On Minimizing the Maximum Color for the 1-2-3 Conjecture

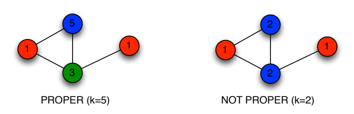
Julien Bensmail, Bi Li, Binlong Li, Nicolas Nisse

Université Côte d'Azur, Inria, CNRS, I3S, France

COATI seminar@home, April 17th, 2020

Vertex Coloration and Edge Labeling

k-Vertex Coloring of $G = (V, E)$: $c : V \to \{1, \cdots, k\}$.
proper: adjacent vertices have distinct colors: for every $uv \in E$, $c(u) \neq c(v)$.

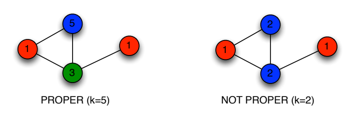


Chromatic number $\chi(G)$

$\chi(G) = \min\{k \mid$ `G has a k proper coloring` $\}$ $\omega(G) \leq \chi(G) \leq \Delta(G) + 1$

Vertex Coloration and Edge Labeling

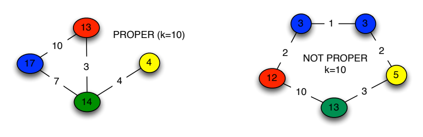
k-Vertex Coloring of $G=(V,E)$: $c: V \to \{1,\cdots,k\}$.
proper: adjacent vertices have distinct colors: for every $uv \in E$, $c(u) \neq c(v)$.



Chromatic number $\chi(G)$

$\chi(G) = \min\{k \mid$ `G has a k proper coloring` $\}$ $\qquad \omega(G) \leq \chi(G) \leq \Delta(G)+1$

k-Edge labeling: $\ell : E \to \{1,\cdots,k\}$
Induced vertex-coloring c_ℓ: $V \to \mathbb{N}$ $\qquad$ for every $v \in V$, $c_\ell(v) = \sum_{u \in N(v)} \ell(uv)$.

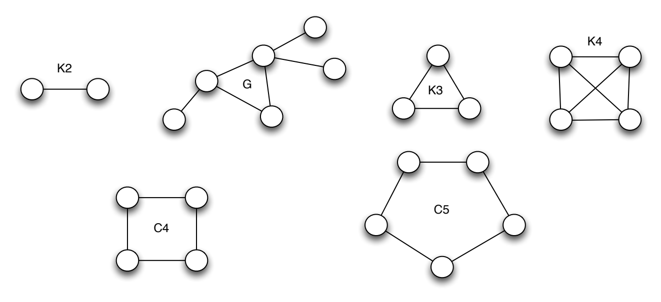


Find a k-labeling inducing proper coloring with $k << \chi(G)$? [Karoński,Luczak,Thomason,04]

Edge Labeling: first examples

k-Edge labeling: $\ell : E \to \{1, \cdots, k\}$

Induced vertex-coloring c_ℓ: $V \to \mathbb{N}$ for every $v \in V$, $c_\ell(v) = \sum_{u \in N(v)} \ell(uv)$.

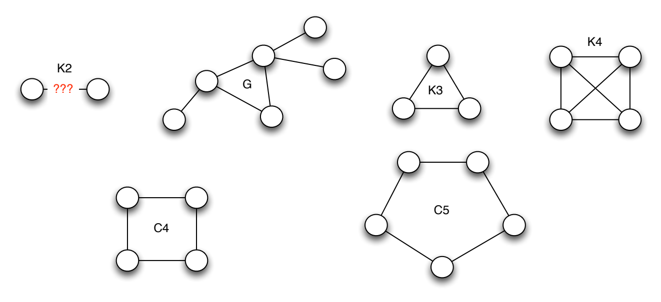


Let's play: for each of the 6 graphs, give a k-labeling inducing a proper coloring

COATIQUIZZ: I count the points!!

Edge Labeling: first examples

k-Edge labeling: $\ell : E \to \{1, \cdots, k\}$
Induced vertex-coloring c_ℓ: $V \to \mathbb{N}$ for every $v \in V$, $c_\ell(v) = \sum_{u \in N(v)} \ell(uv)$.



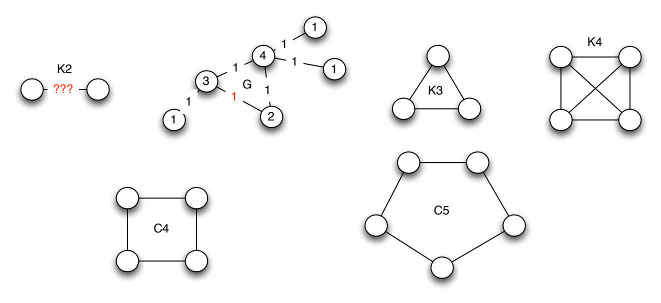
Let's play: for each of the 6 graphs, give a k-labeling inducing a proper coloring

COATIQUIZZ: I count the points!!

- No k-proper labeling of K_2 :(
-

Edge Labeling: first examples

k-Edge labeling: $\ell : E \rightarrow \{1, \cdots, k\}$
Induced vertex-coloring c_ℓ: $V \rightarrow \mathbb{N}$ for every $v \in V$, $c_\ell(v) = \sum_{u \in N(v)} \ell(uv)$.



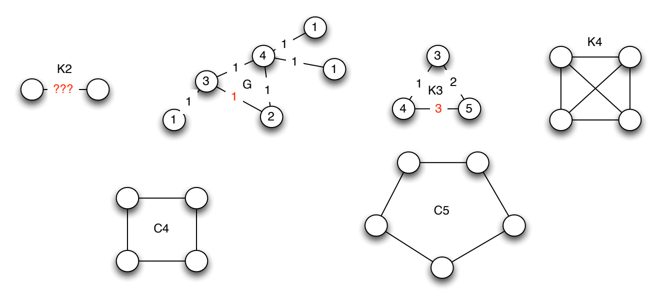
Let's play: for each of the 6 graphs, give a k-labeling inducing a proper coloring
COATIQUIZZ: I count the points!!

- No k-proper labeling of K_2 :(
- locally irregular graph: for every $uv \in E$, $deg(u) \neq deg(v)$.
  locally irregular $\Leftrightarrow$ 1-proper-labeling.

Edge Labeling: first examples

k-Edge labeling: $\ell : E \to \{1, \cdots, k\}$

Induced vertex-coloring c_ℓ: $V \to \mathbb{N}$ for every $v \in V$, $c_\ell(v) = \sum_{u \in N(v)} \ell(uv)$.



Let's play: for each of the 6 graphs, give a k-labeling inducing a proper coloring

COATIQUIZZ: I count the points!!

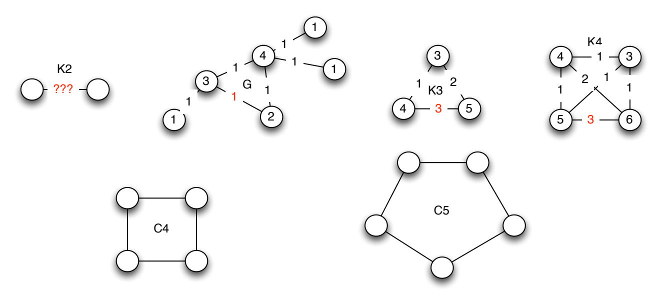
- No k-proper labeling of K_2 :(
- locally irregular graph: for every $uv \in E$, $deg(u) \neq deg(v)$.

locally irregular $\Leftrightarrow$ 1-proper-labeling.

Edge Labeling: first examples

k-Edge labeling: $\ell : E \rightarrow \{1, \cdots, k\}$

Induced vertex-coloring c_ℓ: $V \rightarrow \mathbb{N}$ for every $v \in V$, $c_\ell(v) = \sum_{u \in N(v)} \ell(uv)$.



Let's play: for each of the 6 graphs, give a k-labeling inducing a proper coloring

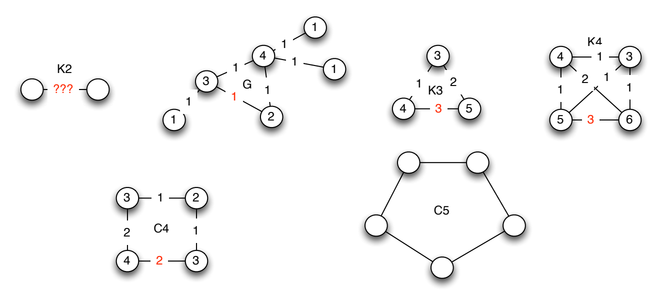
COATIQUIZZ: I count the points!!

- No k-proper labeling of K_2 :(
- locally irregular graph: for every $uv \in E$, $deg(u) \neq deg(v)$.

  locally irregular $\Leftrightarrow$ 1-proper-labeling.

Edge Labeling: first examples

k-Edge labeling: $\ell : E \to \{1, \cdots, k\}$
Induced vertex-coloring c_ℓ: $V \to \mathbb{N}$ for every $v \in V$, $c_\ell(v) = \sum_{u \in N(v)} \ell(uv)$.

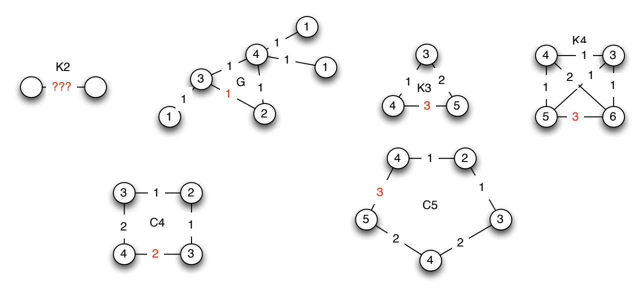


Let's play: for each of the 6 graphs, give a *k*-labeling inducing a proper coloring
COATIQUIZZ: I count the points!!

- No *k*-proper labeling of K_2 :(
- locally irregular graph: for every $uv \in E$, $deg(u) \neq deg(v)$.
  locally irregular $\Leftrightarrow$ 1-proper-labeling.

Edge Labeling: first examples

k-Edge labeling: $\ell : E \rightarrow \{1, \cdots, k\}$
Induced vertex-coloring c_ℓ: $V \rightarrow \mathbb{N}$ for every $v \in V$, $c_\ell(v) = \sum_{u \in N(v)} \ell(uv)$.

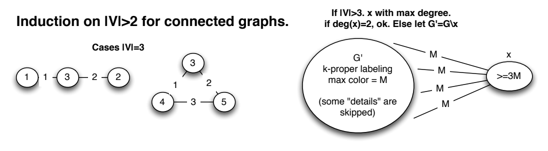


Let's play: for each of the 6 graphs, give a k-labeling inducing a proper coloring
COATIQUIZZ: I count the points!!

- No k-proper labeling of K_2 :(
- locally irregular graph: for every $uv \in E$, $deg(u) \neq deg(v)$.
  locally irregular $\Leftrightarrow$ 1-proper-labeling.

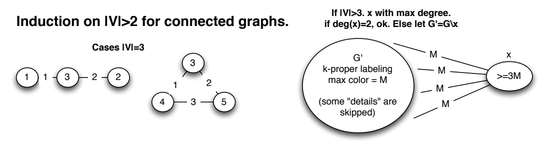
Neighbour-sum-distinguishing chromatic index $\chi_{\sum}(G)$

G has a proper k-labeling (for some $k \in \mathbb{N}$) $\Leftrightarrow$ no connected component of G is K_2.



Neighbour-sum-distinguishing chromatic index $\chi_{\sum}(G)$

G has a proper k-labeling (for some $k \in \mathbb{N}$) $\Leftrightarrow$ no connected component of G is K_2.



$\chi_{\sum}(G) = \min\{k \mid$ `G has a k-proper-labeling` $\}$
well defined for all graphs without K_2 as connected component (nice graphs).

History. For every nice graph G:

$\chi_{\sum}(G) \leq 30$ [Addario-Berry,Dalal,McDiarmid,Reed,Thomason 2007]

$\chi_{\sum}(G) \leq 5$ [Kalkowski,Karoński,Pfender 2010]

1-2-3 Conjecture: for every nice graph G, $\chi_{\sum}(G) \leq 3$ [Karoński,Łuczak,Thomason 2004]

Complexity

Deciding if $\chi_{\sum}(G) \leq 2$ is NP-complete in cubic graphs. [Ahadi,Dehghan,Sadegh 2013]

1-2-3 Conjecture holds in various graph classes

Locally irregular graphs (no adjacent vertices have same degree)

$\chi_{\sum}(G) = 1$ iff G is locally irregular.

Trees with at least 2 edges [Chang,Lu,Wu,Yu 2011]

For every nice tree T, $\chi_{\sum}(T) \leq 2$.

Cycles C_n with $n \geq 3$ vertices

$\chi_{\sum}(C_n) = 2$ if $n \equiv 0 \mod 4$ and $\chi_{\sum}(C_n) = 3$ otherwise.

Complete graphs K_n with $n \geq 3$ vertices

$\chi_{\sum}(K_n) = 3$.

Nice bipartite graphs [Thomassen,Wu,Zhang 2016]

For every nice bipartite graph B, $\chi_{\sum}(B) \leq 3$
Characterization of bipartite graphs B with $\chi_{\sum}(B) = 3$ (odd multi cacti)

Nice d-regular graphs (all vertices of degree d) [Przybylo 2019]

For every nice d regular graph G, $\chi_{\sum}(G) \leq 4$, and $\chi_{\sum}(G) \leq 3$ if $d \geq 10^8$.

Let G be a nice graph and $k \geq \chi_{\sum}(G)$. Find a k-proper labeling of G such that:

Minimize the number of distinct colors of vertices

[Baudon,Bensmail,Hocquard,Senhaji,Sopena 2019]

Minimize the maximum color of the vertices (this talk)

[Bensmail,Li,Li,N.]

Minimize the total sum of the colors of the vertices

[Bensmail,Fioravantes,N., IWOCA 2020]

Equitable labeling

$\ell : E \rightarrow \{1, \cdots, k\}$ is equitable if, for all $i \neq j \leq k$, $|\{e \in E \mid \ell(e) = i\}|$ and $|\{e \in E \mid \ell(e) = j\}|$ differ by at most one.

[Baudon,Pilśniak,Przybylo,Senhaji,Sopena,Wozńia 2017]

[Bensmail,Fioravantes,McInerney,N.]

COATIQUIZZ: Find the intruder (you'll get one point)

Proper labeling with min. maximum sum (color)

Let $k \geq \chi_{\sum}(G)$. k-proper-Edge labeling: $\ell : E \to \{1, \cdots, k\}$.
For every $vw \in E$, $c_\ell(v) = \sum_{u \in N(v)} \ell(uv) \neq \sum_{u \in N(w)} \ell(uw) = c_\ell(w)$.

$mS(G, \ell) = \max_{v \in V} c_\ell(v)$.

Proper labeling with min. maximum sum (color)

Let $k \geq \chi_{\sum}(G)$. k-proper-Edge labeling: $\ell : E \to \{1, \cdots, k\}$.
For every $vw \in E$, $c_\ell(v) = \sum_{u \in N(v)} \ell(uv) \neq \sum_{u \in N(w)} \ell(uw) = c_\ell(w)$.

$mS(G, \ell) = \max_{v \in V} c_\ell(v)$.

Find a k-proper labelling minimizing the maximum vertex-color

$mS_k(G) = \min mS(G, \ell)$ among all k-proper-Edge labeling ℓ.

Here: complexity and bounds when k is fixed in various graph classes.

Proper labeling with min. maximum sum (color)

Let $k \geq \chi_{\sum}(G)$. k-proper-Edge labeling: $\ell : E \to \{1, \cdots, k\}$.
For every $vw \in E$, $c_\ell(v) = \sum_{u \in N(v)} \ell(uv) \neq \sum_{u \in N(w)} \ell(uw) = c_\ell(w)$.

$mS(G, \ell) = \max_{v \in V} c_\ell(v)$.

Find a k-proper labelling minimizing the maximum vertex-color

$mS_k(G) = \min mS(G, \ell)$ among all k-proper-Edge labeling ℓ.

Here: complexity and bounds when k is fixed in various graph classes.

Find a proper labelling minimizing the maximum vertex-color

$mS(G) = \min\{mS_k(G) \mid k \geq \chi_{\sum}(G)\}$.

Is it interesting to increase the maximum edge-label k to decrease the maximum color?
I.e., is there $k \geq \chi_{\sum}(G)$ and a graph G such that $mS_k(G) > mS_{k+1}(G)$?

Min. maximum sum: basic (?) results

$mS_k(G) = \min mS(G,\ell)$ among all k-proper-Edge labeling ℓ.
$mS(G) = \min\{mS_k(G) \mid k \geq \chi_{\sum}(G)\}$.

For every nice graph G with maximum degree Δ:

$\Delta \leq mS(G) \leq \chi_{\sum}(G) \cdot \Delta \leq 5\Delta$ [Kalkowski,Karoński,Pfender 2010]

COATIQUIZZ: Why? (you'll get one point)

Min. maximum sum: basic (?) results

$mS_k(G) = \min mS(G,\ell)$ among all k-proper-Edge labeling ℓ.
$mS(G) = \min\{mS_k(G) \mid k \geq \chi_{\sum}(G)\}$.

For every nice graph G with maximum degree Δ:

$\Delta \leq mS(G) \leq \chi_{\sum}(G) \cdot \Delta \leq 5\Delta$ [Kalkowski,Karoński,Pfender 2010]

COATIQUIZZ: Why? (you'll get one point)

Link with 1-2-3 conjecture:

If every nice graph admits a 3-proper labeling such that every vertex is incident to $O(1)$ edges labeled with 3, then, for every nice graph G, $mS(G) \leq 2\Delta + O(1)$.

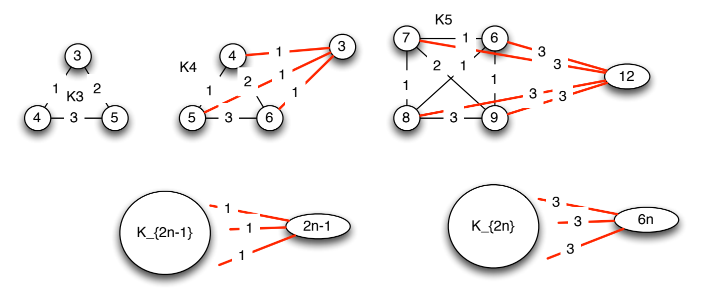
Conjecture? if correct it is mine, otherwise, it is Julien's :)

For every nice graph G with maximum degree Δ, $mS(G) \leq 2\Delta + O(1)$?

Min. maximum sum, first example: cliques

Complete graphs K_n with $n \geq 3$ vertices

$\chi_{\sum}(K_n) = 3$. Classical labeling $\ell \Rightarrow mS(K_{n+1}, \ell) = 3n$ for $n > 2$ even.



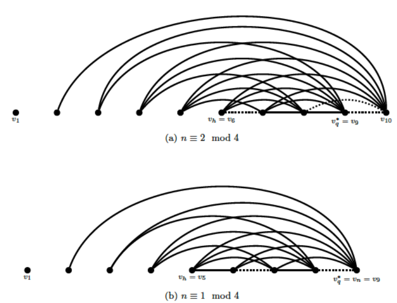
Min. maximum sum, first example: cliques

Complete graphs K_n with $n \geq 3$ vertices

$\chi_{\sum}(K_n) = 3$. Classical labeling $\ell \Rightarrow mS(K_{n+1}, \ell) = 3n$ for $n > 2$ even.

Theorem

$mS(K_n) = 2\Delta$ if $n \equiv 0$ *or* $1 \mod 4$ and $mS(K_n) = 2\Delta + 1$ otherwise.



(a) $n \equiv 2 \mod 4$

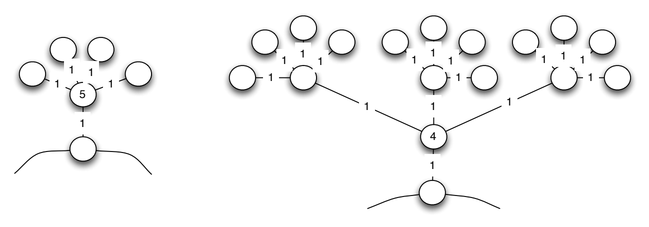
(b) $n \equiv 1 \mod 4$

lower bound by counting argument: all colors must be different and their sum even.

Min. maximum sum: complexity

Theorem

Let $k \geq 2$. Deciding if $mS_k(G) \leq 9$ is NP-hard in bipartite graphs with $\Delta = 9$.

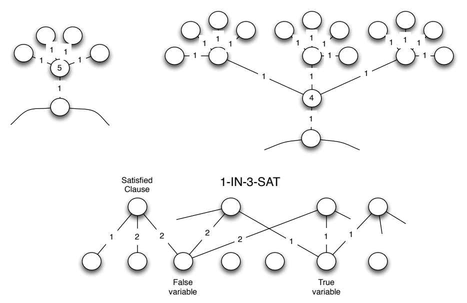


Gadget allowing to have vertices /edges with required colors/labels (here, if we forbid color$>$ 5).

Min. maximum sum: complexity

Theorem

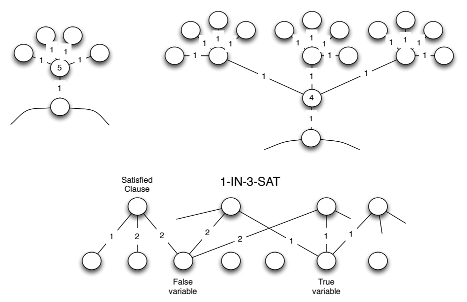
Let $k \geq 2$. Deciding if $mS_k(G) \leq 9$ is NP-hard in bipartite graphs with $\Delta = 9$.



Min. maximum sum: complexity

Theorem

Let $k \geq 2$. Deciding if $mS_k(G) \leq 9$ is NP-hard in bipartite graphs with $\Delta = 9$.



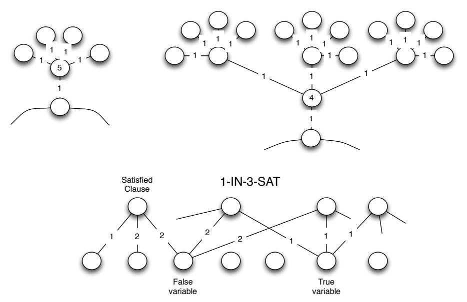
Theorem: (Corollary: deciding $\chi_{\sum}$ is in P when bounded treewidth)

Let $k \geq 2$. Given G and s as inputs, deciding if $mS_k(G) \leq s$ can be solved in time $O(nk^{(t+1)^2}(k\Delta + 1)^{3t+3})$ where t is the treewidth of G.

Min. maximum sum: complexity

Theorem

Let $k \geq 2$. Deciding if $mS_k(G) \leq 9$ is NP-hard in bipartite graphs with $\Delta = 9$.



Theorem: (Corollary: deciding $\chi_{\sum}$ is in P when bounded treewidth)

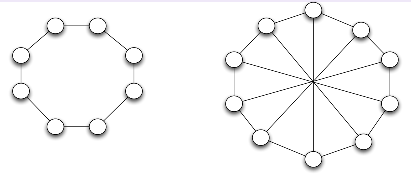
Let $k \geq 2$. Given G and s as inputs, deciding if $mS_k(G) \leq s$ can be solved in time $O(nk^{(t+1)^2}(k\Delta + 1)^{3t+3})$ where t is the treewidth of G.

COATIQUIZZ: what about $mS(G)$??

Min. maximum sum: Bip. graphs with $\chi_{\sum} = 2$

Reminder: $\Delta \leq mS(G) \leq \chi_{\sum}(G) \cdot \Delta$

There are bipartite graphs with $mS(G) = \chi_{\sum}(G) \cdot \Delta = 2\Delta$.

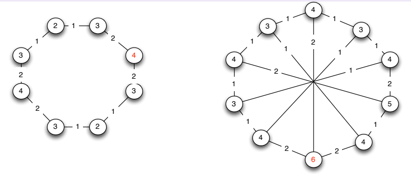


Can you find labeling for these graphs? **COATIQUIZZ: one point!**

Min. maximum sum: Bip. graphs with $\chi_{\sum} = 2$

Reminder: $\Delta \leq mS(G) \leq \chi_{\sum}(G) \cdot \Delta$

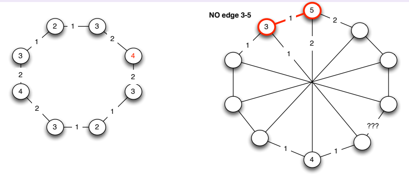
There are bipartite graphs with $mS(G) = \chi_{\sum}(G) \cdot \Delta = 2\Delta$. (for $\Delta \in \{2, 3\}$)



Min. maximum sum: Bip. graphs with $\chi_{\sum} = 2$

Reminder: $\Delta \leq mS(G) \leq \chi_{\sum}(G) \cdot \Delta$

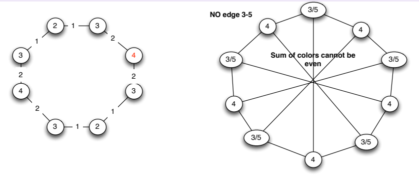
There are bipartite graphs with $mS(G) = \chi_{\sum}(G) \cdot \Delta = 2\Delta$. (for $\Delta \in \{2, 3\}$)



Min. maximum sum: Bip. graphs with $\chi_{\sum} = 2$

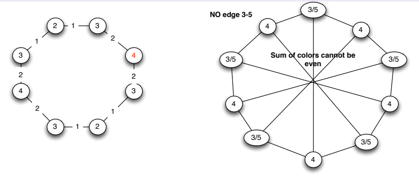
Reminder: $\Delta \leq mS(G) \leq \chi_{\sum}(G) \cdot \Delta$

There are bipartite graphs with $mS(G) = \chi_{\sum}(G) \cdot \Delta = 2\Delta$. (for $\Delta \in \{2, 3\}$)

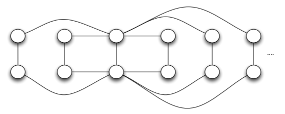


Reminder: $\Delta \leq mS(G) \leq \chi_{\sum}(G) \cdot \Delta$

There are bipartite graphs with $mS(G) = \chi_{\sum}(G) \cdot \Delta = 2\Delta$. (for $\Delta \in \{2, 3\}$)



For all $\Delta, k \geq 2$, there are bipartite graphs with $mS_k(G) \geq \lceil \frac{3\Delta}{2} \rceil$.



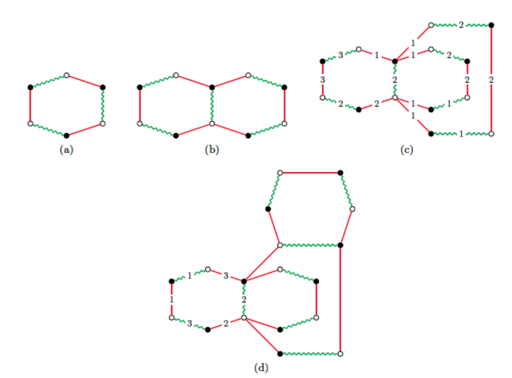
Open: For all $\Delta \geq 2$, are there bipartite graphs with $mS(G) = 2\Delta$?

Min. maximum sum: Bip. graphs with $\chi_{\sum} = 3$

Nice bipartite graphs

[Thomassen,Wu,Zhang 2016]

A bipartite graph B has $\chi_{\sum}(B) = 3$ if and only if B is a odd multi cactus.

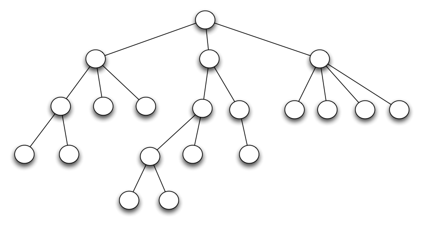


Theorem

For every odd multi cactus B, $\Delta + 1 \leq mS_3(B) \leq \Delta + 2$. Moreover, $mS_k(B)$ can be computed in polynomial time (bounded treewidth).

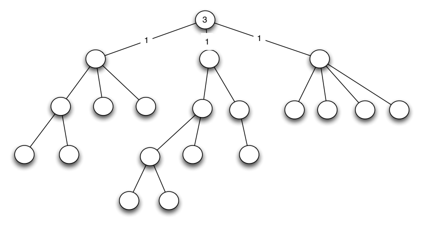
Min. maximum sum: Trees

Reminder: $\chi_{\sum}(T) \leq 2$, so $\Delta \leq mS(T) \leq 2 \cdot \Delta$ for any nice tree T.



Min. maximum sum: Trees

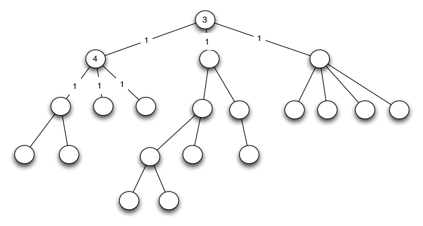
Reminder: $\chi_{\sum}(T) \leq 2$, so $\Delta \leq mS(T) \leq 2 \cdot \Delta$ for any nice tree T.



Greedy algorithm: label child-edges 1,

Min. maximum sum: Trees

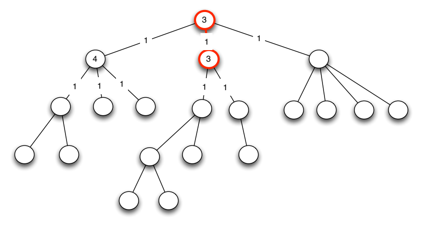
Reminder: $\chi_{\sum}(T) \leq 2$, so $\Delta \leq mS(T) \leq 2 \cdot \Delta$ for any nice tree T.



Greedy algorithm: label child-edges 1,

Min. maximum sum: Trees

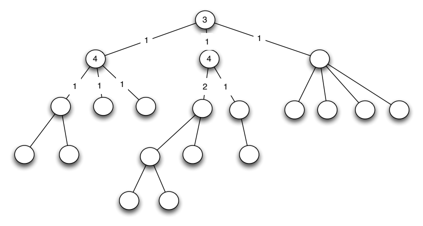
Reminder: $\chi_{\sum}(T) \leq 2$, so $\Delta \leq mS(T) \leq 2 \cdot \Delta$ for any nice tree T.



Greedy algorithm: label child-edges 1, but if conflict: label one child-edge with 2

Min. maximum sum: Trees

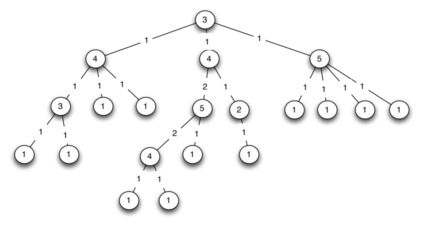
Reminder: $\chi_{\sum}(T) \leq 2$, so $\Delta \leq mS(T) \leq 2 \cdot \Delta$ for any nice tree T.



Greedy algorithm: label child-edges 1, but if conflict: label one child-edge with 2

Min. maximum sum: Trees

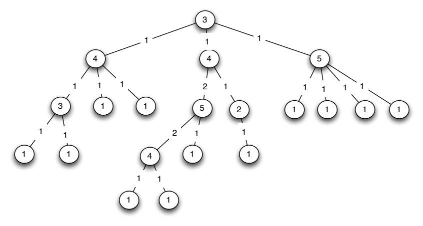
Reminder: $\chi_{\sum}(T) \leq 2$, so $\Delta \leq mS(T) \leq 2 \cdot \Delta$ for any nice tree T.



Greedy algorithm: label child-edges 1, but if conflict: label one child-edge with 2 each vertex incident to at most 2 edges labeled with 2.

Min. maximum sum: Trees

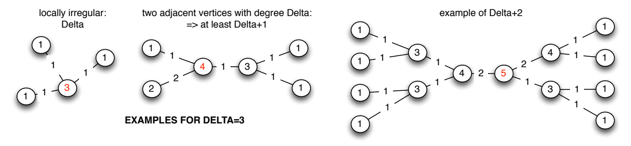
Reminder: $\chi_{\sum}(T) \leq 2$, so $\Delta \leq mS(T) \leq 2 \cdot \Delta$ for any nice tree T.



Greedy algorithm: label child-edges 1, but if conflict: label one child-edge with 2 each vertex incident to at most 2 edges labeled with 2.

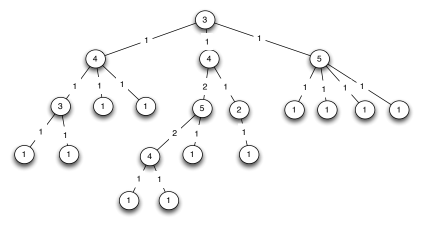
Theorem

For every nice tree T, $\Delta \leq mS(T) \leq \Delta + 2$. All possibilities are reached ($\forall \Delta \geq 3$).



Min. maximum sum: Trees

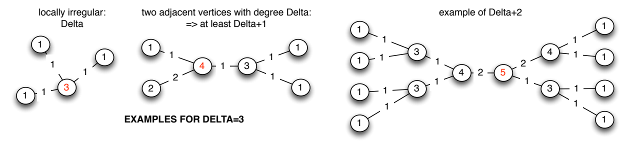
Reminder: $\chi_{\sum}(T) \leq 2$, so $\Delta \leq mS(T) \leq 2 \cdot \Delta$ for any nice tree T.



Greedy algorithm: label child-edges 1, but if conflict: label one child-edge with 2
each vertex incident to at most 2 edges labeled with 2.

Theorem **Open: characterization of "$\Delta + 1$-trees" and "$\Delta + 2$-trees"???**

For every nice tree T, $\Delta \leq mS(T) \leq \Delta + 2$. All possibilities are reached ($\forall \Delta \geq 3$).



Larger label to decrease the maximum color?

COATIQUIZZ:
What do you think? **Points are at stake!!**

- can using larger labels decrease the maximum color? YES/NO?
- if YES, can it decrease the maximum color a lot? how much?

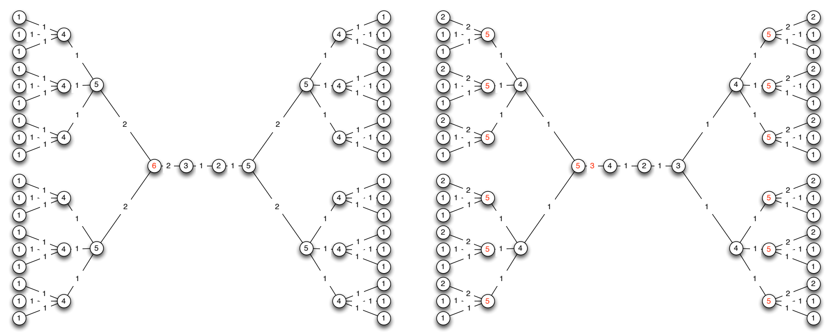
Larger label may decrease max. color in trees

Theorem

For every $k \geq 2$, there exists a tree T_k such that $mS_k(T_k) = mS_{k+1}(T_k) + 1$.

Remark: for all $k, k' \geq 2$ and tree T, $|mS_k(T) - mS_{k'}(T)| \leq 1$ (find a simpler proof?)

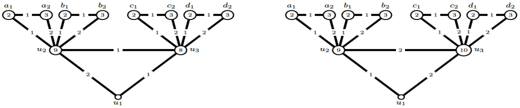
Example for $k = 2$.



Larger label in general graphs

Theorem

For every $\Delta \geq 16$, there exists a graph G with maximum degree Δ verifying $mS_2(G) = 2\Delta$ and $mS_3(G) = \Delta$.



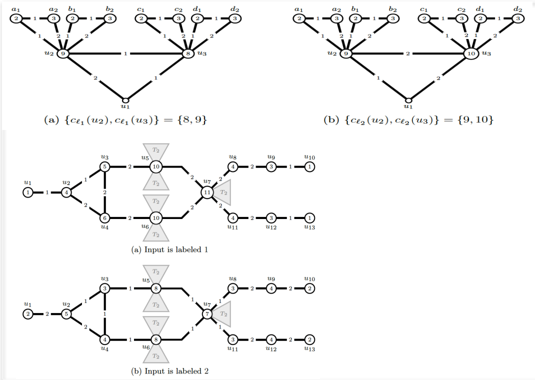
(a) $\{c_{\ell_1}(u_2), c_{\ell_1}(u_3)\} = \{8, 9\}$

(b) $\{c_{\ell_2}(u_2), c_{\ell_2}(u_3)\} = \{9, 10\}$

Larger label in general graphs

Theorem

For every $\Delta \geq 16$, there exists a graph G with maximum degree Δ verifying $mS_2(G) = 2\Delta$ and $mS_3(G) = \Delta$.



(a) $\{c_{\ell_1}(u_2), c_{\ell_1}(u_3)\} = \{8, 9\}$

(b) $\{c_{\ell_2}(u_2), c_{\ell_2}(u_3)\} = \{9, 10\}$

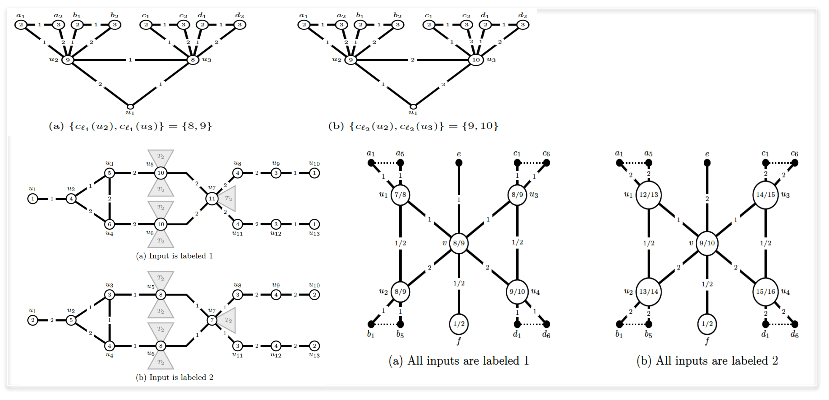
(a) Input is labeled 1

(b) Input is labeled 2

Larger label in general graphs

Theorem

For every $\Delta \geq 16$, there exists a graph G with maximum degree Δ verifying $mS_2(G) = 2\Delta$ and $mS_3(G) = \Delta$.



Further Work

- Are there bipartite graphs B (with $\chi_{\sum}(B) = 2$) such that $mS(B) = 2\Delta$?
- Characterization of nice trees T with $mS(T) = \Delta + 1$ or $mS(T) = \Delta + 2$?
- Computation of $mS(G)$ in bounded treewidth graphs?
- Is it true that $mS(G) \leq 2\Delta + O(1)$ for all nice graphs G?
- Other graph classes (more difficult since $\chi_{\sum}$ is not well understood in it), e.g., planar graphs...

More generally:

- 1-2-3 conjecture?