Cop and robber games when the robber can hide and ride

Jérémie Chalopin[1] Victor Chepoi[1] Nicolas Nisse[2] Yann Vaxès[1]

[1] Lab. Informatique Fondamentale, Univ. Aix-Marseille, CNRS, Marseille, France

[2] MASCOTTE, INRIA, I3S, CNRS, UNS, Sophia Antipolis, France

8^{th} French Combinatorial Conf., Orsay, June 28^{th} 2010

Cops & robber games [Nowakowski and Winkler; Quilliot, 83]

Initialization:

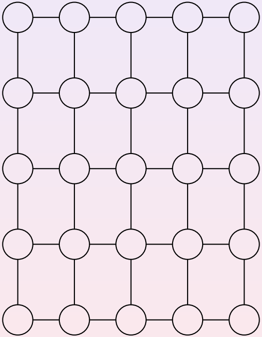
1. $\mathcal{C}$ places the cops;
2. $\mathcal{R}$ places the robber.

Step-by-step:

- each cop traverses at most 1 edge;
- the robber traverses at most 1 edge.

Robber captured:

A cop occupies the same vertex as the robber.



Cops & robber games [Nowakowski and Winkler; Quilliot, 83]

Initialization:

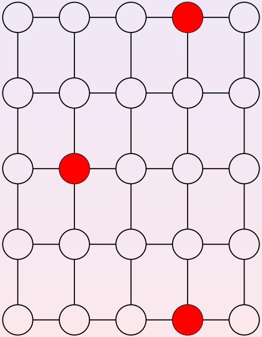
1. $\mathcal{C}$ places the cops;
2. $\mathcal{R}$ places the robber.

Step-by-step:

- each cop traverses at most 1 edge;
- the robber traverses at most 1 edge.

Robber captured:

A cop occupies the same vertex as the robber.



Cops & robber games [Nowakowski and Winkler; Quilliot, 83]

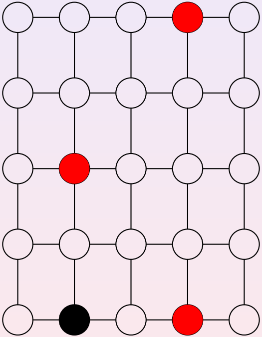
Initialization:

1. $\mathcal{C}$ places the cops;
2. $\mathcal{R}$ places the robber.

Step-by-step:

- each cop traverses at most 1 edge;
- the robber traverses at most 1 edge.

Robber captured:
A cop occupies the same vertex as the robber.



Cops & robber games [Nowakowski and Winkler; Quilliot, 83]

Initialization:

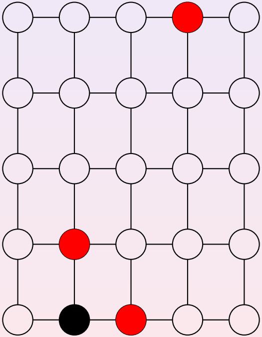
1. $\mathcal{C}$ places the cops;
2. $\mathcal{R}$ places the robber.

Step-by-step:

- each cop traverses at most 1 edge;
- the robber traverses at most 1 edge.

Robber captured:

A cop occupies the same vertex as the robber.



Cops & robber games [Nowakowski and Winkler; Quilliot, 83]

Initialization:

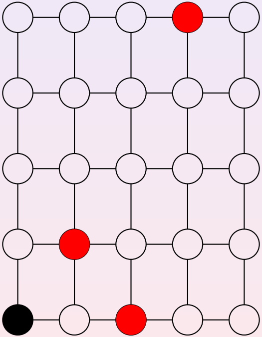
1. $\mathcal{C}$ places the cops;
2. $\mathcal{R}$ places the robber.

Step-by-step:

- each cop traverses at most 1 edge;
- the robber traverses at most 1 edge.

Robber captured:

A cop occupies the same vertex as the robber.



Cops & robber games [Nowakowski and Winkler; Quilliot, 83]

Initialization:

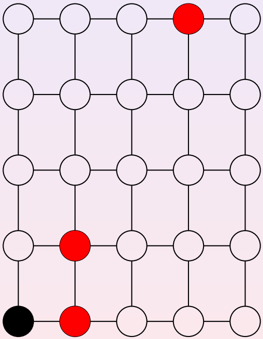
1. $\mathcal{C}$ places the cops;
2. $\mathcal{R}$ places the robber.

Step-by-step:

- each cop traverses at most 1 edge;
- the robber traverses at most 1 edge.

Robber captured:

A cop occupies the same vertex as the robber.



Cops & robber games [Nowakowski and Winkler; Quilliot, 83]

Initialization:

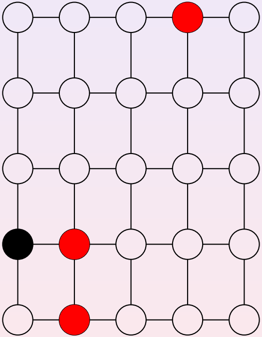
1. $\mathcal{C}$ places the cops;
2. $\mathcal{R}$ places the robber.

Step-by-step:

- each cop traverses at most 1 edge;
- the robber traverses at most 1 edge.

Robber captured:

A cop occupies the same vertex as the robber.



Cops & robber games [Nowakowski and Winkler; Quilliot, 83]

Initialization:

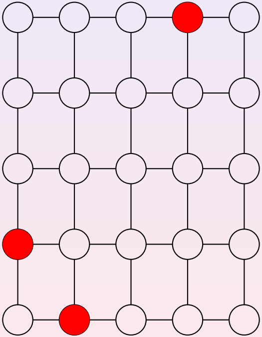
1. $\mathcal{C}$ places the cops;
2. $\mathcal{R}$ places the robber.

Step-by-step:

- each cop traverses at most 1 edge;
- the robber traverses at most 1 edge.

Robber captured:

A cop occupies the same vertex as the robber.



Cop number

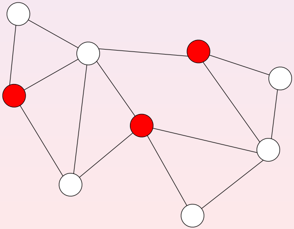
Easy cases for the Cops

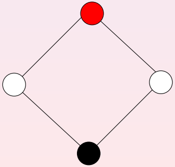
- n cops in n-node graphs
- k cops in a graph with dominating set $\leq k$

$\longrightarrow$ Robber's dead

An easy case for the Robber

in a C_4 against a single cop





Cop number

Easy cases for the Cops

- n cops in n-node graphs
- k cops in a graph with dominating set $\leq k$

$\rightarrow$ Robber's dead

An easy case for the Robber

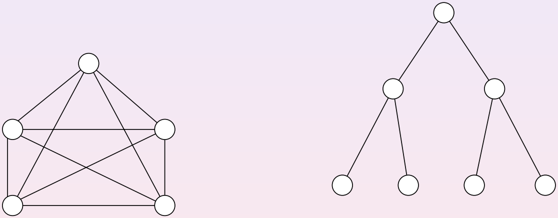
in a C_4 against a single cop

Minimize the number of Cops

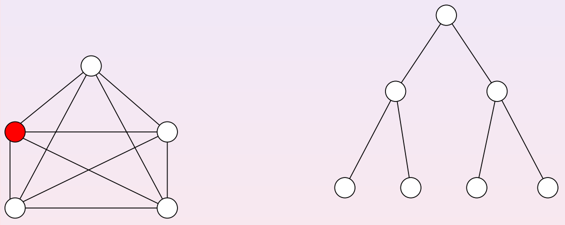
Capture the robber using as few cops as possible
Given a graph G: the minimum called cop-number, **cn**(G).

Simple Examples

Cop number in cliques and trees

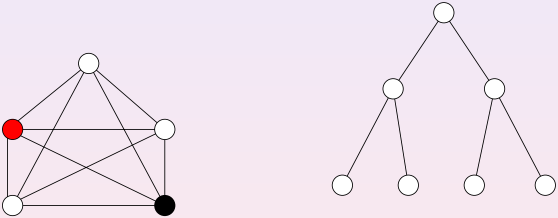


Cop number in cliques and trees



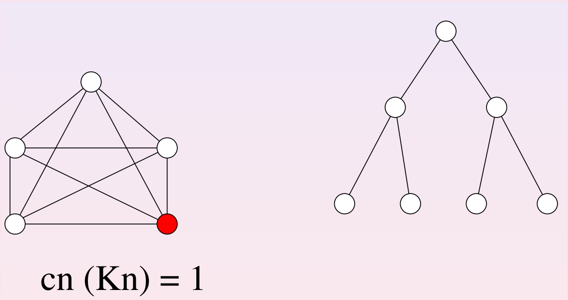
Simple Examples

Cop number in cliques and trees



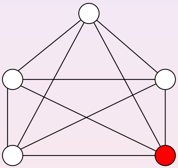
Simple Examples

Cop number in cliques and trees

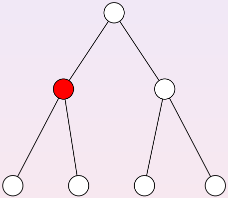


Simple Examples

Cop number in cliques and trees

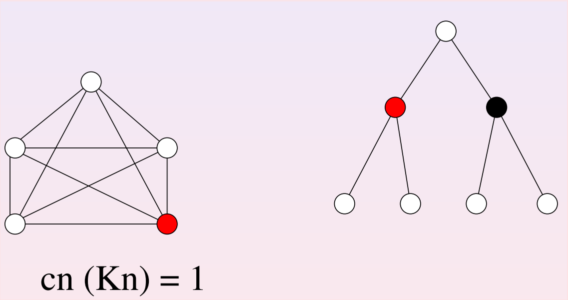


$\mathrm{cn}\ (K_n) = 1$



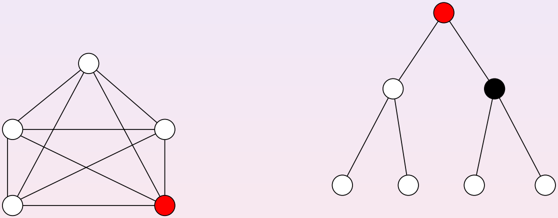
Simple Examples

Cop number in cliques and trees



Simple Examples

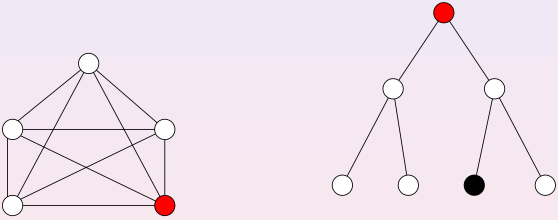
Cop number in cliques and trees



cn (Kn) = 1

Simple Examples

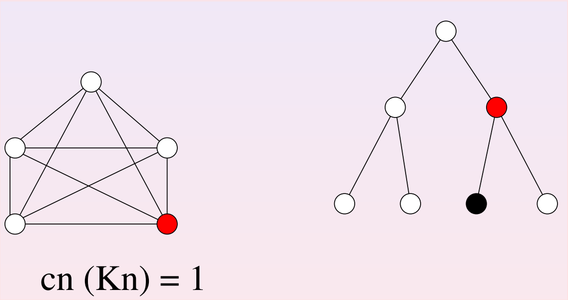
Cop number in cliques and trees



cn (Kn) = 1

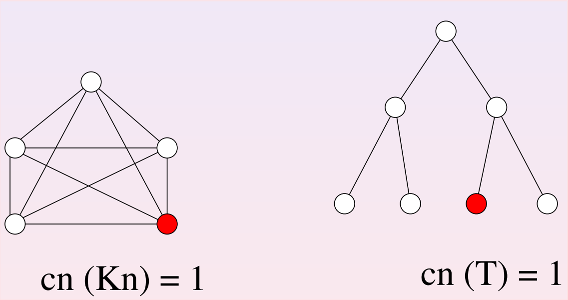
Simple Examples

Cop number in cliques and trees



Simple Examples

Cop number in cliques and trees



State of art: characterization and complexity

- **Characterization** of cop-win graphs $\{G \mid \mathbf{cn}(G) = 1\}$.
  [Nowakowski & Winkler, 83; Quilliot, 83; Chepoi, 97]

- **Algorithms:** $O(n^k)$ to decide if $\mathbf{cn}(G) \leq k$.
  [Hahn & MacGillivray, 06]

- **Complexity:** Computing the cop-number is EXPTIME-complete. [Goldstein & Reingold, 95]
  - in directed graphs;
  - in undirected graphs if initial positions are given.

State of art: lower bound

- For any graph G with girth ≥ 5 and min degree $\geq d$, $cn(G) \geq d$. [Aigner & Fromme, 84]
- $\mathbf{cn}(G) \geq d^t$, where
  $d+1 =$ minimum degree, girth $\geq 8t-3$. [Frankl, 87]
  ($\Rightarrow$ there are n-node graphs G with $\mathbf{cn}(G) \geq \Omega(\sqrt{n})$)
- For any k, n, it exists a k-regular graph G with $cn(G) \geq n$ [Andreae, 84]

State of art: upper bound

- **Planar graph** G: $\mathbf{cn}(G) \leq 3$.
  [Aigner & Fromme, 84]
- **Bounded genus graph** G with genus g:
  $cn(G) \leq 3/2g + 3$ [Schröder, 01]
- **Minor free graph** G excluding a minor H:
  $cn(G) \leq |E(H \setminus \{x\})|$, where x is any non-isolated vertex of H [Andreae, 86]
- **General upper bound** For any connected graph G,
  $cn(G) \leq O(n/log(n))$ [Chiniforooshan, 08]
  recently improved [Lu & Peng 09, Scott & Sudakov 10]

State of art: upper bound

- **Planar graph** G: $\mathbf{cn}(G) \leq 3$.
  [Aigner & Fromme, 84]
- **Bounded genus graph** G with genus g:
  $cn(G) \leq 3/2g + 3$ [Schröder, 01]
- **Minor free graph** G excluding a minor H:
  $cn(G) \leq |E(H \setminus \{x\})|$, where x is any non-isolated vertex of H [Andreae, 86]
- **General upper bound** For any connected graph G,
  $cn(G) \leq O(n/log(n))$ [Chiniforooshan, 08]
  recently improved [Lu & Peng 09, Scott & Sudakov 10]

Conjecture: For any connected graph G, $cn(G) \leq O(\sqrt{n})$.

Faster protagonists [Fomin,Golovach,Kratochvil,N.,Suchan, TCS 2010]

Speed $=$ max number of edges traversed in 1 step: $speed_{\mathcal{R}} \geq speed_{\mathcal{C}} = 1$
$\mathbf{cn}_s(G)$ min number of cops to capture a robber with speed s in G

Computational hardness

Computing $\mathbf{cn}_s$ for any $s \geq 1$ is NP-hard; the parameterized version is $W[2]$-hard. For $s \geq 2$, it is true already on split graphs.

Fast robber in interval graphs

robber with speed $s \geq 1$, $\mathbf{cn}_s(G) \leq function(s)$
$\Rightarrow$ algorithm in time $O(n^{function(s)})$

Cop-number is unbounded in planar graphs

$\forall s > 1$, $\forall n$: then $\mathbf{cn}_s(Grid_n) = \Omega(\sqrt{\log n})$.
$\forall H$ planar with an induced subgraph $Grid_{\Omega(2^{k^2})}$, $\mathbf{cn}(H) \geq k$.

Three variants we consider

When cops and robber can ride

$s = speed_{\mathcal{R}} \geq speed_{\mathcal{C}} = s'$

When the robber can hide (witness) [Clarke DM'08]

The robber is visible only every k steps.

When the cops can shoot (radius of capture)[BCP TCS'10]

Robber captured when at distance k from a cop.

Three variants we consider

When cops and robber can ride

$s = speed_{\mathcal{R}} \geq speed_{\mathcal{C}} = s'$ $\mathcal{CWFR}(s, s')$

When the robber can hide (witness) [Clarke DM'08]

The robber is visible only every k steps. $\mathcal{CWW}(k)$

When the cops can shoot (radius of capture)[BCP TCS'10]

Robber captured when at distance k from a cop. $\mathcal{CWRC}(k)$

Problem: Characterization of cop-win graphs

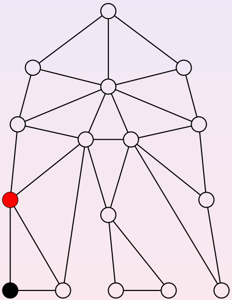
(cop-win graph: in which one cop always captures the robber)

Let us go back to a slow robber

- **Characterization** of cop-win graphs $\{G \mid \mathbf{cn}(G) = 1\}$.
  [Nowakowski & Winkler, 83; Quilliot, 83; Chepoi, 97]

Theorem: $cn(G) = 1$ iff

$V(G) = \{v_1, \cdots, v_n\}$ and
for any $i < n$, there is $j > i$ s.t.
$N[v_i] \subseteq N[v_j]$ in the subgraph
induced by $v_i, \cdots, v_n$



Trees, chordal graphs, bridged graphs (...) are cop win.

One fast cop vs. a fast robber

Theorem [Nowakowski & Winkler, 83; Quilliot, 83]

$G \in \mathcal{CWFR}(1,1)$ iff $V(G) = \{v_1, \cdots, v_n\}$, $\forall i < n$, $\exists j > i$, s.t. $N_1(v_i, G) \cap X_i \subseteq N_1(v_j, G_i)$ with $X_i = \{v_i, \cdots, v_n\}$

Theorem Characterization of $\mathcal{CWFR}(s, s')$

$G \in \mathcal{CWFR}(s, s')$ iff $V(G) = \{v_1, \cdots, v_n\}$, $\forall i < n$, $\exists j > i$, s.t. $N_s(v_i, G \setminus \{v_j\}) \cap X_i \subseteq N_{s'}(v_j)$ with $X_i = \{v_i, \cdots, v_n\}$

Cop-win graphs and hyperbolicity

Theorem — Characterization of $\mathcal{CWFR}(s, s')$

$G \in \mathcal{CWFR}(s, s')$ iff $V(G) = \{v_1, \cdots, v_n\}$, $\forall i < n$, $\exists j > i$, s.t. $N_s(v_i, G \setminus \{v_j\}) \cap X_i \subseteq N_{s'}(v_j)$ with $X_i = \{v_i, \cdots, v_n\}$

Any graph G is δ-hyperbolic for some $\delta \geq 0$
the smaller δ, the closer the metric of G is to the metric of a tree.

Theorem — Hyperbolicity helps the cop

$\forall s' > 2\delta \geq 0$, and G a δ-hyperbolic graph, $G \in \mathcal{CWFR}(2s', s' + \delta)$

Theorem — Cop-win "leads" to hyperbolicity

If $s \geq 2s'$, then any $G \in \mathcal{CWFR}(s, s')$ is $(s-1)$-hyperbolic.

Question: $\forall s > s'$, any $G \in \mathcal{CWFR}(s, s')$ is $f(s)$-hyperbolic?

One slow cop vs. a fast robber

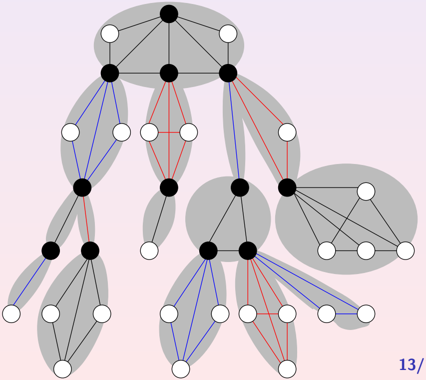
We consider $\mathcal{C}$ with speed one $\qquad \mathcal{CWFR}(s) = \mathcal{CWFR}(s,1)$

$G \in \mathcal{CWFR}(s)$ iff $V(G) = \{v_1, \cdots, v_n\}$, $\forall i < n$, $\exists j > i$,
s.t. $N_s(v_i, G \setminus \{v_j\}) \cap X_i \subseteq N_1(v_j)$ with $X_i = \{v_i, \cdots, v_n\}$

Characterization of $\mathcal{CWFR}(s)$

$G \in \mathcal{CWFR}(s)$ iff G is
Case $s = 1$**:** dismantable
Case $s = 2$**:** dually-chordal
Case $s \geq 3$**:** a "big brother graph"

G is a *big brother graph* if each block (maximal 2-connected comp.) is dominated by its articulation point with its parent-block.

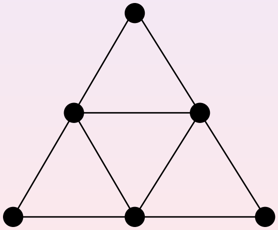


The witness version

When the robber can hide (witness) [Clarke DM'08]

The robber is visible only every k steps.

$\mathcal{CWW}(k) = \{G | \ \mathcal{C}$ wins against $\mathcal{R}$ visible every k steps $\}$



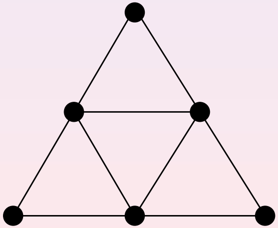
The witness version

Relationship with fast robber $\mathcal{CWFR}(s) \subseteq \mathcal{CWW}(s)$

Lemma invisibility is weaker than speed

$\forall s \geq 2$, $\mathcal{CWFR}(s) \subset \mathcal{CWW}(s)$

diameter 2, no dominating vertex, 2-connected $\Rightarrow G \notin \mathcal{CWFR}(s)$, $s \geq 2$.



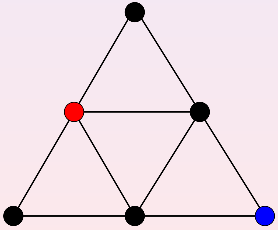
The witness version

Relationship with fast robber $\mathcal{CWFR}(s) \subseteq \mathcal{CWW}(s)$

Lemma invisibility is weaker than speed

$\forall s \geq 2$, $\mathcal{CWFR}(s) \subset \mathcal{CWW}(s)$

diameter 2, no dominating vertex, 2-connected $\Rightarrow G \notin \mathcal{CWFR}(s)$, $s \geq 2$.



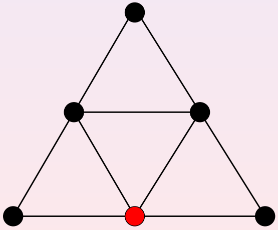
The witness version

Relationship with fast robber $\mathcal{CWFR}(s) \subseteq \mathcal{CWW}(s)$

Lemma invisibility is weaker than speed

$\forall s \geq 2$, $\mathcal{CWFR}(s) \subset \mathcal{CWW}(s)$

diameter 2, no dominating vertex, 2-connected $\Rightarrow G \notin \mathcal{CWFR}(s)$, $s \geq 2$.



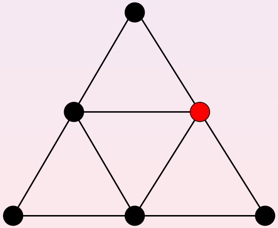
The witness version

Relationship with fast robber $\mathcal{CWFR}(s) \subseteq \mathcal{CWW}(s)$

Lemma invisibility is weaker than speed

$\forall s \geq 2$, $\mathcal{CWFR}(s) \subset \mathcal{CWW}(s)$

diameter 2, no dominating vertex, 2-connected $\Rightarrow G \notin \mathcal{CWFR}(s)$, $s \geq 2$.



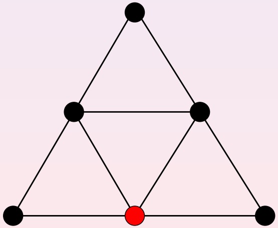
The witness version

Relationship with fast robber $\mathcal{CWFR}(s) \subseteq \mathcal{CWW}(s)$

Lemma invisibility is weaker than speed

$\forall s \geq 2$, $\mathcal{CWFR}(s) \subset \mathcal{CWW}(s)$

diameter 2, no dominating vertex, 2-connected $\Rightarrow G \notin \mathcal{CWFR}(s)$, $s \geq 2$.



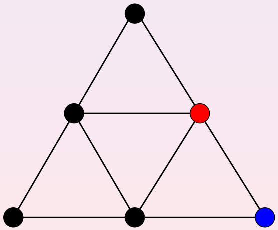
The witness version

Relationship with fast robber $\mathcal{CWFR}(s) \subseteq \mathcal{CWW}(s)$

Lemma invisibility is weaker than speed

$\forall s \geq 2$, $\mathcal{CWFR}(s) \subset \mathcal{CWW}(s)$

diameter 2, no dominating vertex, 2-connected $\Rightarrow G \notin \mathcal{CWFR}(s)$, $s \geq 2$.



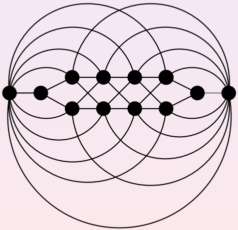
$G \in \mathcal{CWW}(s)$ importance of edge-separator

The witness version

Recall that more speed does not help agains one speed-1 cop
$\forall s \geq 3, \mathcal{CWFR}(3) = \mathcal{CWFR}(s)$ here: different behaviour

Lemma less visibility helps $\mathcal{R}$

$\forall k \geq 1$, there are graphs in $\mathcal{CWW}(k) \setminus \mathcal{CWW}(k+1)$



Question: $\mathcal{CWW}(k+1) \subset \mathcal{CWW}(k)$?

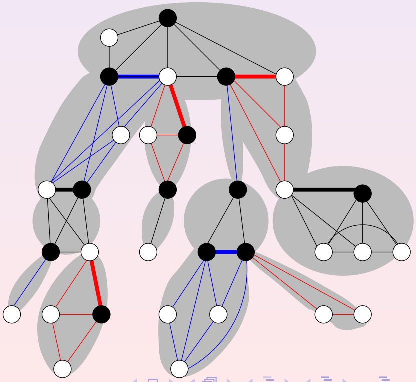
The big two-brother graphs

$\mathcal{CWW} = \{G | \forall k, \mathcal{C} \text{ wins vs. } \mathcal{R} \text{ visible every } k \text{ steps } \} = \bigcap_k \mathcal{CWW}(k)$

Theorem

$\mathcal{CWW}$ is the class of the big two-brother graphs

G is a big two-brother graph if $\exists y \in V$ or $xy \in E$ s.t. x or y dominated a connected comp. C of $G \setminus \{x, y\}$ and $G \setminus C$ is a big two-brother graph



$\mathcal{CWW}(2)$

Lemma — Necessity

If $G \in \mathcal{CWW}(2)$ **then**

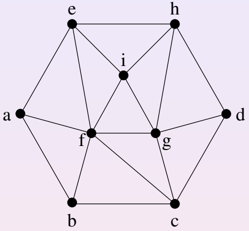
$V = \{v_1, \cdots, v_n\}$, and $\forall i$, $\exists xy \in E(G_{i+1})$ (possibly $x = y$) $N_2(v_i, G \setminus xy) \cap G_i \subseteq N_1(y)$

Lemma — Sufficiency

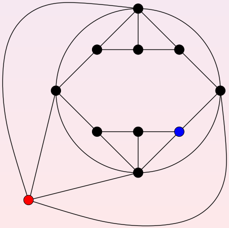
If $V = \{v_1, \cdots, v_n\}$, and $\forall i$, $\exists xy \in E(G_{i+1})$ (possibly $x = y$) $N_2(v_i, G \setminus xy) \cap G_i \subseteq N_1(y)$
and, if $x \neq y$, then
$N_2(v_i, G \setminus y) \cap G_i \subseteq N_2(x, G \setminus y)$

then $G \in \mathcal{CWW}(2)$

Not sufficient



Not necessary



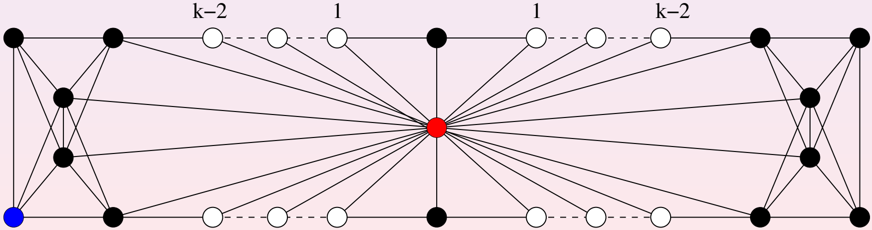
$\mathcal{CWW}(k)$ when k odd

Lemma — Sufficiency for any k odd

If $V = \{v_1, \cdots, v_n\}$, and $\forall i$, $\exists xy \in E(G_{i+1})$ (possibly $x = y$) $N_k(v_i, G \setminus xy) \cap G_i \subseteq N_1(y)$

then $G \in \mathcal{CWW}(k)$

But it is not necessary...



Finally, let's advantage the cop :(

When the cops can shoot (radius of capture)[BCP TCS'10]

Robber captured when at distance k from a cop.

$\mathcal{CWRC}(k) = \{G | \ \mathcal{C}$ wins when capturing at dist. $\leq k\}$

Theorem bipartite graphs in $\mathcal{CWRC}(1)$

A bipartite graph G is in $\mathcal{CWRC}(1)$ iff $V = \{v_1, \cdots, v_n\}$ s.t. $\{v_{n-1}, v_n\} \in E$ and $\forall i, \exists j > i$, $\{v_j, v_i\} \notin E$ and $N(v_i, G_i) \subseteq N_1(v_j)$

Characterization of $\mathcal{CWRC}(k)$ seems harder, even for $k = 1$...

Perspectives

In case $speed_{\mathcal{R}} = speed_{\mathcal{C}} = 1$

- G of **genus** $g \Rightarrow \mathbf{cn}(G) \leq \frac{3}{2}g + 3$. [Schröder, 01]
  Conjecture: G of genus $g \Rightarrow \mathbf{cn}(G) \leq g + 3$.
- General upper bound for cn ?
  Conjecture: $cn(G) \leq O(\sqrt{n})$.

In case $speed_{\mathcal{R}} > speed_{\mathcal{C}}$

- $\Omega(\sqrt{\log(n)}) \leq \mathbf{cn}(Square_n) \leq O(n)$. Exact value?
- What about other graphs'classes ?

Full characterization of cop-win graphs when witness?
Characterization of cop-win graphs when the cop can "shoot"?