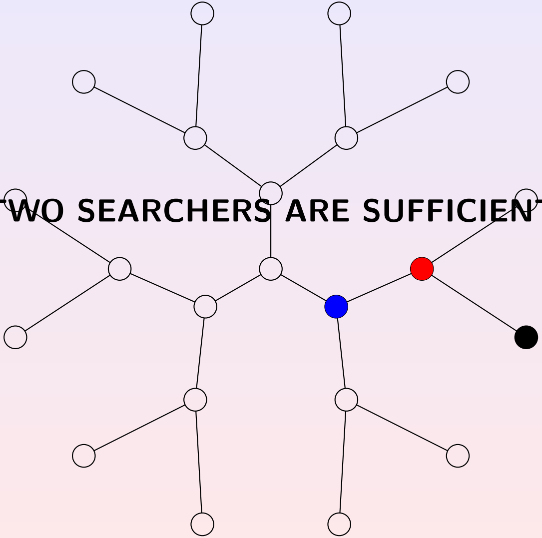
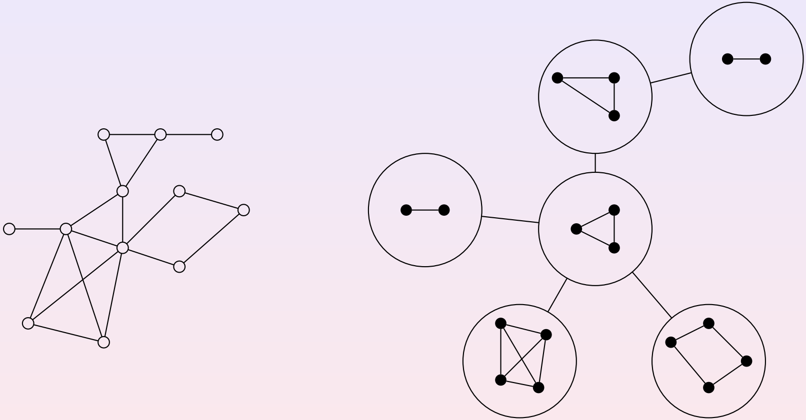


TWO SEARCHERS ARE SUFFICIENT

Tree Decomposition and Visible Search



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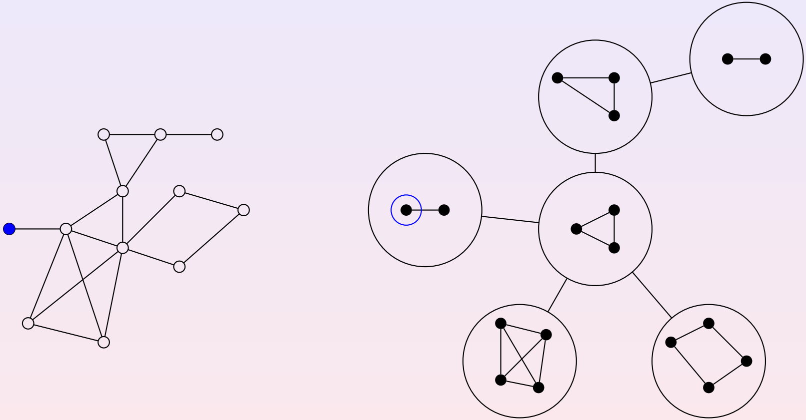

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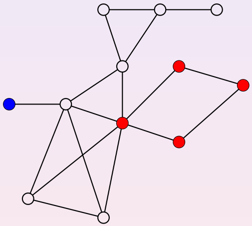

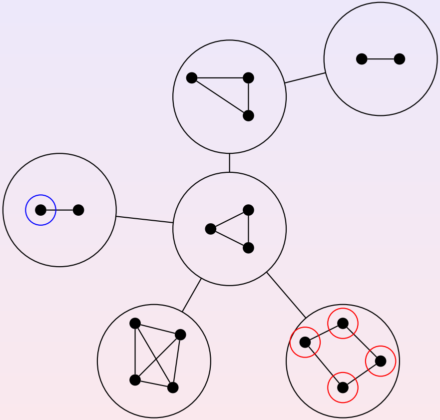

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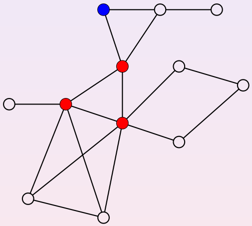

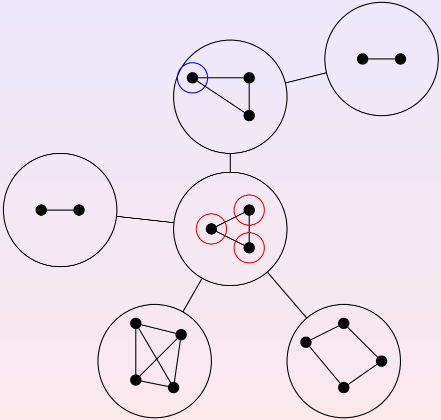

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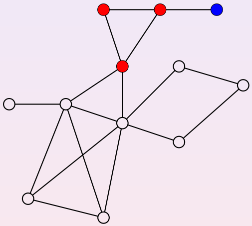

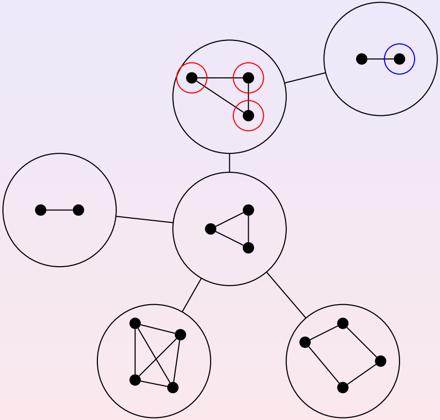

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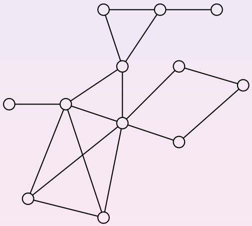

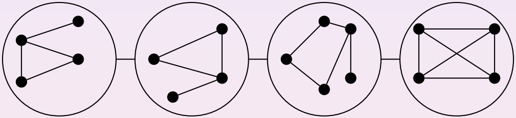

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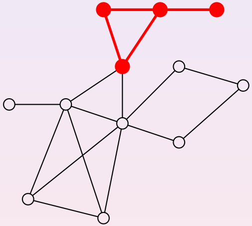

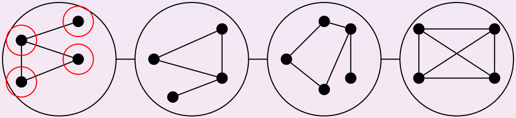

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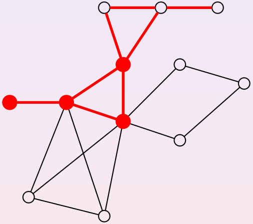

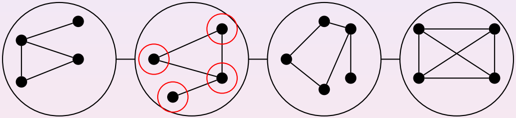

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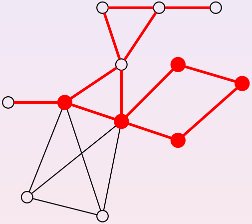
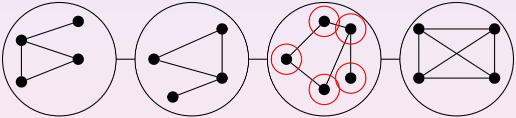




Related Work

$\mathbf{s}(G) = \mathbf{pw}(G) + 1$

- **Kirousis and Papadimitriou**, Theor. Comp. Sc., 1986
  Searching and Pebbing.
- **Ellis, Sudborough and Turner**. Inf. and Comp., 1994
  The vertex separation and search number of a graph

$\mathbf{s}_{visible}(G) = \mathbf{tw}(G) + 1$

- **Seymour and Thomas**, J. Combin. Theory, 1993
  Graph searching and min-max theorem for treewidth.

Nondeterministic Graph Searching

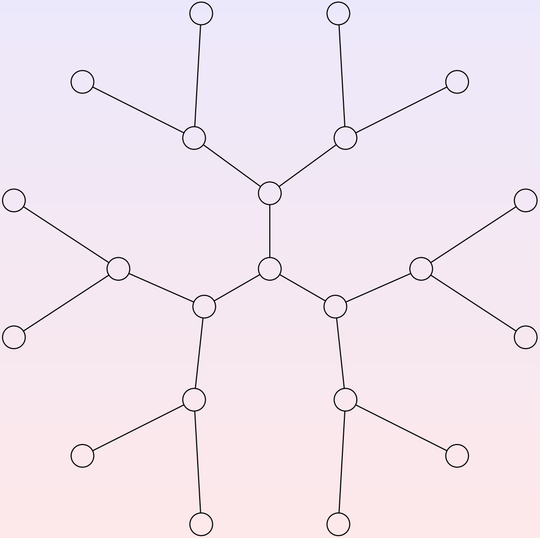
Query operation

- oracle ;
- tradeoff between number of searchers and number of query steps ;
- q-limited nondeterministic search number, $s_q(G)$.

Link with graph searching

- $s(G) = s_0(G)$;
- $s_{visible}(G) = s_\infty(G)$.

Limited Graph searching with 2 queries



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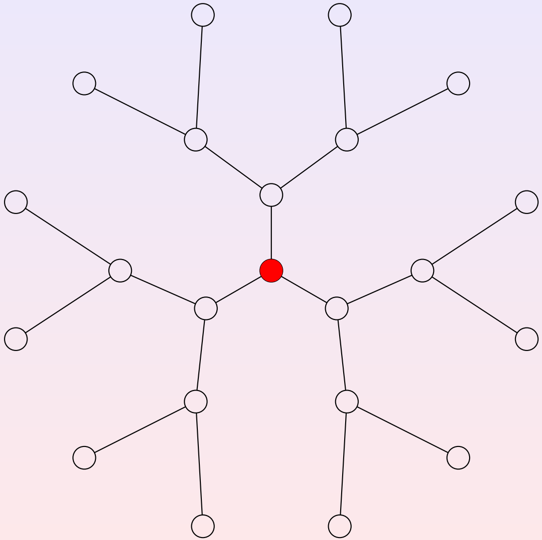

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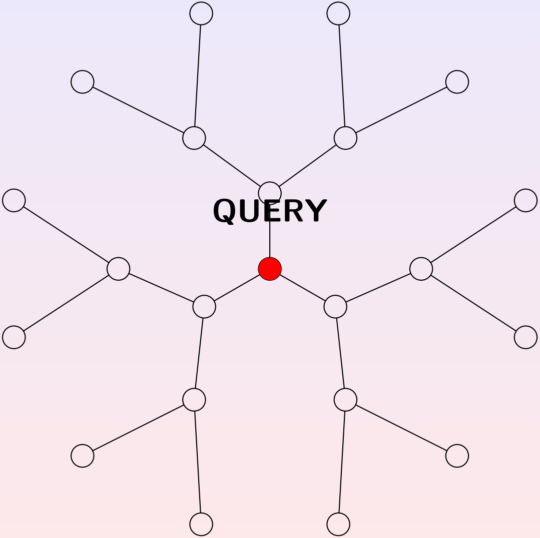

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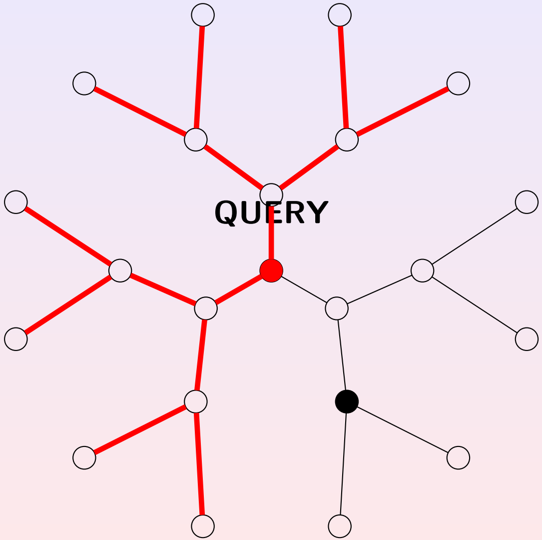

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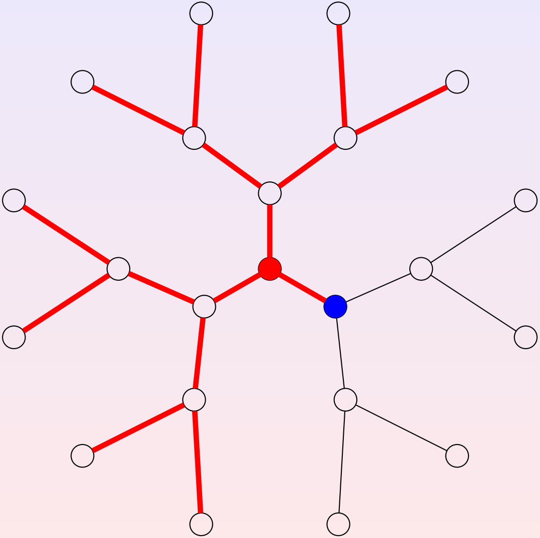

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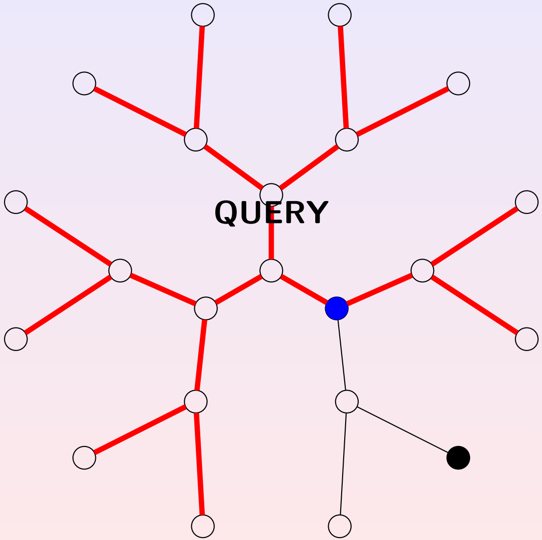

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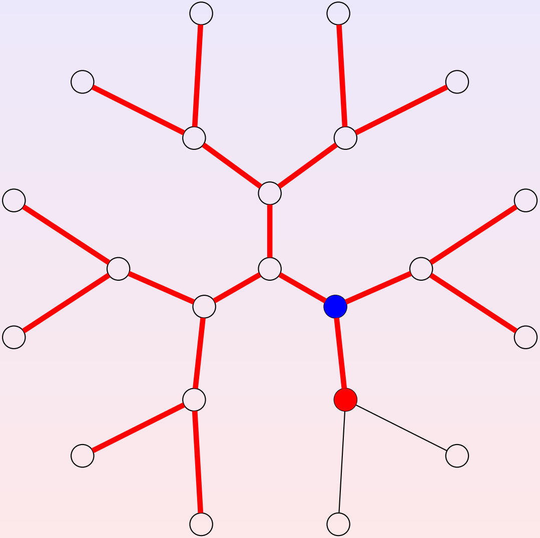
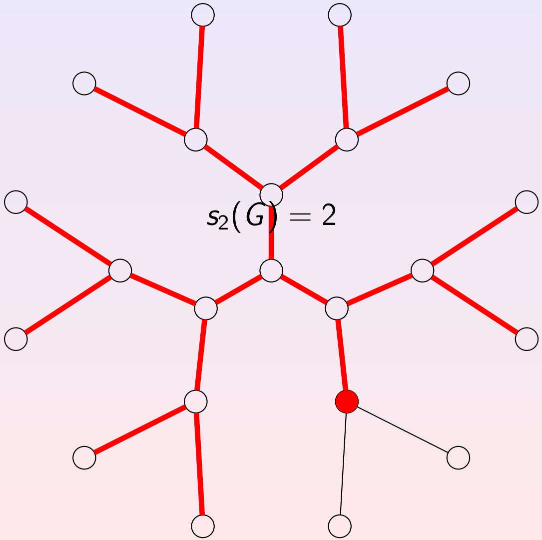
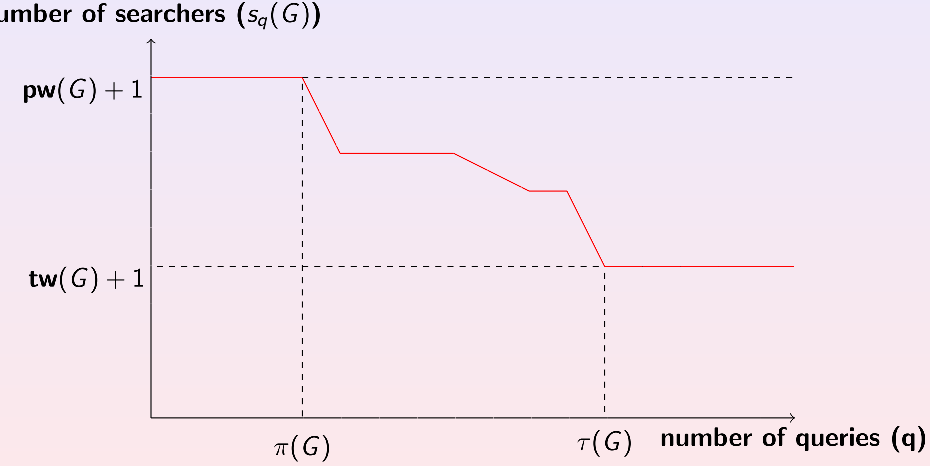


Limited Graph searching with 2 queries



Controlled Amount of Nondeterminism



Branched Tree Decomposition

q-branched treewidth, $tw_q(G)$.

- rooted tree decomposition ;
- branching node (at least two children) ;
- every path from the root to a leaf contains at most q branching nodes.

- path decomposition = 0-branched tree decomposition $\mathbf{pw}(G) = \mathbf{tw}_0(G)$;
- tree decomposition = ∞-branched tree decomposition $\mathbf{tw}(G) = \mathbf{tw}_\infty(G)$;

Main Results (1)

Branched Treewidth vs. Limited Graph Searching

Theorem 1 : For any $q \geq 0$, for any graph G,
$\mathbf{s}_q(G) = \mathbf{tw}_q(G) + 1$.

NP-Completeness

Computing whether $\mathbf{tw}_q(G) \leq k$ is NP-complete for any q.

Main Results (2)

Exact Exponential Algorithm

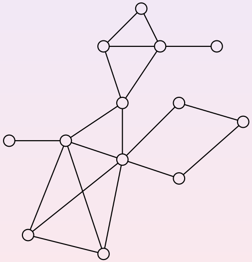
Theorem 2 : There exists an algorithm that, for any graph G and any $q \geq 0$, computes $\mathbf{tw}_q(G)$ and an optimal q-branched tree decomposition of G.

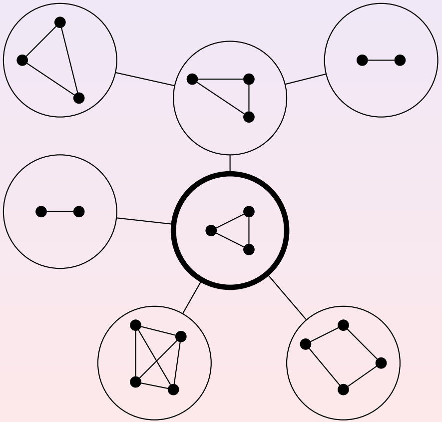
Bounding the Nondeterminism

Theorem 3 : For any $q \geq 1$, for any graph G, $\mathbf{tw}_{q-1}(G) \leq 2\ \mathbf{tw}_q(G)$.

Theorem 1 : $\mathbf{s}_q(G) = \mathbf{tw}_q(G) + 1$

- place searchers at vertices of the root ;
- perform a query at any branching node we meet.





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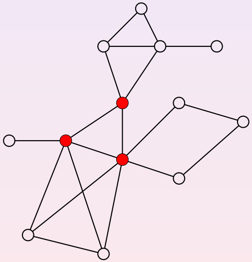

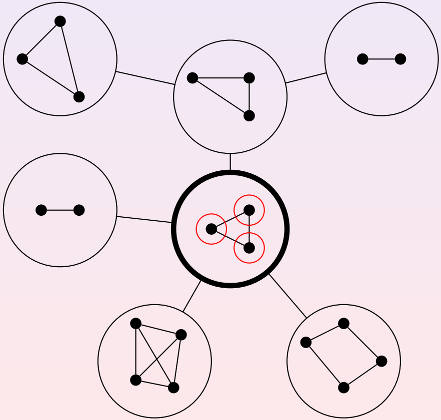

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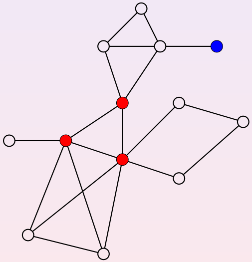

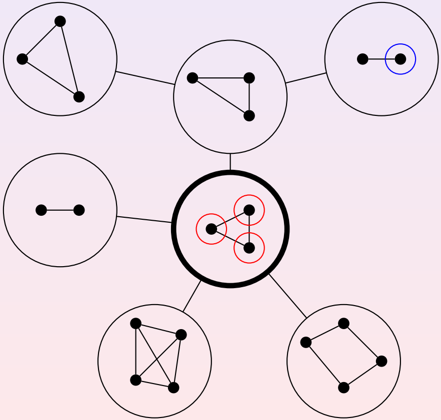

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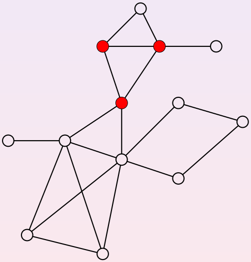

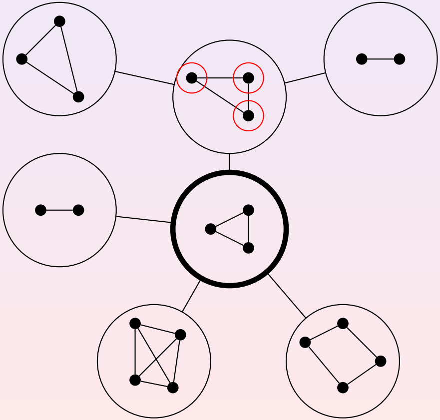

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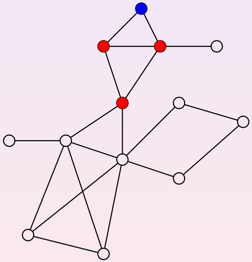
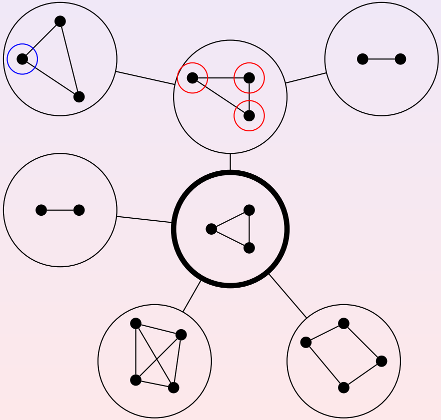




Exact Exponential Algorithm

Theorem 2

There exists an algorithm that, for any graph G and any $q \geq 0$, computes $\mathbf{tw}_q(G)$ and an optimal q-branched tree decomposition of G.

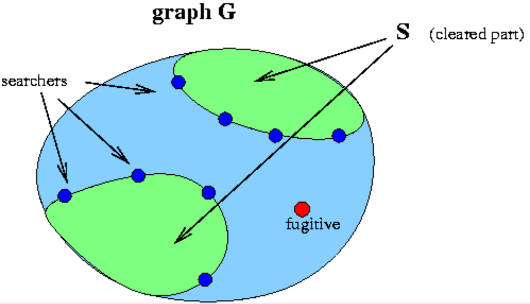
Thanks to Theorem 1, the following is sufficient :

For any k and any graph G, our algorithm computes the minimum q such that there exists a non-deterministic search strategy of G, using k searchers and q queries and the corresponding strategy.

Exact Exponential Algorithm

Configuration digraph H

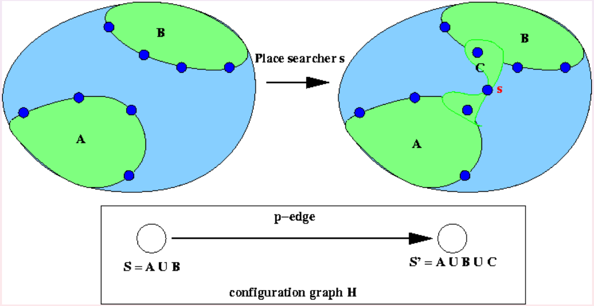
- a **vertex** S of H is a *set of clear vertices* of G *reachable* by a search program using k searchers.
  $V(H) = \{S \subseteq V(G)\ s.t.\ |\delta(S)| \leq k\}$;



Exact Exponential Algorithm

Configuration digraph H

- a **directed edge** $\{S, S'\}$ of H is a *search step* that allows to reach S' from S.
  place edges and *query* edges ;



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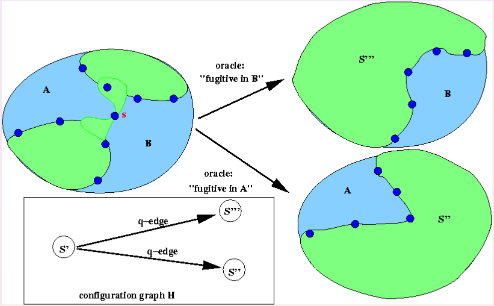
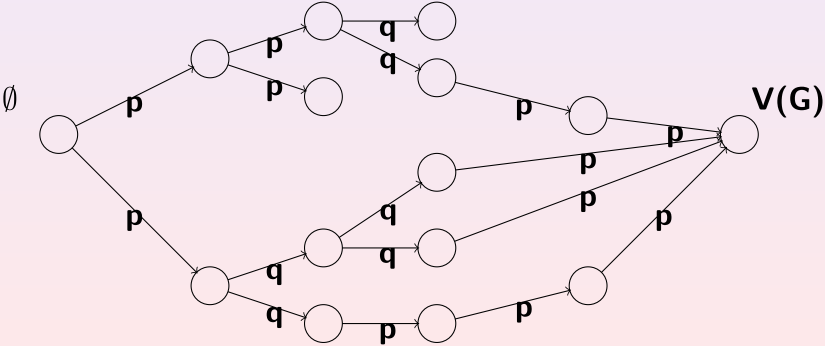


Exact Exponential Algorithm

Principle of the Algorithm

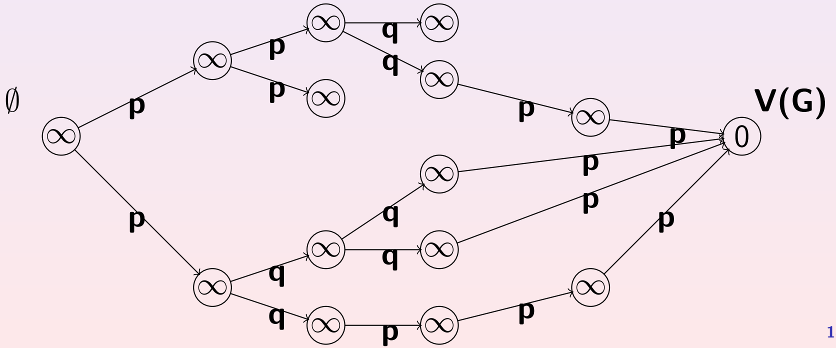
- To find a path in H from $S = \emptyset$ to $S = V(G)$. Correspondence with a search strategy.
- Labelling of vertices of the configuration graph.



Exact Exponential Algorithm

Principle of the Algorithm

- *label*(S)= number of queries required to reach $V(G)$;
- Propagation of the labelling from $label(V(G)) = 0$;
  $label(\emptyset)$ is the minimum q such that $\mathbf{s}_q(G) = k$;



Exact Exponential Algorithm

Rules of labelling

- (S, S') p-edge : $label(S) = \min(label(S), label(S'))$;
- (S, S_i) q-edges :
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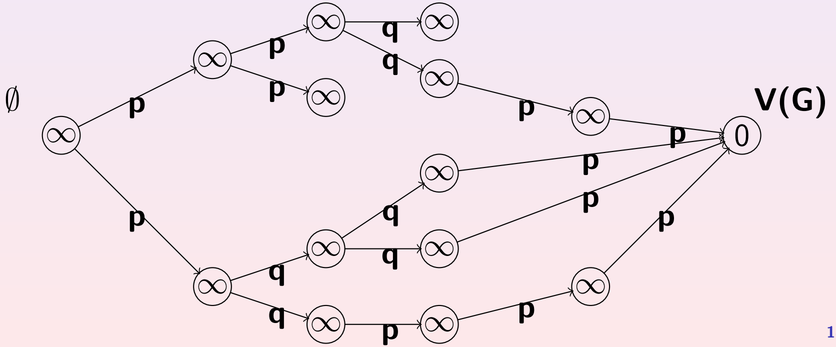

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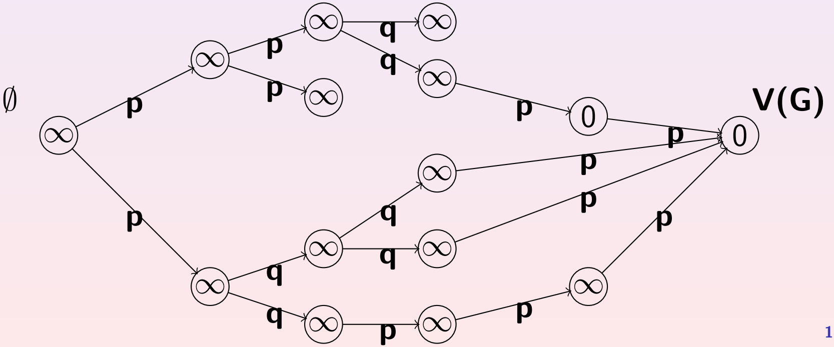

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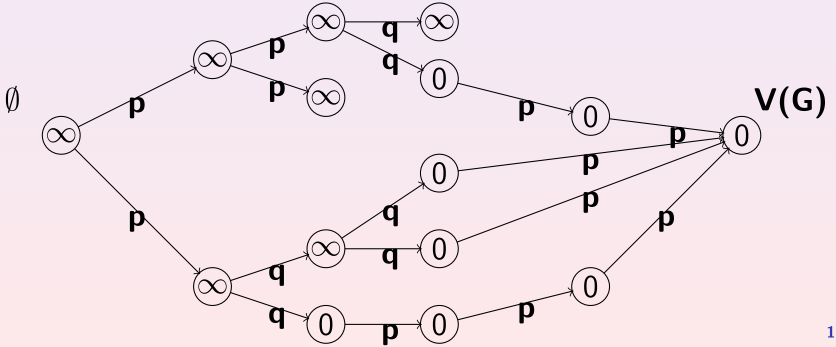

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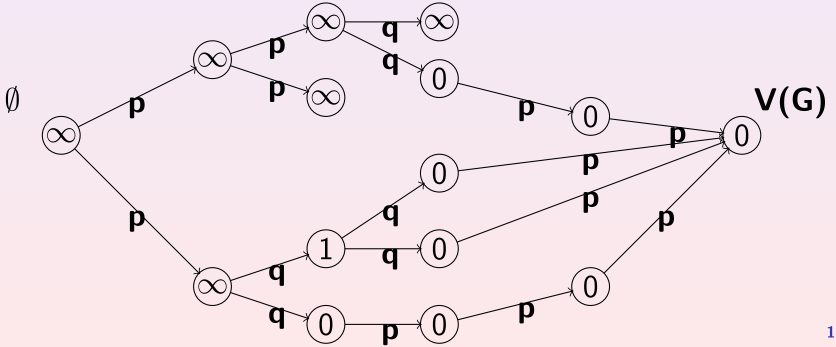

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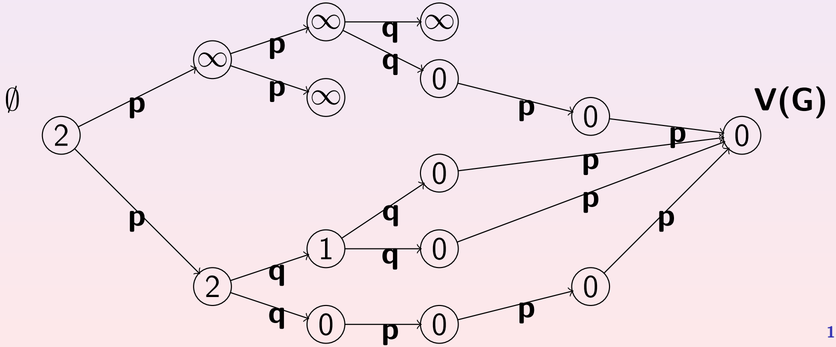

Conclusion and Further Work

New variant of search game

- nondeterministic search game ;
- unify pathwidth and treewidth.

Open Problems

- role of recontamination ;
- design of an $O(c^n)$-time exact algorithm ($c < 2$**) ;**
- **design of a linear algorithm for trees.**