Cop and robber games when the robber can hide and ride

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General Problem

Capture an intruder in a network
- $C$ plays with a team of cops
- $R$ plays with one robber

Cops’ goal:
- $C$: Capture the robber using $k$ cops ("few");
- The minimum called cop-number, $\text{cn}(G)$.

Robber’s goal:
- $R$: Perpetually evade $k$ cops ("many");
- The maximum equal $\text{cn}(G) - 1$. 
Initialization:

1. $C$ places the cops;
2. $R$ places the robber.

Step-by-step:

- each cop traverses at most 1 edge;
- the robber traverses at most 1 edge.

Robber apprehended:
A cop occupies the same vertex as the robber.
**Cops & robber games** [Nowakowski and Winkler; Quilliot, 83]

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Simple Examples

Cop number in cliques and trees
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Cop and robber games when the robber can hide and ride
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\[ \text{cn (Kn)} = 1 \]
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Cop number in cliques and trees

\[
\text{cn (Kn)} = 1 \quad \text{cn (T)} = 1
\]
State of art: $cn(G) = 1$

- **Characterization** of cop-win graphs $\{ G \mid cn(G) = 1 \}$.
  [Nowakowski & Winkler, 83; Quilliot, 83; Chepoi, 97]

**Theorem:** $cn(G) = 1$ iff

$V(G) = \{ v_1, \ldots , v_n \}$ and for any $i < n$, there is $j > i$ s.t.

$N[v_i] \subseteq N[v_j]$ in the subgraph induced by $v_i, \ldots , v_n$

Trees, chordal graphs, bridged graphs (...) are cop win.
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Trees, chordal graphs, bridged graphs (…) are cop win.
State of art: complexity

- **Algorithms:** $O(n^k)$ to decide if $\text{cn}(G) \leq k$. [Hahn & MacGillivray, 06]

  $\text{cn}(G) \leq k$ iff the configurations’ graph with $k$ cops is copwin.

- **Complexity:** Computing the cop-number is EXPTIME-complete. [Goldstein & Reingold, 95]
  - in directed graphs;
  - in undirected graphs if initial positions are given.
For any graph $G$ with girth $\geq 5$ and min degree $\geq d$, $cn(G) \geq d$. [Aigner & Fromme, 84]

$cn(G) \geq d^t$, where $d + 1 = \text{minimum degree}$, girth $\geq 8t - 3$. [Frankl, 87] ($\Rightarrow$ there are $n$-node graphs $G$ with $cn(G) \geq \Omega(\sqrt{n})$)

For any $k, n$, it exists a $k$-regular graph $G$ with $cn(G) \geq n$ [Andreae, 84]
State of art: upper bound

- **Planar graph** $G$: $cn(G) \leq 3$.  
  [Aigner & Fromme, 84]

- **Bounded genus graph** $G$ with genus $g$:  
  $cn(G) \leq 3/2g + 3$ [Schröder, 01]

- **Minor free graph** $G$ excluding a minor $H$:  
  $cn(G) \leq |E(H \setminus \{x\})|$, where $x$ is any non-isolated vertex of $H$ [Andreae, 86]

- **General upper bound**
  For any connected graph $G$, $cn(G) \leq O(n/\log(n))$  
  [Chiniforooshan, 08]
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**Conjecture**: For any connected graph $G$, $cn(G) \leq O(\sqrt{n})$.  

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Faster protagonists [Fomin, Golovach, Kratochvil, N., Suchan, TCS 2010]

Speed = \( \max \) number of edges traversed in 1 step: \( \text{speed}_R \geq \text{speed}_C = 1 \)

\( \text{cn}_s(G) \) min number of cops to capture a robber with speed \( s \) in \( G \)

Computational hardness

Computing \( \text{cn}_s \) for any \( s \geq 1 \) is NP-hard; the parameterized version is \( \text{W}[2] \)-hard. For \( s \geq 2 \), it is true already on split graphs.

Fast robber in interval graphs

Robber with speed \( s \geq 1 \), \( \text{cn}_s(G) \leq \text{function}(s) \)

\( \Rightarrow \) algorithm in time \( O(n^{\text{function}(s)}) \)

Cop-number is unbounded in planar graphs

\( \forall s > 1, \forall n: \text{then } \text{cn}_s(n \times n\text{grid}) = \Omega(\sqrt{\log n}). \)

\( \forall H \text{ planar with an induced subgraph } \text{Square}_{2f}(k), \text{cn}(H) \geq k. \)
## Three variants we consider

When cops and robber can **ride**

\[ s = \text{speed}_R \geq \text{speed}_C = s' \]

When the robber can **hide** (witness) [Clarke DM 08]

The robber is visible only every \( k \) steps.

When the cops can **shoot** (radius of capture)

Robber captured when at distance \( k \) from a cop.

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**Problem:** Characterization of cop-win graphs

(cop-win graph: in which one cop always captures the robber)
Three variants we consider

When cops and robber can ride

\[ s = \text{speed}_R \geq \text{speed}_C = s' \]

\[ CWFR(s, s') \]

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The robber is visible only every \( k \) steps.

\[ CWWW(k) \]

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\[ CWRC(k) \]

Problem: Characterization of cop-win graphs

(cop-win graph: in which one cop always captures the robber)
One fast cop vs. a fast robber

\[ \text{CWFR}(s, s') = \{ G \mid \text{C with speed } s' \text{ wins against } R \text{ with speed } s \} \]

**Theorem** [Nowakowski & Winkler, 83; Quilliot, 83]

\[ G \in \text{CWFR}(1, 1) \text{ iff } G \text{ dismantable, i.e., } V(G) = \{ v_1, \ldots, v_n \}, \]
\[ \forall i < n, \exists j > i, \text{ s.t. } N_1(v_i, G_i) \subseteq N_1(v_j, G_i) \text{ with } G_i = G[v_i, \ldots, v_n] \]

Roughly,

if \( R \) on \( v_i \) and \( C \) on \( v_j \)

\( R \) must go to a smaller vertex

How to generalize?
One fast cop vs. a fast robber

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How to generalize?

\[ N_s(v_i, G_i \setminus \{v_j\}) \subseteq N_s'(v_j, G_i) \]

?? No

\[ N_s(v_i, G \setminus \{v_j\}) \cap G_i \subseteq N_s'(v_j) \]
## One fast cop vs. a fast robber

### Theorem Characterization of $\text{CWFR}(s, s')$

$G \in \text{CWFR}(s, s')$ iff $V(G) = \{v_1, \ldots, v_n\}$, $\forall i < n$, $\exists j > i$, s.t. $N_s(v_i, G \setminus \{v_j\}) \cap X_i \subseteq N_{s'}(v_j)$ with $X_i = \{v_i, \ldots, v_n\}$

For the proof: more general game with $X \subseteq V(G)$

**X-game:** $\mathcal{C}$ and $\mathcal{R}$ occupy only $X$ but can pass through $V(G)$

### Theorem X-game, $X \subseteq V(G)$

$G \in X\text{-CWFR}(s, s')$ iff $X = \{v_1, \ldots, v_{|X|}\}$, $\forall i < n$, $\exists j > i$, s.t. $N_s(v_i, G \setminus \{v_j\}) \cap X_i \subseteq N_{s'}(v_j)$ with $X_i = \{v_i, \ldots, v_{|X|}\}$

proof by induction on $|X|$
One fast cop vs. a fast robber

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$X$-game: $C$ and $R$ occupy only $X$ but can pass through $V(G)$

Theorem $X$-game, $X \subseteq V(G)$

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proof by induction on $|X|$
Assume \( G \in X-\mathcal{CWFR}(s, s') \)

Consider a "longuest" sequence for \( \mathcal{R} \). \( v_1 \in X \)

one-to-last vertex occupied by \( \mathcal{R} \).

\( y \in X \) the vertex occupied by \( \mathcal{C} \) at the same moment.

\( \mathcal{R} \) caugth after next move

\[ \Rightarrow N_s(v_1, G \setminus \{y\}) \cap X \subseteq N_{s'}(y) \]

Remains to show

\( G \in X_2-\mathcal{CWFR}(s, s') \)

\( (X_2 = \{v_2, \cdots, |X|\}) \)
Assume $G \in X-CWFR(s, s')$
Consider a ”longuest” sequence for $R$.

We have $N_s(v_1, G \setminus \{y\}) \cap X \subseteq N_{s'}(y)$
We show that $G \in X_2-CWFR(s, s')$

$\sigma$ winning strategy in $X$-game (positional game)
$\sigma : X \times X \to X, (\text{pos. C}, \text{pos. R}) \to \text{next pos. C}$
Assume $G \in X$-$CWFR(s, s')$

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$\sigma$ winning strategy in $X$-game (positional game)

$\sigma : X \times X \rightarrow X, (\text{pos. } C, \text{pos. } R) \rightarrow \text{next pos. } C$

$\sigma'$ winning strategy in $X_2$-game (with one bit of memory)

Roughly, $C$ acts as in $X$ but if it was in $v_1$

$\approx \sigma'(c, r, 0) = (\sigma(c, r), 0)$ if $\sigma(c, r) \neq v_1$, $\sigma'(c, r, 0) = (y, 1)$ otherwise
Sketch of proof

\[ N_s(v_i, G \setminus \{v_j\}) \cap X_i \subseteq N'_{s'}(v_j) \]

- If \( G \in X-C\mathcal{WFR}(s, s') \) then \( X \) admits the desired ordering
- Assume \( X \) admits the desired ordering

Then \( X_2 \) admits the desired ordering
By induction, \( G \in X_2-C\mathcal{WFR}(s, s') \)

\( \sigma' \) winning strategy in \( X_2 \)-game \( \rightarrow \) Build \( \sigma \) in \( X \)-game

Roughly, \( C \) follows \( \sigma' \) but if \( R \) in \( v_1 \).
In the latter case, \( C \) captures \( R \) or acts as if \( R \) in \( y \).

\[ \approx \sigma(c, r) = \sigma'(c, r) \text{ if } r, c \neq v_1, \text{ and } \sigma'(c, v_1) = \sigma'(c, y) \text{ otherwise} \]

We prove that \( \sigma \) is winning \( \quad \text{(technical part)} \)
Cop-win graphs and hyperbolicity

**Theorem** Characterization of $\mathcal{CWFR}(s, s')$

$G \in \mathcal{CWFR}(s, s')$ iff $V(G) = \{v_1, \ldots, v_n\}$, $\forall i < n$, $\exists j > i$, s.t. $N_s(v_i, G \setminus \{v_j\}) \cap X_i \subseteq N_{s'}(v_j)$ with $X_i = \{v_i, \ldots, v_n\}$

$G$ is $\delta$-hyperbolic if $\forall u, v, x, y \in V(G)$, the 2 larger distances among $d(u, v) + d(x, y), d(u, x) + d(v, y), d(u, y) + d(v, x)$ differ by at most $2\delta$
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Theorem Hyperbolicity helps the cop

$\forall r > 2\delta \geq 0$, and $G$ a $\delta$-hyperbolic graph, $G \in \mathcal{CWFR}(2r, r + \delta)$

Theorem Cop-win ”leads” to hyperbolicity

If $s \geq 2s'$, then any $G \in \mathcal{CWFR}(s, s')$ is $(s - 1)$-hyperbolic.

Question: $\forall s > s'$, any $G \in \mathcal{CWFR}(s, s')$ is $f(s)$-hyperbolic?
One slow cop vs. a fast robber

We consider $C$ with speed one

$CWFR(s) = CWFR(s, 1)$

Characterization of $CWFR(s)$

$G \in CWFR(s)$ iff $G$ is

Case $s = 1$: dismantable
Case $s = 2$: dually-chordal
Case $s \geq 3$: a ”big brother graph”

$G$ is a big brother graph if each block (maximal 2-connected comp.) is dominated by its articulation point with its parent-block.
More speed does not help $R$ vs. a slow cop

$\forall s \geq 3, \ G \in CWFR(s) \text{ iff } G \text{ is a big brother graph.}$

$G \text{ big brother } \Rightarrow G \in CWFR(\infty) \subseteq \cdots \subseteq CWFR(3)$
More speed does not help $\mathcal{R}$ vs. a slow cop

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Let $G \in CWFR(3)$

induction on the number of blocks

- if $G$ is 2-connected

induction on $|V(G)|$

$V = \{v_1, \cdots, v_n\}$, $N_3(v_1, G \setminus \{y\}) \subseteq N_1(y)$ and $G \setminus \{v_1\} \in CWFR(3)$

We prove $G$ is dominated by $y$

either $G \setminus \{v_1\}$ is 2–connected dominated by $b$
More speed does not help $\mathcal{R}$ vs. a slow cop

$\forall s \geq 3, G \in CWFR(s)$ iff $G$ is a big brother graph.

Let $G \in CWFR(3)$

induction on the number of blocks

- if $G$ is 2-connected
  induction on $|V(G)|$
  \[ V = \{v_1, \ldots, v_n\}, N_3(v_1, G \setminus \{y\}) \subseteq N_1(y) \text{ and } G \setminus \{v_1\} \in CWFR(3) \]

We prove $G$ is dominated by $y$

or $G \setminus \{v_1\}$ is not 2-connected and $y$ is the dominating articulation point
More speed does not help $R$ vs. a slow cop

$\forall s \geq 3$, $G \in CWFR(s)$ iff $G$ is a big brother graph.

$G$ big brother $\Rightarrow G \in CWFR(\infty) \subseteq \cdots \subseteq CWFR(3)$

Let $G \in CWFR(3)$

induction on the number of blocks

- if $G$ is 2-connected $\Rightarrow$ dominated

- if $G$ is not 2-connected

If $B$ a leaf-block dominated by articulation point

$G \setminus B \in CWFR(3)$ because retract

$\Rightarrow G \setminus B$ is big brother $\Rightarrow G$ is big brother
More speed does not help $R$ vs. a slow cop

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$G$ big brother $\Rightarrow G \in CWFR(\infty) \subseteq \cdots \subseteq CWFR(3)$

Let $G \in CWFR(3)$

induction on the number of blocks
- if $G$ is 2-connected $\Rightarrow$ dominated
- if $G$ is not 2-connected
  If $B$ a leaf-block dominated by articulation point
  $G \setminus B \in CWFR(3)$ because retract
  $\Rightarrow G \setminus B$ is big brother $\Rightarrow G$ is big brother

If not, escape strategy for $R$
More speed does not help $\mathcal{R}$ vs. a slow cop

$\forall s \geq 3, \ G \in CWFR(s) \iff G$ is a big brother graph.

$G$ big brother $\Rightarrow G \in CWFR(\infty) \subseteq \cdots \subseteq CWFR(3)$

Let $G \in CWFR(3)$

induction on the number of blocks
- if $G$ is 2-connected $\Rightarrow$ dominated
- if $G$ is not 2-connected
  If $B$ a leaf-block dominated by articulation point $G \setminus B \in CWFR(3)$ because retract
  $\Rightarrow G \setminus B$ is big brother $\Rightarrow G$ is big brother
  If not, escape strategy for $\mathcal{R}$
More speed does not help $\mathcal{R}$ vs. a slow cop

∀$s \geq 3$, $G \in CWFR(s)$ iff $G$ is a big brother graph.

$G$ big brother $\Rightarrow$ $G \in CWFR(\infty) \subseteq \cdots \subseteq CWFR(3)$

Let $G \in CWFR(3)$
induction on the number of blocks
- if $G$ is 2-connected $\Rightarrow$ dominated
- if $G$ is not 2-connected
  If $B$ a leaf-block dominated by articulation point
  $G \setminus B \in CWFR(3)$ because retract
  $\Rightarrow$ $G \setminus B$ is big brother $\Rightarrow$ $G$ is big brother

If not, escape strategy for $\mathcal{R}$
The witness version

\[ CWW(k) = \{ G | C \text{ wins against } R \text{ visible every } k \text{ steps } \} \]
\[ CWFR(s) \subseteq CWW(s) \]

Equality ??
$CW\forall(k) = \{ G | C \text{ wins against } R \text{ visible every } k \text{ steps } \}$

$CWFR(s) \subseteq CW\forall(s)$

Equality ??

NO: $G$ with diameter 2 and no dominating vertex

$\Rightarrow G \notin CWFR(s)$ for any $s \geq 2$, but $G \in CW\forall(s)$ for any $k \geq 1$
The witness version

\[ \text{CWW}(k) = \{ G \mid \text{C wins against R visible every } k \text{ steps} \} \]

\[ \text{CWFR}(s) \subseteq \text{CWW}(s) \quad \text{equality ??} \]

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The witness version

\[ \mathcal{CWWW}(k) = \{ G \mid C \text{ wins against } R \text{ visible every } k \text{ steps} \} \]

\[ \mathcal{CWFR}(s) \subseteq \mathcal{CWWW}(s) \]

**NO:** \( G \) with diameter 2 and no dominating vertex

\[ \Rightarrow G \not\in \mathcal{CWFR}(s) \text{ for any } s \geq 2, \text{ but } G \in \mathcal{CWWW}(s) \text{ for any } k \geq 1 \]
$CWW(k) = \{ G \mid \text{C wins against R visible every } k \text{ steps} \}$

$CWFR(s) \subseteq CWW(s)$

**NO:** $G$ with diameter 2 and no dominating vertex

$\Rightarrow G \notin CWFR(s)$ for any $s \geq 2$, but $G \in CWW(s)$ for any $k \geq 1$
The witness version

\[ C_{\text{WW}}(k) = \{ G \mid C \text{ wins against } R \text{ visible every } k \text{ steps} \} \]

\[ C_{\text{WFR}}(s) \subseteq C_{\text{WW}}(s) \]

Equality ??

**NO:** \( G \) with diameter 2 and no dominating vertex

\[ G \notin C_{\text{WFR}}(s) \text{ for any } s \geq 2, \text{ but } G \in C_{\text{WW}}(s) \text{ for any } k \geq 1 \]

Importance of edge-separator
The witness version

\[ CWW(k) = \{ G \mid C \text{ wins against } R \text{ visible every } k \text{ steps} \} \]

**Lemma** visibility is weaker than speed

\[ \forall s \geq 2, \ CWFR(s) \subset CWW(s) \]

**Lemma** less visibility helps \( R \)

\[ \forall k \geq 1, \text{ there are graphs in } CWFR(k) \setminus CWFR(k + 1) \]
The witness version

\[ CWW(k) = \{ G \mid C \text{ wins against } R \text{ visible every } k \text{ steps} \} \]

**Lemma**

visibility is weaker than speed

\[ \forall s \geq 2, \ CWR(s) \subset CWW(s) \]

**Lemma**

less visibility helps \( R \)

\[ \forall k \geq 1, \text{ there are graphs in } CWR(k) \setminus CWR(k + 1) \]
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\[ CWW(k) = \{ G \mid C \text{ wins against } R \text{ visible every } k \text{ steps} \} \]

**Lemma** visibility is weaker than speed

\( \forall s \geq 2, CWFR(s) \subset CWW(s) \)

**Lemma** less visibility helps \( R \)

\( \forall k \geq 1, \text{ there are graphs in } CWFR(k) \setminus CWFR(k + 1) \)
The witness version

$$CW\mathcal{W}(k) = \{ G \mid C \text{ wins against } R \text{ visible every } k \text{ steps} \}$$

**Lemma**

Visibility is weaker than speed

$$\forall s \geq 2, \ CW\mathcal{F}R(s) \subset CW\mathcal{W}(s)$$

**Lemma**

Less visibility helps R

$$\forall k \geq 1, \text{ there are graphs in } CW\mathcal{F}R(k) \setminus CW\mathcal{F}R(k + 1)$$
The witness version

\[ CWW(k) = \{ G \mid C \text{ wins against } R \text{ visible every } k \text{ steps} \} \]

**Lemma** visibility is weaker than speed

\[ \forall s \geq 2, \ CWF_R(s) \subset CWW(s) \]

**Lemma** less visibility helps \( R \)

\[ \forall k \geq 1, \text{ there are graphs in } CWF_R(k) \setminus CWF_R(k + 1) \]
The witness version

\[ CWW(k) = \{ G \mid C \text{ wins against } R \text{ visible every } k \text{ steps} \} \]

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**Question:** \( CWFR(k + 1) \subset CWFR(k) \)?
The big two-brother graphs

\[ \mathcal{WWW} = \{ G | \forall k, C \text{ wins vs. } R \text{ visible every } k \text{ steps } \} = \bigcap_k \mathcal{WWW}(k) \]

**Theorem**

\( \mathcal{WWW} \) is the class of the big two-brother graphs

\( G \) is a big two-brother graph if

\[ \exists y \in V \text{ or } xy \in E \text{ s.t. } x \text{ or } y \]

-dominated a connected comp. \( C \) of \( G \setminus \{x, y\} \) and \( G \setminus C \) is a big two-brother graph
**The big two-brother graphs**

\[ CWW = \{ G \mid \forall k, C \text{ wins vs. } R \text{ visible every } k \text{ steps } \} = \cap_k CWW(k) \]

**Theorem**

*CWW* is the class of the big two-brother graphs

*big two-brother* ⇒ *CWW* (easy)

G is a **big two-brother graph** if

\[ \exists y \in V \text{ or } xy \in E \text{ s.t. } x \text{ or } y \text{ dominated a connected comp. } C \text{ of } G \setminus \{x, y\} \text{ and } G \setminus C \text{ is a big two-brother graph} \]

*big two-brother* ⇒ *CWW* (easy)
The big two-brother graphs

$$\mathcal{CWW} = \{ G | \forall k, \, C \text{ wins vs. } R \text{ visible every } k \text{ steps } \} = \bigcap_k \mathcal{CWW}(k)$$

**Theorem**

$\mathcal{CWW}$ is the class of the big two-brother graphs

$G$ is a big two-brother graph if

\[ \exists y \in V \text{ or } xy \in E \text{ s.t. } x \text{ or } y \text{ dominated a connected comp. } C \text{ of } G \setminus \{x, y\} \text{ and } G \setminus C \text{ is a big two-brother graph} \]

* big two-brother $\Rightarrow$ $\mathcal{CWW}$ (easy)

* $\mathcal{CWW}$ $\Rightarrow$ big two-brother

\[ \forall k, \, G \in \mathcal{CWW}(k^2) \text{ without degree-1 vertex then, } \exists v \in V, \, xy \in E, \, N_k(v, G \setminus xy) \subseteq N_1(y) \]
Lemma Necessity

If \( G \in \text{CWW}(2) \) then

\[ V = \{v_1, \ldots, v_n\}, \text{ and } \forall i, \exists xy \in E(G_{i+1}) \]

(possibly \( x = y \)) \( N_2(v_i, G \setminus xy) \cap G_i \subseteq N_1(y) \)

Lemma Sufficiency

If \( V = \{v_1, \ldots, v_n\}, \text{ and } \forall i, \exists xy \in E(G_{i+1}) \)

(possibly \( x = y \)) \( N_2(v_i, G \setminus xy) \cap G_i \subseteq N_1(y) \)

and, if \( x \neq y \), then

\( N_2(v_i, G \setminus y) \cap G_i \subseteq N_2(x, G \setminus y) \)

then \( G \in \text{CWW}(2) \)
Lemma

If \( V = \{v_1, \cdots, v_n\} \), and \( \forall i, \exists xy \in E(G_{i+1}) \) (possibly \( x = y \))
\( N_k(v_i, G \setminus xy) \cap G_i \subseteq N_1(y) \)

then \( G \in \mathcal{CWW}(k) \)

Sufficiency for any \( k \) odd

But it is not necessary...
Sketch of proof

If \( V = \{v_1, \ldots, v_n\}, \forall i, \exists xy \in E(G_{i+1}) \) (possibly \( x = y \)) \( N_k(v_i, G \setminus xy) \cap G_i \subseteq N_1(y) \)
then \( G \in CWW(k) \)

Use of configuration’s graph

Procedure:

1. init: all config. unmarked
2. mark all \((c, r)\) s.t. \( r \in N_1(c) \) with 1
3. while possible,
   mark \((c, r)\) with the minimum \( \ell + 1 \) s.t. \( \exists y \in N_1(c) \) and \( x \in N_1(y) \subseteq r \) s.t.
   \( \forall z \in N_k(r, G \setminus xy), (y, z) \) marked \( \leq \ell \)

If all config. marked \( \Rightarrow G \in CWW(k) \)
If desired ordering \( \Rightarrow \) all config. marked
Finally, let’s advantage the cop :) 

\[ \text{CWRC}(k) = \{ G \mid \text{C wins when capturing at dist. } \leq k \} \]

**Theorem**

A bipartite graph \( G \) is in \( \text{CWRC}(1) \) iff 

\[
V = \{ v_1, \ldots, v_n \} \text{ s.t. } \\
\{ v_{n-1}, v_n \} \in E \text{ and } \forall i, \exists j > i, \{ v_j, v_i \} \notin E \text{ and } N(v_i, G_i) \subseteq N_1(v_j)
\]

Characterization of \( \text{CWRC}(k) \) seems harder, even for \( k = 1 \)...
Perspectives

In case $speed_R = speed_C = 1$

- $G$ of genus $g \Rightarrow cn(G) \leq \frac{3}{2}g + 3$. [Schröder, 01]
  Conjecture: $G$ of genus $g \Rightarrow cn(G) \leq g + 3$.

- General upper bound for $cn$?
  for any connected graph $G$, $cn(G) \leq O(n/\log n)$.
  [Chiniforooshan, 08]
  Conjecture: $cn(G) \leq O(\sqrt{n})$.

In case $speed_R > speed_C$

- $\Omega(\sqrt{\log(n)}) \leq cn(Square_n) \leq O(n)$. Exact value?
- What about other graphs’classes?

Full characterization of cop-win graphs when witness?
Characterization of cop-win graphs when the cop can ”shoot”?

J. Chalopin, V. Chepoi, N. Nisse, Y. Vaxès
Thank you

Any questions?