Tree-decompositions with metric properties on the bags

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Based on works with D. Coudert, G. Ducoffe, A. Kosowski, S. Legay, B. Li and K. Suchan

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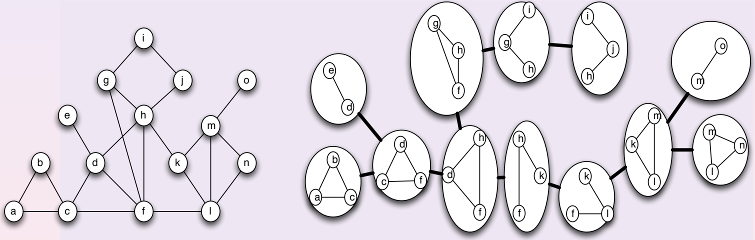
Tree/Path-Decompositions [Robertson and Seymour 83]

Representation of a graph as a Tree preserving connectivity properties



Tree T + family $\mathcal{X} = (X_t)_{t \in V(T)}$ of "bags" (set of vertices of G)
Important: intersection of two adjacent bags = separator of G

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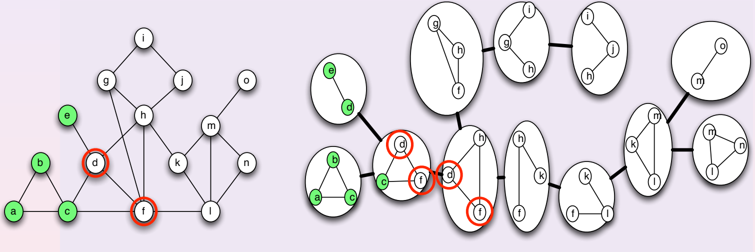

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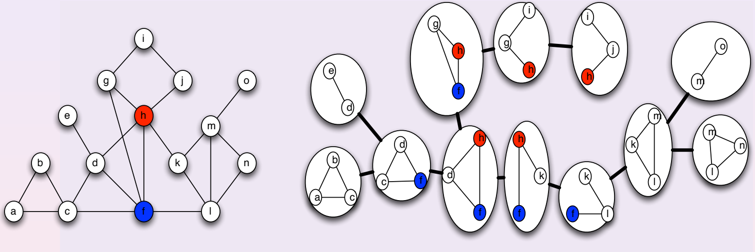

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Important: intersection of two adjacent bags = separator of G

- $\bigcup_{t \in V(T)} X_t = V(G)$;
- for any $uv \in E(G)$, there exists a bag X_t containing u and v;
- for any $v \in V(G)$, $\{t \in V(T) \mid v \in X_t\}$ induces a subtree.

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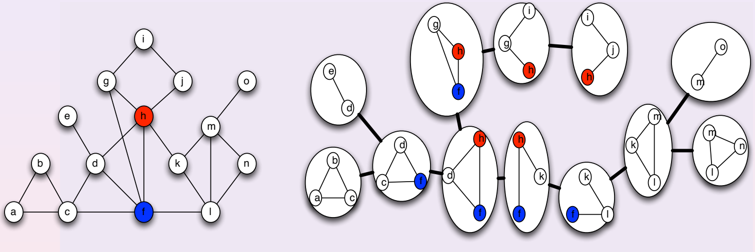

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Width of $(T, \mathcal{X})$: size of largest bag (minus 1)
Treewidth of a graph G, $tw(G)$: min width over all tree-decompositions.

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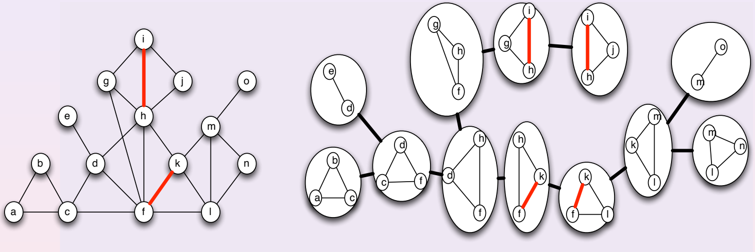

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Tree-decomposition $\Leftrightarrow$ (clique tree of) triangulation of G (bags are maximal cliques)
Treewidth+1 $\Leftrightarrow$ Minimum $\omega(H)$ among any chordal supergraph H of G

Many important Algorithmic Applications of *tw*

Brief reminder on Computational Complexity

P. Class of Problems that can be solved by a **deterministic** algorithm in **polynomial-time** $O(n^c)$ (in the size n of the input)

NP. Class of Problems that can be solved by a **non deterministic** algorithm in **polynomial-time** $O(n^c)$ (in the size n of the input)

NP-hard. Class of Problems "**as hard as the hardest problems in NP**". Essentially: cannot be solved in polynomial-time unless P=NP

FPT. (**Fixed Parameter Tractable**) Unformally: $k \in \mathbb{N}$ be a **fixed** parameter. Let G be an input of size n, and Π a graph property

$\Pi(G) \leq n$? is FPT if it can be solved by a deterministic algorithm in time $f(k)n^c$ for some computable function f.

example: vertex cover (NP-hard). $vc(G) \leq k$? solvable in time:

- $2^{O(n)}$ if k is part of the input
- $O(2^k n)$ if k is fixed.

Many important Algorithmic Applications of *tw*

- cornerstone of Graph Minors Theorem [Robertson and Seymour 1983-2004]

  $\Rightarrow$ any graph property ($\Pi(G) \leq k$) that is closed under minor is FPT in k

- problems expressible in MSOL solvable in polynomial time in graphs of bounded treewidth (dynamic programming) [Courcelle, 90]

  any such problem is FPT in *tw*

- design of sub-exponential algorithms in some graph classes (e.g., planar, bounded genus, H-minor-free...)

  (**bi-dimensionality**) [Demaine *et al.* 04]

- design of FPT algorithms (meta-kernelization/protrusions)

  [Fomin *et al.* 09]

Main Problem: Computing tree-decomposition

Deciding if $tw(G) \leq k$? $\Rightarrow$ Very hard!

Related Work

Exact algorithms

- NP-hard if k part of the input [Arnborg,Corneil,Prokurowski 87]
- FPT: algorithm in $O(2^{k^3} n)$ [Bodlaender,Kloks 96]
- "practical" algorithms only for graph with treewidth ≤ 4 e.g., [Sanders 96]
- Branch & Bound algorithms (for small graphs) [Bodlaender *et al.* 12] [Coudert,Mazauric,N. 14]

Approximation algorithms

- 5-approximation in time $O(2^k n)$ [Bodlaender *et al.* 13]
- $\sqrt{\log OPT}$-approximation in polynomial-time (SDP) [Feige *et al.* 05]
- assuming Small Set Expansion Conjecture, no poly-time constant-ratio approximation [Wu,Austrin,Pitassi,Liu 14]
- 3/2-approximation in planar graphs (*complexity?*) [Seymour,Thomas 93]

Heuristics

- Mainly based on local complementations of edges (minimum fill-in: perfect elimination ordering of vertices) [Bodlaender *et al.*]

In this talk ⇒ We want to compute it anyway!

Instead of constraining the size of bags ⇒ constraint bags' properties

1. bags' structure
2. bags' diameter: treelength
3. bags' radius: tree-breadth

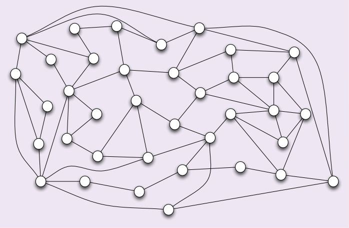
and relationships between them...

First alternative: constraint bags' structure

Simple algorithm

[Kosowski,Li,N.,Suchan, Algorithmica 2015]

- increase an induced path P
  until $N[P]$ (nodes of P and their neighbors) separates the graph
- "apply recursively to the connected components"



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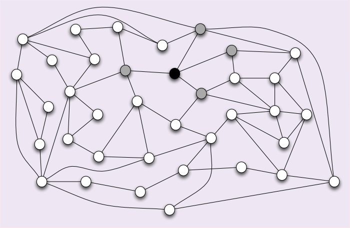

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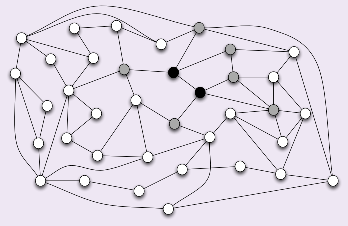

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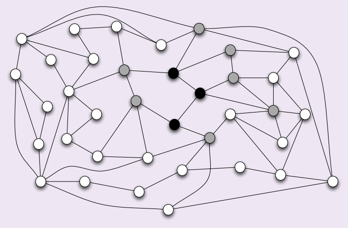

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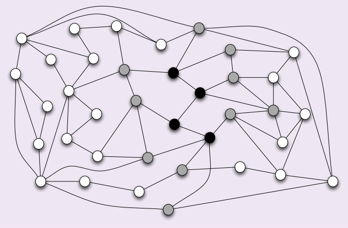

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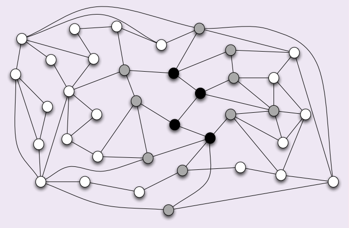

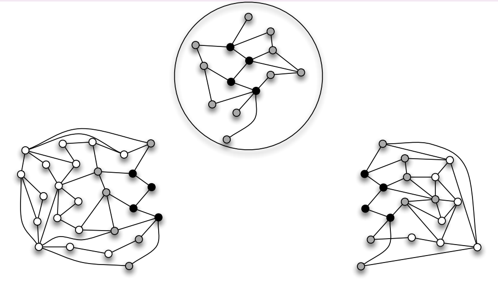

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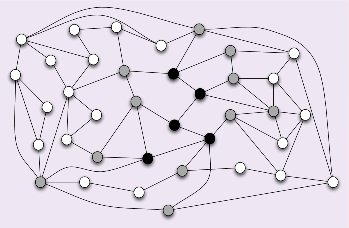

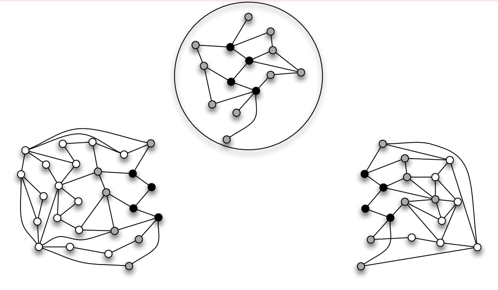

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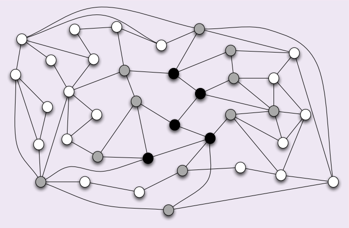

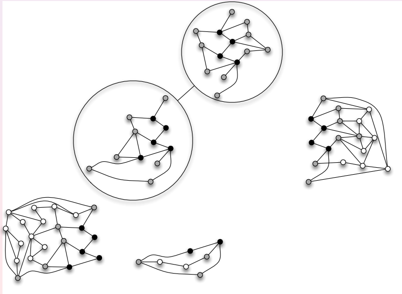

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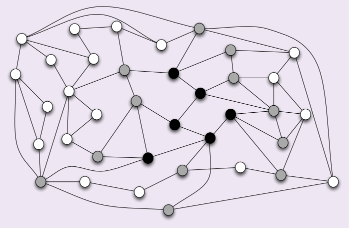

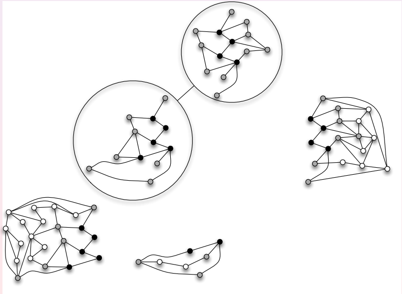

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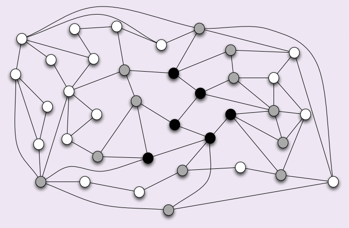
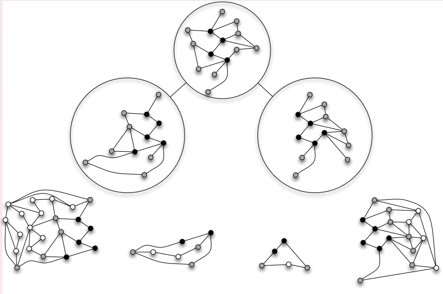




First alternative: constraint bags' structure

chordality, $ch(G)$: max. length of induced cycle in G
k-chordal $\Leftrightarrow ch(G) \leq k$

Bags have *induced dominating path*

[Kosowski,Li,N.,Suchan, Algorithmica 2015]

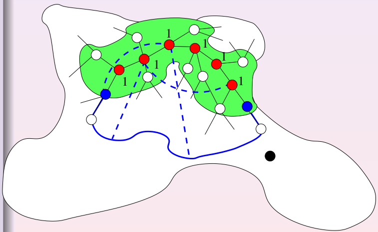
- efficient algorithm $O(m^2)$ in m-edge graphs

  based on DFS from u_0 such that all paths from u_0 are induced

  - either returns an induced cycle larger than k,
  - or compute a tree-decomposition with each bag being the closed neighborhood of an induced path of length $\leq k-1$.

- achieves a tree-decomposition of width $O(\Delta k)$ for k-chordal graphs

(improves [Bodlaender,Thilikos'97] result)



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- **further work:** implement it and analyze its performance in practice

  How to improve it? (guide the DFS)

Bags have *bounded chromatic number* [Seymour 16]

using similar techniques: triangle-free k-chordal graphs have a tree-decomposition with bags with chromatic number $O(k)$

2nd approach: constraint bags' metric properties

Treelength [Dourisboure,Gavoille 07]

Length of $(T, \mathcal{X})$: largest diameter (in G) of a bag $\max_{u,v \in B} dist_G(u,v)$
Treelength of a graph G, $t\ell(G)$: min length over all tree-decompositions.

Applications

- upper bound on the hyperbolicity
- PTAS for TSP in bounded tree-length graphs [Krauthgamer,Lee 06]
- compact routing, greedy routing with low additive stretch [Dourisboure 05] [Boguna,Papadopoulos,Krioukov 10]

2nd approach: constraint bags' metric properties

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Treelength of a graph G, $t\ell(G)$: min length over all tree-decompositions.

Complexity of tree-length and tree-breadth

Bad news

- NP-hard to decide if $t\ell(G) \leq 2$ (not FPT) [Lokshtanov 10]

Good news

- efficient 3-approximation for $t\ell(G)$ in time $O(nm)$ (using a kind of LexBFS: Lex M) [Dourisboure,Gavoille 07]
- **open question:** better approx?

Approximation for tree-length

[Dourisboure,Gavoille 07]

LexM "kind of LexBFS" [Rose,Tarjan,Lueker 76]
with specific tie-break rules

LexM returns:

- ordering $(v_1, \cdots, v_n)$ of V
- "BFS" rooted in v_1
  $\forall i$, $\exists$ a shortest path from v_1 to v_i with all internal nodes $< i$
- minimal triangulation H of G s.t. $v_i v_j \in E(H) \setminus E(G)$ iff there exists a path from v_i to v_j with all internal nodes $> \max\{i, j\}$

Algorithm LexM
Input: A graph $G = (V, E)$
Result: A supergraph of G: $H = (V, E' \cup E)$
begin
  Assign empty label to all vertices of G and empty set to E';
  for *i from n downto* 1 **do**
    Select:
      pick an unnumbered vertex u with largest label;
      Assign to u the number i: $\alpha(i) = u$;
    Update:
      for *each unnumbered vertex* v *such that there is a chain* $u = w_1, w_2, \ldots, w_{p+1} = v$ *wi*
      w_j *unnumbered and* $\text{label}(w_j) < \text{label}(v)$ *for all* $j \in \{2, \ldots, p\}$ **do**
        add i to label(v);
        add$\{u, v\}$ to E';
      end
  end
end

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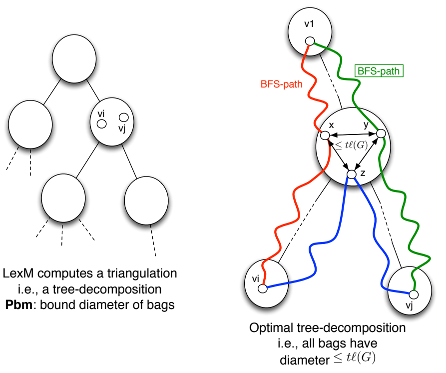

Theorem [Dourisboure,Gavoille 07]

LexM returns a tree-decomposition with each bag has diameter at most $3t\ell(G)$.

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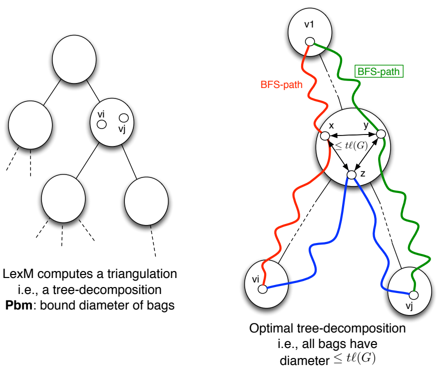

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Can approximation for tree-length be used for treewidth?

Relationship between treewidth and treelength

Treewidth and Treelength are not comparable in general.

- $2 = tw(C_n) < t\ell(C_n) = \lceil n/3 \rceil$ in any cycle C_n $(n > 6)$
- $n - 1 = tw(K_n) > t\ell(K_n) = 1$ in any clique K_n $(n > 2)$

Are these two cases (large cycles and large cliques) the only two problems?

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Are these two cases (large cycles and large cliques) the only two problems?

Somehow YES... [Coudert,Ducoffe,N., SIDMA 2016]

- $t\ell(G) \leq \lfloor \frac{\ell(G)}{2} \rfloor tw(G)$ with $\ell(G)$ the length of a largest **isometric** cycle in G
- $tw(G) = O(g^{3/2}) t\ell(G)$ with g the **genus** of G

A subgraph H of G is **isometric** if, for any $u, v \in V(H)$, $dist_H(u,v) = dist_G(u,v)$.
(i.e., the distances do not increase in H)

Sketch of $t\ell(G) \leq \lfloor \frac{\ell(G)}{2} \rfloor (tw(G) - 1)$

To prove the Theorem, first we prove

Lemma 1: For any minimal separator S of G, $diam(S) \leq \lfloor \frac{\ell(G)}{2} \rfloor (|S| - 1)$

Sketch of $t\ell(G) \leq \lfloor \frac{\ell(G)}{2} \rfloor (tw(G) - 1)$

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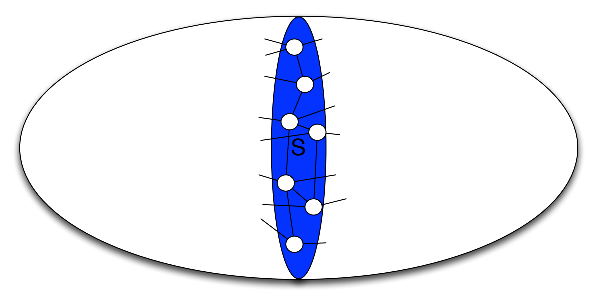
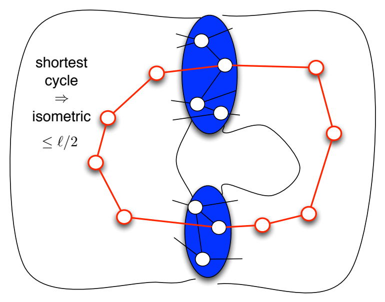


Clear if S is connected: $diam(S) \leq |S| - 1 \leq \lfloor \frac{\ell(G)}{2} \rfloor (|S| - 1)$

Sketch of $t\ell(G) \leq \lfloor \frac{\ell(G)}{2} \rfloor (tw(G) - 1)$

Lemma 1: For any minimal separator S of G, $diam(S) \leq \lfloor \frac{\ell(G)}{2} \rfloor (|S| - 1)$

For this purpose: When at least 2 components: join them by paths of length $\leq \lfloor \frac{\ell(G)}{2} \rfloor$



"easy" if S has 2 connected components

Sketch of $t\ell(G) \leq \lfloor \frac{\ell(G)}{2} \rfloor (tw(G) - 1)$

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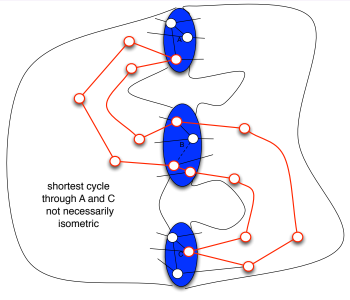

does not "work" anymore for more components :(

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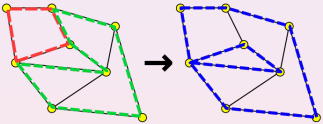
Tool:

Cycle space: $\mathcal{C}(G)$ **set of Eulerian subgraphs of** G

[folklore] Any Eulerian subgraph can be obtained as symmetric difference Δ of cycles.



Theorems:[?] $(\mathcal{C}(G), \Delta)$ is a vector space

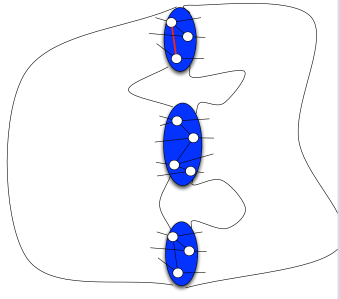
- with dimension $m - n + 1$,
- $\exists$ a basis with cycles of length $\leq \ell(G)$

Let $\mathcal{G}_\ell$ be the set of graphs such that $(\mathcal{C}(G), \Delta)$ has a basis of cycles of length $\leq \ell$.

Sketch of $t\ell(G) \leq \lfloor\frac{\ell(G)}{2}\rfloor(tw(G)-1)$

Lemma 1: For any minimal separator S of G, $diam(S) \leq \lfloor\frac{\ell(G)}{2}\rfloor(|S|-1)$

For this purpose: When at least 2 components: join them by paths of length $\leq \lfloor\frac{\ell(G)}{2}\rfloor$



lemma 2: $\mathcal{G}_\ell$ stable by contraction

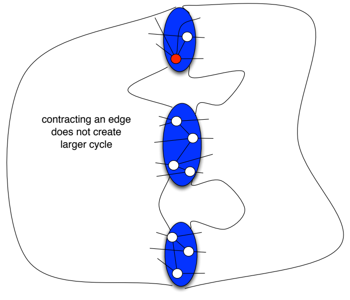
proof: $G \in \mathcal{G}_\ell$ for $\ell = \ell(G)$

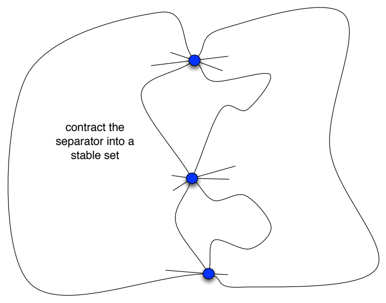
- $\mathcal{B} = C_1, \cdots, C_d$ cycle basis
  (cycles of length $\leq \ell(G)$)
  $d = dim(\mathcal{C}(G)) = m - n + 1$
- $e \in E(G)$ ($k = \#$ triangles containing e)
  $t = dim(\mathcal{C}(G/e)) = d - k$
- $C'_1, \cdots, C'_t$ obtained from $\mathcal{B}$ by contracting e and removing the triangles
- prove that $C'_1, \cdots, C'_t$ is linearly independent
  $\Rightarrow$ it is a basis of cycles of length $\leq \ell$
- $G/e \in \mathcal{G}_\ell$

Sketch of $t\ell(G) \leq \lfloor \frac{\ell(G)}{2} \rfloor (tw(G) - 1)$

Lemma 1: For any minimal separator S of G, $diam(S) \leq \lfloor \frac{\ell(G)}{2} \rfloor (|S| - 1)$

For this purpose: When at least 2 components: join them by paths of length $\leq \lfloor \frac{\ell(G)}{2} \rfloor$





By Lemma 2, contracting each component of S, the obtained graph remains in $\mathcal{G}_\ell$ and S remains a minimal separator

Sketch of $t\ell(G) \leq \lfloor \frac{\ell(G)}{2} \rfloor (tw(G) - 1)$

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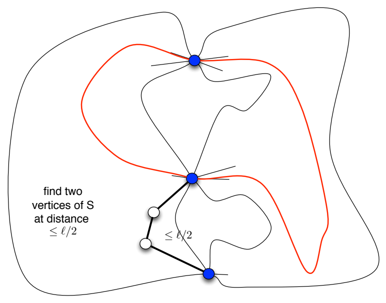

Lemma 3: If S is a minimal stable-set separator, $\exists$ 2 vertices of S at distance $\leq \lfloor \frac{\ell(G)}{2} \rfloor$

proof: take a cycle C "crossing" the separator; $C = \oplus_i C_i$, where the C_i's are cycles (of the basis) of length $\leq \ell$. One of them "crosses" the separator

Sketch of $t\ell(G) \leq \lfloor \frac{\ell(G)}{2} \rfloor (tw(G) - 1)$

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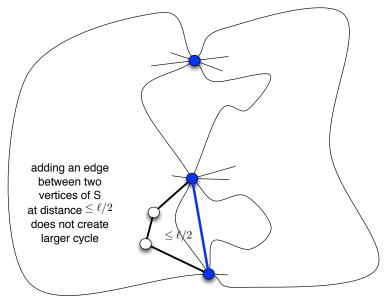

Last step of the proof

We can add an edge between 2 vertices of S at distance $\leq \ell/2$ (Lemma 3).
It does not create larger cycle (proof using cycle space) and S still a minimal separator.
# of connected components of S have been reduced $\Rightarrow$ proceed recursively

Sketch of $t\ell(G) \leq \lfloor \frac{\ell(G)}{2} \rfloor (tw(G) - 1)$

Lemma 1: For any minimal separator S of G, $diam(S) \leq \lfloor \frac{\ell(G)}{2} \rfloor (|S| - 1)$

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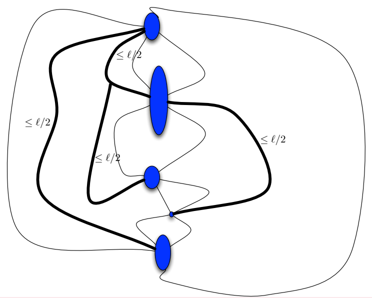


The result follows !

Sketch of $t\ell(G) \leq \lfloor \frac{\ell(G)}{2} \rfloor (tw(G) - 1)$

Lemma 1: For any minimal separator S of G, $diam(S) \leq \lfloor \frac{\ell(G)}{2} \rfloor (|S| - 1)$

Proof of the Theorem

Now take any tee-decomposition resulting from a minimal triangulation for any bag B and $u, v \in B$, if $\{u, v\} \notin E$, u and v belong to a minimal separator

[Bouchitté, Todinca 01]

Lemma 1 $\Rightarrow$ the diameter of the bag is $\leq \lfloor \frac{\ell(G)}{2} \rfloor (|B| - 1)$

Sketch of $tw(G) = O(g^{3/2})t\ell(G)$ Apply bi-dimensionality

if $tw(G)$ "small" OK

otherwise:

Theorem [Demain,Hajiaghayi,Thilikos 06]

Let G be a graph with genus g and $tw(G) > 4k(g+1)$ with $k \geq 12g$, then G contains a $(k-12g, g)$-gridoid as a contraction.

a (k, g)-gridoid is partially triangulated $(k \times k)$-grid in with g extra edges

proof of our result

- $t\ell((k, g)$-gridoid$) \geq$ function of k and g
- hence, $t\ell(G) \geq$ function of k and g (since $t\ell$ close under contraction)
- $\Rightarrow$ if G (with genus g) has "large" treewidth, it has "large" tree-length

3rd alternative: Tree-breadth

[Dragan,Abu-Ata 14]

Breadth of $(T,\mathcal{X})$: largest excentricity (in G) of a bag $\min_{v \in G} \max u \in B dist_G(u,v)$
Tree-breadth of a graph G, $tb(G)$: min breadth over all tree-decompositions.

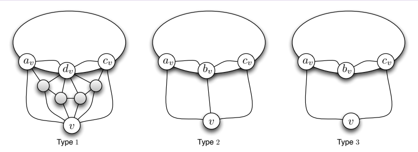
$tb(G) \leq t\ell(G) \leq 2tb(G)$ for any graph G

Complexity of tree-breadth was open

- **Good news:** efficient 3-approximation for $tb(G)$ [Dragan,Abu-Ata 14]

Our results [Ducoffe,Legay,N. 16]

- **Bad news:** NP-hard to decide if $tb(G) \leq 1$ (not FPT)
- Deciding $tb(G) \leq 1$ can be solved in polynomial-time in



- bipartite graphs (linear time)
- $K_{3,3}$ minor-free graphs ($O(n^3m)$ time)

Further work

Evaluation of our algorithms for treewidth

implement the DFS-algorithm for treewidth in real large networks

turn our results on treewidth/treelength into a constructive algorithm
i.e., how to use an algo that approx treelength to approx treewidth?

Computing Treelength and Tree-breadth

better approximation for treelength? for tree-breadth?

complexity of deciding $tb(G) \leq k$ or $t\ell(G) \leq k$ in planar graphs?

Other interesting bag's properties to be investigated?

Design an efficient algorithm to compute "good" tree-decompositions in practice

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谢谢