Minimum lethal sets in grids and tori under 3-neighbour bootstrap percolation

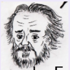
F. Benevides[1,2], J-C. Bermond[2], H. Lesfari[2], N. Nisse[2]

1. Universidade Federal do Ceara, Fortaleza, Brazil
2. Université Côte d'Azur, Inria, CNRS, I3S, France

MAC (Moving and Computing), Pisa, September 21st 2022

What is it about and why?

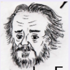
John Horton Conway FRS (26 Dec. 1937 - 11 April 2020) was an English mathematician active in the theory of *finite groups*, *knot theory*, *number theory*, *combinatorial game theory* and *coding theory*. He also made contributions to many branches of **recreational mathematics**, most notably the invention of the *cellular automaton* called the Game of Life. (Wikipedia)



(J. Conway by F. Havet)

What is it about and why?

John Horton Conway FRS (26 Dec. 1937 - 11 April 2020) was an English mathematician active in the theory of *finite groups*, *knot theory*, *number theory*, *combinatorial game theory* and *coding theory*. He also made contributions to many branches of **recreational mathematics**, most notably the invention of the *cellular automaton* called the Game of Life. (Wikipedia)



(J. Conway by F. Havet)

AFFAIRE DE LOGIQUE –N° 1141

Le jeu de la viralité (hommage à John Conway)

Confinés chez eux, Alice et Bob ont imaginé, sur un échiquier de 8 cases sur 8, des jeux simulant la propagation d'un virus. Lors du premier jeu, Bob pose un certain nombre de pions, représentant des virus, sur des cases de l'échiquier. La suite est automatique : si une case vide touche par au moins deux de ses côtés des cases contenant un pion, Bob pose un pion dans cette case. L'objectif : remplir l'échiquier tout entier.

1. Combien Bob doit-il au moins poser de pions au départ pour l'atteindre ? Trouver une configuration initiale permettant de réussir avec ce nombre minimum de pions.

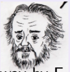
Alice imagine alors un jeu identique, mais où la case vide doit toucher au moins trois cases contenant un pion pour qu'elle puisse poser un pion dans cette case. L'objectif est le même : remplir l'échiquier.

2. Quel nombre minimum de pions de départ permet de réussir ? Trouver une configuration correspondante.

(Le Monde, enigma in tribute to J. Conway, April 16^{th} 2020)

What is it about and why?

John Horton Conway FRS (26 Dec. 1937 - 11 April 2020) was an English mathematician active in the theory of *finite groups*, *knot theory*, *number theory*, *combinatorial game theory* and *coding theory*. He also made contributions to many branches of **recreational mathematics**, most notably the invention of the *cellular automaton* called the Game of Life. (Wikipedia)



(J. Conway by F. Havet)

AFFAIRE DE LOGIQUE – N° 1141

Le jeu de la viralité (hommage à John Conway)

Confinés chez eux, Alice et Bob ont imaginé, sur un échiquier de 8 cases sur 8, des jeux simulant la propagation d'un virus. Lors du premier jeu, Bob pose un certain nombre de pions, représentant des virus, sur des cases de l'échiquier. La suite est automatique : si une case vide touche par au moins deux de ses côtés des cases contenant un pion, Bob pose un pion dans cette case. L'objectif : remplir l'échiquier tout entier.

1. Combien Bob doit-il au moins poser de pions au départ pour l'atteindre ? Trouver une configuration initiale permettant de réussir avec ce nombre minimum de pions.

Alice imagine alors un jeu identique, mais où la case vide doit toucher au moins trois cases contenant un pion pour qu'elle puisse poser un pion dans cette case. L'objectif est le même : remplir l'échiquier.

2. Quel nombre minimum de pions de départ permet de réussir ? Trouver une configuration correspondante.

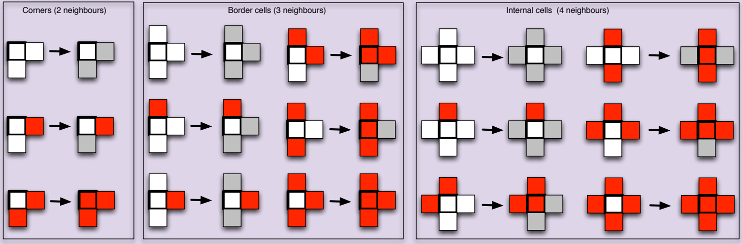
(Le Monde, enigma in tribute to J. Conway, April 16^{th} 2020)

[F. Benevides, J-C. Bermond, H. Lesfari, N. Nisse. Minimum lethal sets in grids and tori under 3-neighbour bootstrap percolation. Eur. J. of Comb. 2022+]

Warm-up: 2-neighbour bootstrap percolation in grids

Rules of the game: given an $n \times n$ grid (vertices, or cells, are depicted by squares)

Each cell can be either **contaminated** (colored) or **healthy** (white).
A contaminated cell remains contaminated forever.
A healthy cell becomes contaminated if it has <u>at least</u> 2 contaminated neighbours.



(grey cells are undetermined (contaminated or healthy))

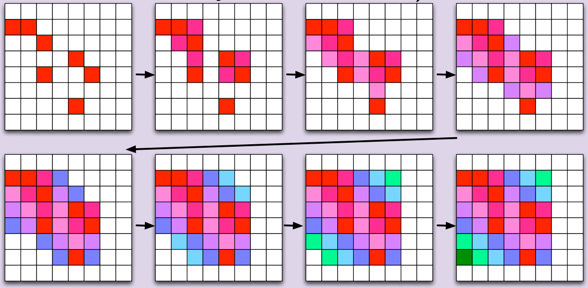
Warm-up: 2-neighbour bootstrap percolation in grids

Rules of the game: given an $n \times n$ grid (vertices, or cells, are depicted by squares)

Each cell can be either **contaminated** (colored) or **healthy** (white).
A contaminated cell remains contaminated forever.
A healthy cell becomes contaminated if it has at least 2 contaminated neighbours.

Given some initially contaminated cells, do all cells eventually become contaminated?

Example (with $n = 8$ and 7 initially contaminated cells) :



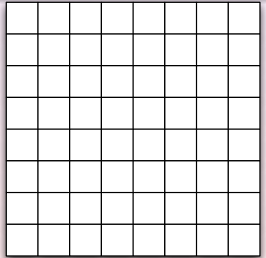
Warm-up: 2-neighbour bootstrap percolation in grids

Rules of the game: given an $n \times n$ grid (vertices, or cells, are depicted by squares)

Each cell can be either **contaminated** (colored) or **healthy** (white).
A contaminated cell remains contaminated forever.
A healthy cell becomes contaminated if it has at least 2 contaminated neighbours.

Lethal set: set of initially contaminated cells eventually contaminating the whole grid

Question: Minimum size of a lethal set in the $n \times n$ grid?



https://scratch.mit.edu/projects/517390361/fullscreen/

(click on the green flag, choose your language and then choose "contamination" game)

Warm-up: 2-neighbour bootstrap percolation in grids

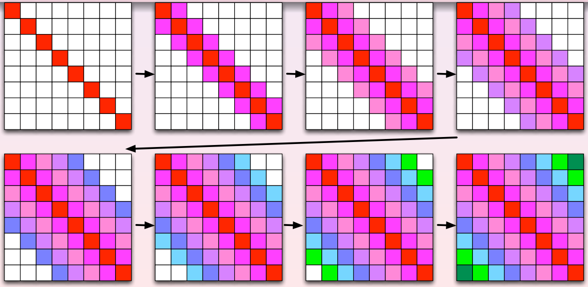
Rules of the game: given an $n \times n$ grid (vertices, or cells, are depicted by squares)

Each cell can be either **contaminated** (colored) or **healthy** (white).
A contaminated cell remains contaminated forever.
A healthy cell becomes contaminated if it has at least 2 contaminated neighbours.

Lethal set: set of initially contaminated cells eventually contaminating the whole grid

Question: Minimum size of a lethal set in the $n \times n$ grid?

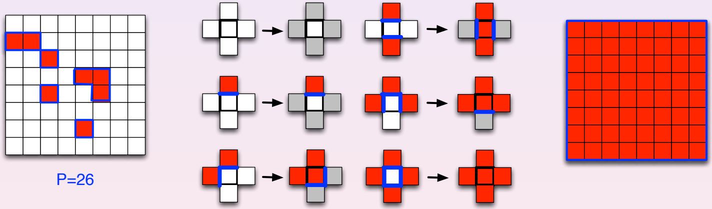
n initially contaminated cells are sufficient. **Is it necessary?**



Warm-up: 2-neighbour bootstrap percolation in grids

Theorem [Bollobás] : The minimum number of initially contaminated cells to eventually contaminate the whole $n \times n$ grid equals n.

n initially contaminated cells are sufficient (e.g., diagonal).
Elegant idea : Notion of "perimeter" P of the contaminated component.



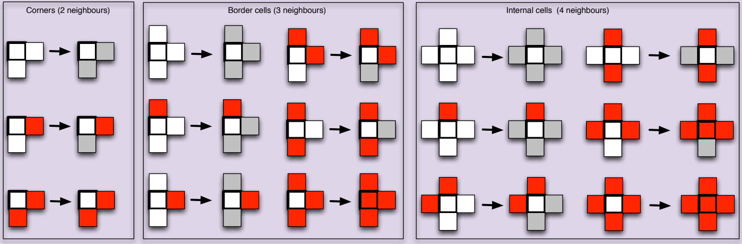
If x cells are initially contaminated: initially $P \leq 4x$.
Through the game, the perimeter P cannot increase!
Eventually (if the whole grid is contaminated) : $P = 4n$. $\Rightarrow x \geq n$

I.e., any lethal set in the $n \times n$ grid has size at least n.

3-neighbour bootstrap percolation in grids

Rules of the (next) game: given an $n \times n$ grid

Each cell can be either **contaminated** (colored) or **healthy** (white).
A contaminated cell remains contaminated forever.
A healthy cell becomes contaminated if it has at least 3 contaminated neighbours.



Lethal set: set of initially contaminated cells eventually contaminating the whole grid

Question: Min. size g_n of a lethal set in the $n \times n$ grid ?

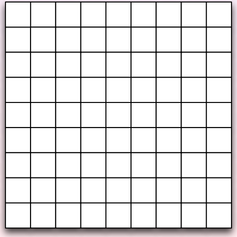
3-neighbour bootstrap percolation in grids

Rules of the (next) game: given an $n \times n$ grid

Each cell can be either **contaminated** (colored) or **healthy** (white).
A contaminated cell remains contaminated forever.
A healthy cell becomes contaminated if it has at least 3 contaminated neighbours.

Lethal set: set of initially contaminated cells eventually contaminating the whole grid

Question: Min. size g_n of a lethal set in the $n \times n$ grid ?



https://scratch.mit.edu/projects/517390361/fullscreen/

(In the "contamination" game, change parameter "Contamination rate" to 3)

3-neighbour bootstrap percolation in grids/graphs

Rules of the (next) game: given any graph G

Each vertex can be either **contaminated** (colored) or **healthy** (white).
A contaminated vertex remains contaminated forever.
A healthy vertex becomes contaminated if it has at least 3 contaminated neighbours.

Lethal set: set of initially contaminated cells eventually contaminating the whole grid (resp., any graph G)

Question: Min. size g_n (resp., $g(G)$) of a lethal set in the $n \times n$ grid (resp., in G)?

Related work

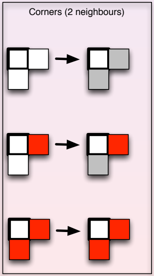
- in bounded max. degree graphs, deciding if $g(G) \leq k$ is NP-complete [Chen 09]
- $\lceil \frac{n^2+2n}{3} \rceil \leq g_n \leq \frac{n^2+4n}{3} + O(1)$ [Pete 97]
- $\lceil \frac{n^2+2n+4}{3} \rceil \leq g_n$ if n even (with equality if $n \equiv 2 \pmod 6$)
  $\lceil \frac{n^2+2n}{3} \rceil \leq g_n$ if n odd [Bollobás 06]

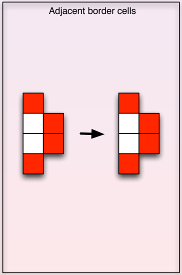
3-neighbour bootstrap percolation: basic properties

Rules of the game: given an $n \times n$ grid

Each cell can be either **contaminated** (colored) or **healthy** (white).
A contaminated cell remains contaminated forever.
A healthy cell becomes contaminated if it has at least 3 contaminated neighbours.

- any lethal set must contain all corners;
- for any two adjacent cells u and v on a border, any lethal set must contain at least one of u and v.





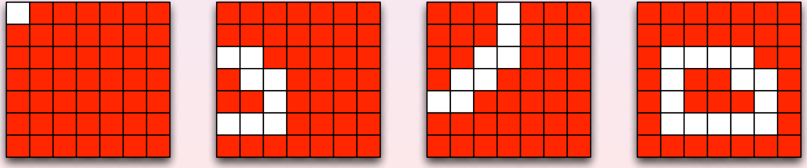
3-neighbour bootstrap percolation: basic properties

Rules of the game: given an $n \times n$ grid

Each cell can be either **contained** (colored) or **healthy** (white).
A contaminated cell remains contaminated forever.
A healthy cell becomes contaminated if it has at least 3 contaminated neighbours.

More generally, if $L \subseteq V$ is a lethal set, then

- each connected component induced by $V \setminus L$ "touches" at most one border;
- each connected component induced by $V \setminus L$ has at most one border vertex;
- $V \setminus L$ induces an acyclic graph.



3-neighbour bootstrap percolation: lower bounds in grids

Rules of the game: given an $n \times n$ grid

Each cell can be either **contaminated** (colored) or **healthy** (white).
A contaminated cell remains contaminated forever.
A healthy cell becomes contaminated if it has at least 3 contaminated neighbours.

Perimeter argument: let L be a lethal set

- each cell of L counts for at most 4 in the initial perimeter;
- if n is even, for each border, there are at least two adjacent vertices in L (counting together for at most 6 in the initial perimeter);
- the perimeter of the contaminated component decreases by at least 2 at every step.

g_n = min. size of a lethal set in the $n \times n$ grid.

Theorem: $g_n \geq LB_n$ where [Bollobás 06]

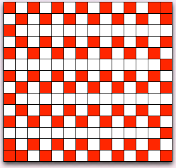
$$LB_n = \begin{cases} \left\lceil \frac{n^2+2n}{3} \right\rceil & \text{if } n \text{ is odd,} \\ \left\lceil \frac{n^2+2n+4}{3} \right\rceil & \text{if } n \text{ is even.} \end{cases}$$

3-neighbour bootstrap percolation: first results in grids

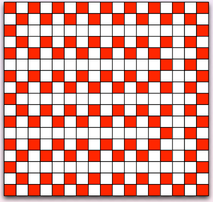
Rules of the game: given an $n \times n$ grid

Each cell can be either **contaminated** (colored) or **healthy** (white).
A contaminated cell remains contaminated forever.
A healthy cell becomes contaminated if it has at least 3 contaminated neighbours.

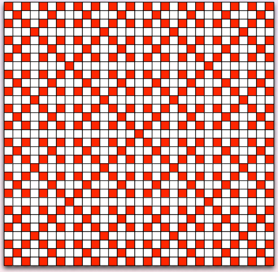
Some optimal initial configurations:



n = 2 (mod 6)



n = 5 (mod 6)



n = 2^p-1 (recursive construction)

Lemma 1: optimality in some cases

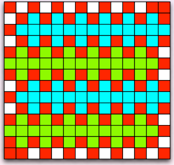
If $n \equiv 2 \pmod 6$ [Bollobás 06] or $n \equiv 5 \pmod 6$ or $n = 2^p - 1$, then $g_n = LB_n$.

3-neighbour bootstrap percolation: first results in grids

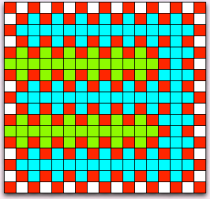
Rules of the game: given an $n \times n$ grid

Each cell can be either **contaminated** (colored) or **healthy** (white).
A contaminated cell remains contaminated forever.
A healthy cell becomes contaminated if it has at least 3 contaminated neighbours.

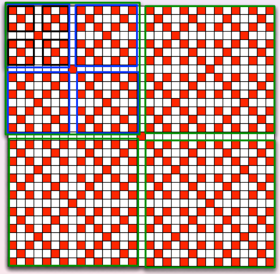
Some optimal initial configurations:



n = 2 (mod 6)



n = 5 (mod 6)



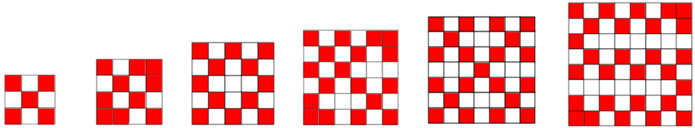
n = 2^p-1 (recursive construction)

Lemma 1: optimality in some cases (general patterns)

If $n \equiv 2 \pmod 6$ [Bollobás 06] or $n \equiv 5 \pmod 6$ or $n = 2^p - 1$, then $g_n = LB_n$.

3-neighbour bootstrap percolation: base cases in grids

Claim: for every $n \leq 8$, $g_n = LB_n$.
($LB_3 = 5, LB_4 = 10, LB_5 = 12, LB_6 = 18, LB_7 = 21$ et $LB_8 = 28$)

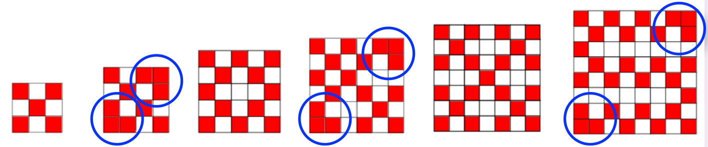


(a) $n = 3$ (b) $n = 4$ (c) $n = 5$ (d) $n = 6$ (e) $n = 7$ (f) $n = 8$

Optimal initial configurations

3-neighbour bootstrap percolation: base cases in grids

Claim: for every $n \leq 8$, $g_n = LB_n$.
$(LB_3 = 5, LB_4 = 10, LB_5 = 12, LB_6 = 18, LB_7 = 21$ et $LB_8 = 28)$

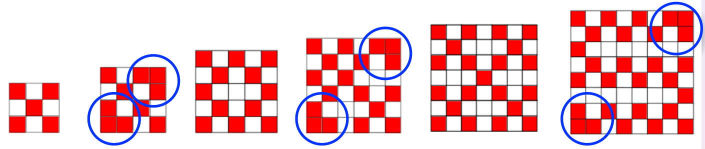


(a) $n = 3$ (b) $n = 4$ (c) $n = 5$ (d) $n = 6$ (e) $n = 7$ (f) $n = 8$

Optimal initial configurations (note the corners in even cases)

3-neighbour bootstrap percolation: base cases in grids

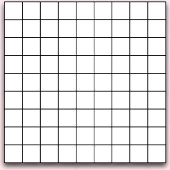
Claim: for every $n \leq 8$, $g_n = LB_n$.
($LB_3 = 5, LB_4 = 10, LB_5 = 12, LB_6 = 18, LB_7 = 21$ et $LB_8 = 28$)

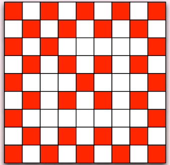


(a) $n = 3$ (b) $n = 4$ (c) $n = 5$ (d) $n = 6$ (e) $n = 7$ (f) $n = 8$

Optimal initial configurations (note the corners in even cases)

A first "strange" case: $n = 9$ $LB_9 = 33$, best known 34?





3-neighbour bootstrap percolation: by computer

Using Mixed Integer Linear Programming

$$\text{Variable } c_{v,t} = \begin{cases} 1 & \text{if } v \in V \text{ contaminated at step t} \\ 0 & \text{otherwise.} \end{cases}$$

Remark: need to fix a maximum number T of steps :(

In general graph $G = (V, E)$, with contamination if $\geq r$ contaminated neighbours

$$\text{Minimize } \sum_{v \in V} c_{v,0} \quad (1)$$

$$\text{Subject to } \sum_{v \in V} c_{v,T} = |V| \quad (2)$$

$$c_{v,t} \leq c_{v,t-1} + \frac{1}{r} \sum_{w \in N(v)} c_{w,t-1} \qquad \forall v \in V, t \in \{1, ..., T\} \quad (3)$$

$$c_{v,t} \in \{0, 1\} \qquad \forall v \in V, t \in \{1, ..., T\} \quad (4)$$

Using Mixed Integer Linear Programming

$$\text{Variable } c_{v,t} = \begin{cases} 1 & \text{if } v \in V \text{ contaminated at step t} \\ 0 & \text{otherwise.} \end{cases}$$

Remark: need to fix a maximum number T of steps :(

In general graph $G = (V, E)$, with contamination if $\geq r$ contaminated neighbours

Minimize	$\sum_{v\in V} c_{v,0}$		(1)
Subject to	$\sum_{v\in V} c_{v,T} = \lvert V\rvert$		(2)
"no decontamination"	$c_{v,t-1} \leq c_{v,t}$	$\forall v \in V, t \in \{1, ..., T\}$	(3)
	$c_{v,t} \leq c_{v,t-1} + \frac{1}{r} \sum_{w\in N(v)} c_{w,t-1}$	$\forall v \in V, t \in \{1, ..., T\}$	(4)
"corners"	$c_{v,0} = 1$	$\forall v \in V, d(v) < r$	(5)
"borders"	$c_{u,0} + c_{v,0} \geq 1$	$\forall \{u, v\} \in E, d(u) = d(v) = r$	(6)
"close neighbourhood"	$c_{v,0} + \sum_{w\in S} c_{w,0} \leq r$	$\forall v \in V, d(v) \geq r, \forall S \subseteq N(v), \lvert S\rvert = r$	(7)
	$c_{v,t} \in \{0, 1\}$	$\forall v \in V, t \in \{1, ..., T\}$	(8)

Even with more cuts, not efficient for 9×9 grid :(

Using Mixed Integer Linear Programming

$$\text{Variable } c_{v,t} = \begin{cases} 1 & \text{if } v \in V \text{ contaminated at step t} \\ 0 & \text{otherwise.} \end{cases}$$

Remark: need to fix a maximum number T of steps :(

More constraints in the specific case of grids and $r = 3$:

$$
\begin{aligned}
&\text{Minimize } \sum_{v\in V} c_{v,0} && && (1)\\
&\text{Subject to } \sum_{v\in V} c_{v,T} = |V| && && (2)\\
&c_{v,t-1} \le c_{v,t} && \forall v \in V, t \in \{1,...,T\} && (3)\\
&c_{v,t} \le c_{v,t-1} + \frac{1}{r} \sum_{w\in N(v)} c_{w,t-1} && \forall v \in V, t \in \{1,...,T\} && (4)\\
&c_{v,0} = 1 && \forall v \in V, d(v) < r && (5)\\
&c_{u,0} + c_{v,0} \ge 1 && \forall \{u,v\} \in E, d(u) = d(v) = r && (6)\\
&c_{v,0} + \sum_{w\in S} c_{w,0} \le r && \forall v \in V, d(v) \ge r, \forall S \subseteq N(v), |S| = r && (7)\\
&c_{v,t} \in \{0,1\} && \forall v \in V, t \in \{1,...,T\} && (8)
\end{aligned}
$$

Definition of the subgrid induced by initially healthy cells (new binary variables)

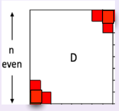
$$
\begin{aligned}
y_e + c_{u,0} + c_{v,0} &\ge 1 && \forall e = \{u,v\} \in E(G') && (9)\\
2y_e + c_{u,0} + c_{v,0} &\le 2 && \forall e = \{u,v\} \in E(G') && (10)\\
y_e &\in \{0,1\} && \forall e = \{u,v\} \in E(G') && (11)
\end{aligned}
$$

It must be acyclic (new fractional variables)

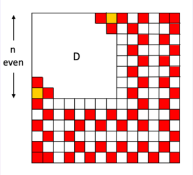
$$
\begin{aligned}
z_{e,u} + z_{e,v} &= y_e && \forall e = \{u,v\} \in E(G') && (12)\\
\sum_{\substack{e'=\{w,v\}\\ w\in N(v)}} z_{e',v} &< 1 && \forall v \in V(G') && (13)\\
z_{e,u}, z_{e,v} &\ge 0 && \forall e = \{u,v\} \in E(G') && (14)
\end{aligned}
$$

New solved cases $n \in \{9, 13\}$: $g_9 = LB_9 + 1 = 34$ and $g_{13} = LB_{13} + 1 = 66$.

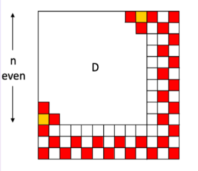
3-neighbour bootstrap percolation: main result in grids



Configuration with LB_n contaminated cells for the n*n grid



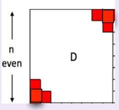
Configuration with LB_{n+6} contaminated cells for the (n+6)*(n+6) grid



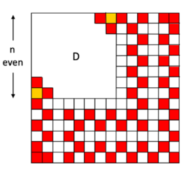
Configuration with $LB_{n+3}+1$ contaminated cells for the (n+3)*(n+3) grid

Lemma: $g_n = LB_n$ if n even, and $g_n \leq LB_n + 1$ if n odd. (induction on n even)

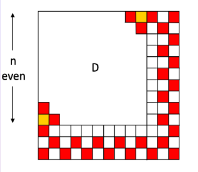
3-neighbour bootstrap percolation: main result in grids



Configuration with LB_n contaminated cells for the n*n grid



Configuration with LB_{n+6} contaminated cells for the (n+6)*(n+6) grid



Configuration with $LB_{n+3}+1$ contaminated cells for the (n+3)*(n+3) grid

Theorem [Benevides, Bermond, Lesfari, N., 2021]

- $g_n = LB_n$ if n even, and $LB_n \leq g_n \leq LB_n + 1$ if n odd;
- moreover, $g_n = LB_n$ if $n \equiv 5 \pmod 6$ or $n = 2^p - 1$;
- and, $g_n = LB_n + 1$ if $n \in \{9 \text{ (ugly case analysis)}, 13 \text{ (result of the MILP)}\}$.

n	1	2	3	4	5	6	7	8	**9**	10	11	12	**13**	14	15	16
LB_n	1	3	5	10	12	18	21	28	**33**	42	48	58	**65**	76	85	98
g_n	-	-	-	-	-	-	-	-	**34**	-	-	-	**66**	-	-	-

n	17	18	19	20	21	22	23	24	25	26	27	28
LB_n	108	122	133	148	161	178	192	210	225	244	261	282
g_n	-	-	≤ 134	-	≤ 162	-	-	-	≤ 226	-	≤ 262	-

3-neighbour bootstrap percolation in square tori

$t_n =$ min. size of a lethal set in the $n \times n$ torus.

Previous work (irreversible dynamic monopolies) [Flocchini *et al.* 04]

For any $n \geq 1$, $t_n \geq LBT_n = \lceil \frac{n^2+1}{3} \rceil$.

Proof: $I \subseteq V$ any lethal set, $H = V \setminus I$. m_{IH} # of edges between I and H, m_H # of edges induced by H. Recall that H induces a forest. Hence, $m_H \leq |H| - 1$. Moreover (degree), $m_{IH} = 4|H| - 2m_H \leq 4|I|$. □

Moreover, $t_n \leq \lceil \frac{n}{3} \rceil (n+1)$.

3-neighbour bootstrap percolation in square tori

t_n = min. size of a lethal set in the $n \times n$ torus.

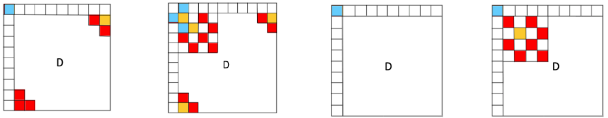
Previous work (irreversible dynamic monopolies) [Flocchini *et al.* 04]

For any $n \geq 1$, $t_n \geq LBT_n = \lceil \frac{n^2+1}{3} \rceil$.

Proof: $I \subseteq V$ any lethal set, $H = V \setminus I$. m_{IH} # of edges between I and H, m_H # of edges induced by H. Recall that H induces a forest. Hence, $m_H \leq |H| - 1$. Moreover (degree), $m_{IH} = 4|H| - 2m_H \leq 4|I|$. □

Moreover, $t_n \leq \lceil \frac{n}{3} \rceil (n+1)$.

Theorem: For any $n \geq 1$, $t_n = LBT_n$ [Benevides, Bermond, Lesfari, N., 2021]



(a) $n \equiv 3$ [6] (b) $n \equiv 1, 5$ [6] (c) $n \equiv 0$ [6] (d) $n \equiv 2, 4$ [6]

where D is a lethal set of the $(n-1) \times (n-1)$-grid of size $LB_{n-1} + 1$ if $n \equiv 2, 4$ (mod 6) and of size LB_{n-1} otherwise. (removing yellow cells and adding blue cells)

Further work

Case of $n \times n$ grids: many remining unknown cases

$g_{19} \in \{133, 134\}$, $g_{21} \in \{161, 162\}$, $g_{25} \in \{225, 226\}$, $g_{27} \in \{261, 262\}$...

More generally, what if n odd and

$n \not\equiv 5 \pmod 6$ and $n \neq 2^p - 1$ and $n \notin \{9, 13\}$?

Tradeoff between number of initially contaminated cells and contamination time?

(could be useful for the MILP)

Case of rectangular grids/tori?

(seems quite similar to the square case, but maybe not...)

Higher dimensions?

...

Thanks for your attention !

Further work

Case of $n \times n$ grids: many remining unknown cases

$g_{19} \in \{133, 134\}$, $g_{21} \in \{161, 162\}$, $g_{25} \in \{225, 226\}$, $g_{27} \in \{261, 262\}$...

More generally, what if n odd and

$$n \not\equiv 5 \pmod 6 \text{ and } n \neq 2^p - 1 \text{ and } n \notin \{9, 13\}?$$

BREAKING NEWS (?): if $n \not\equiv 5 \pmod 6$ odd, then $g_n = LB_n$ iff $n = 2^p - 1$

Abel Romer, *Tight Bounds on 3-Neighbor Bootstrap Percolation*, Master thesis, Quest University Canada, **2022-08-31**. `https://dspace.library.uvic.ca/handle/1828/14170`

Tradeoff between number of initially contaminated cells and contamination time?

(could be useful for the MILP)

Case of rectangular grids/tori?

(seems quite similar to the square case, but maybe not...)

Higher dimensions?

...

Thanks for your attention !