On some positional games in graphs

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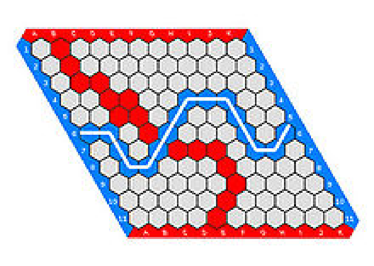
Two well known 2-Player games



Hex

Alice and Bob alternately select cells.

Alice wins if red sides connected.
Bob wins otherwise.

Tic Tac Toe

Alice($\circ$) and Bob($\times$) alternately select cells

First one to align 3 cells wins.

Pictures from Wikipedia

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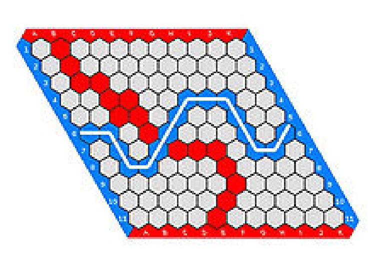

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Maker-Breaker

Always one winner.

Maker-Maker

Either Alice wins or draw.

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Positional games

[Erdös, Selfridge 73 ; Beck 08]

Combinatorial games: typically sequential games with perfect information

Positional games (2-Player game)

- Finite set X of elements; *(vertices of an hypergraph)*
- Family $\mathcal{F}$ of subsets of X; *(winning hyperedges)*

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Waiter-Client: Alice chooses 2 vertices, Bob selects one, Alice the other;
Biais: Alice selects $a \geq 1$ vertices and Bob selects $b \geq a$ vertices;

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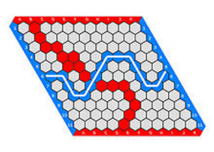

Board: (arbitrary large) hexagonal grid.
Winning sets: all paths "between top and bottom".
Maker-Breaker / **Maker-Maker**

Theorem: Alice has a winning strategy

Proof: Strategy-stealing argument.

By contradiction Bob has a winning stragegy. Alice initially selects any vertex (it "does not hurt"). Then, she plays as a second player using Bob's winning strategy.

Open problem: Alice's winning strategy only known for small grids (9×9, computer proof).

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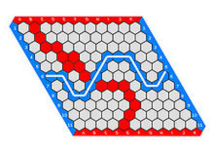

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Remark: Hex (initial configuration) is PSPACE-complete [Reisch 81]

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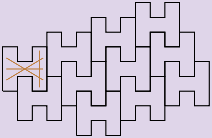

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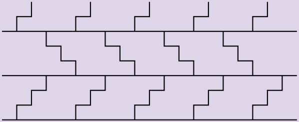

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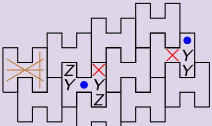

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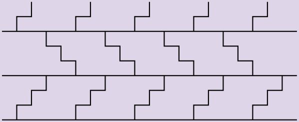

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Positional Games / Complexity

Consider a game $(X, \mathcal{F})$ in Maker-Breaker convention.

Erdös-Selfridge criterion [Erdös, Selfridge 73]

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POS CNF: set of variables $X = \{x_1, \ldots, x_n\}$, $\mathcal{F} = \{C_1, \cdots, C_m\} \subseteq 2^X$ and
CNF formula $\phi = C_1 \wedge \cdots \wedge C_m$ (all variables in positive form).

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Complexity of deciding the outcome

PSPACE-complete

- Clauses/hyperedges of size 11 [Schaefer 78], of size 6 [Rahman, Watson 21]

Polynomial

- Clauses of size 2 (graphs: trivial), of size 3 [Galliot, Gravier, Sivignon 23+]

Positional Games “arising” from graphs

Game $(V, \mathcal{F})$ where $\mathcal{F}$ “reflects some structures” of a graph $G = (V, E)$

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Generalized Hex (Shannon switching game on vertices)

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Maker-Breaker Domination game [Duchêne, Gledel, Parreau, Renault 18]

$G = (V, E)$, $\mathcal{F}$: set of all dominating sets.

- PSPACE-complete in bipartite and split graphs;
- Polynomial in trees and cographs; **Open:** Interval, bounded tw...
- Study on length of the game.

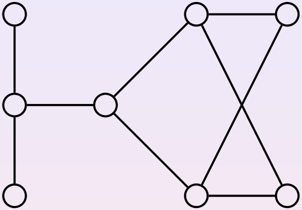
Largest Connected Subgraph game

joint works with J. Bensmail, F. Fioravantes, F. Mc Inerney and N.Oijid.

Largest con. subgraph game [Bensmail, Fioravantes, Mc Inerney, N., 2021]

- Alice and Bob alternately colour uncoloured vertices, until all vertices coloured.

- **Scoring game:** Player with the **largest** (# nodes) **connected subgraph** of their colour wins.



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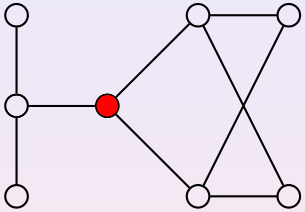

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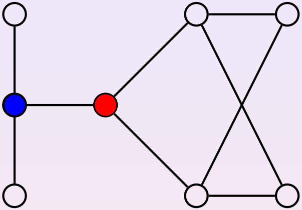

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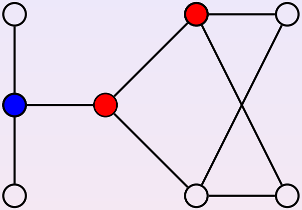

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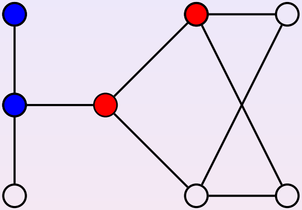

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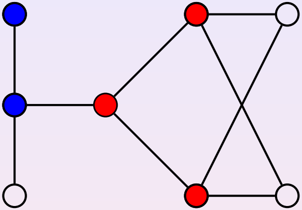

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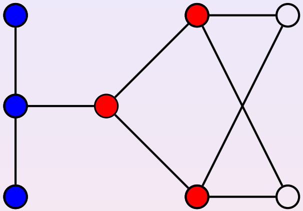

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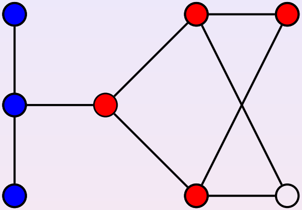

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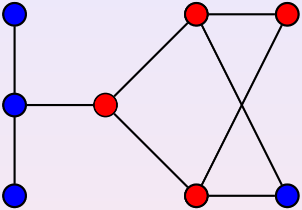

Alice wins $4 - 3$

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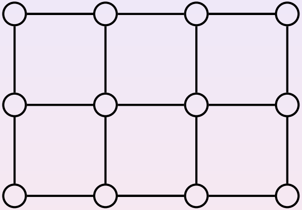

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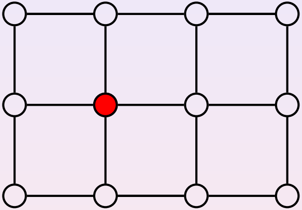

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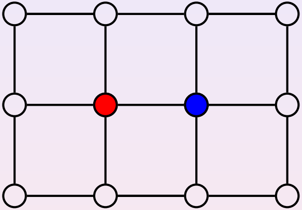

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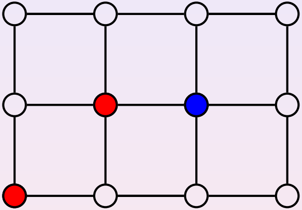

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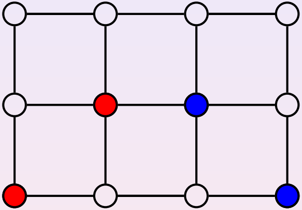

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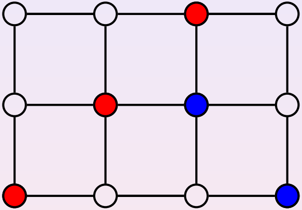

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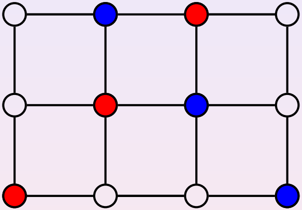

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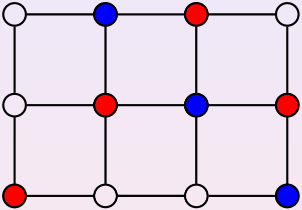

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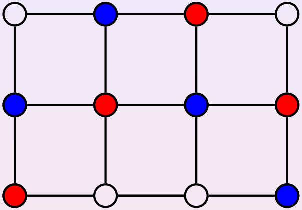

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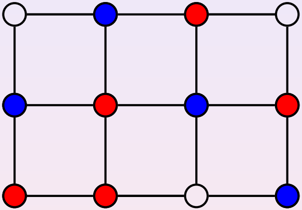

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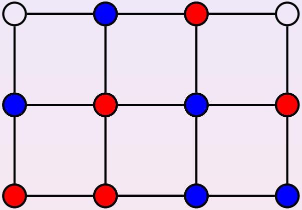

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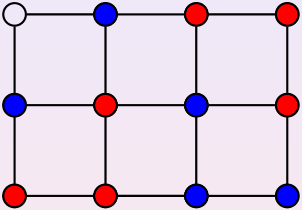

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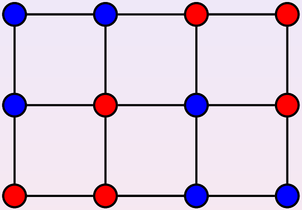

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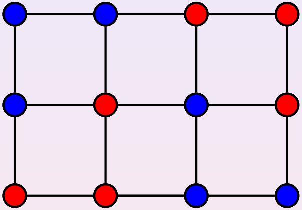


Draw 3 – 3

Theorem

Either Alice has a winning strategy, or Bob can ensure a draw.

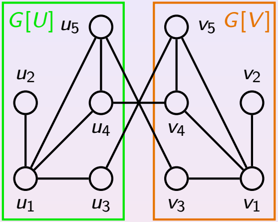
Complexity of LCSG

[Bensmail, Fioravantes, Mc Inerney, N., 2021]

Reflection graph

Any graph G for which $V(G)$ can be **partitioned** into $U = (u_1, \ldots, u_n)$ and $V = (v_1, \ldots, v_n)$ such that:

1. $f\colon U \rightarrow V$ with $f(u_i) = v_i$ for all $i \in \{1, \ldots, n\}$, is an **isomorphism** between $G[U]$ and $G[V]$;
2. for any two $i, j \in \{1, \ldots, n\}$, if $\mathrm{u_i v_j} \in \mathrm{E(G)}$, then $\mathrm{u_j v_i} \in \mathrm{E(G)}$.



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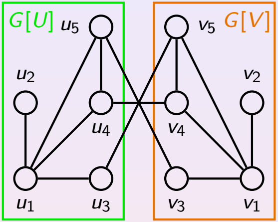

Theorem

In any reflection graph, Bob can force a draw.
Recognising reflection graphs is GI-hard.

Complexity of LCSG

[Bensmail, Fioravantes, Mc Inerney, N., 2021]

Reflection graph

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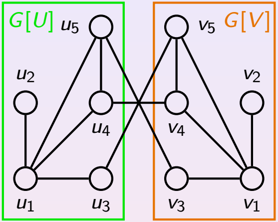

Theorem

In any reflection graph, Bob can force a draw.
Recognising reflection graphs is GI-hard.

Theorem

Deciding if Alice wins the LCSG is PSPACE-complete, even for bipartite graphs with diameter at most 5.

LCSG in simple graph classes

[Bensmail, Fioravantes, Mc Inerney, N., 2021]

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Alice wins

LCSG in simple graph classes

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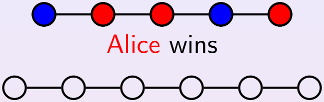

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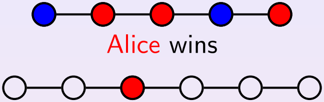

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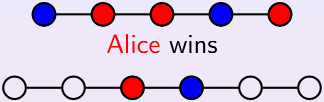

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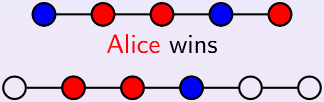

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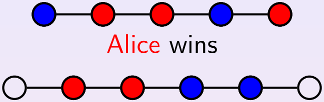

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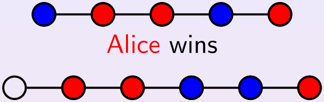

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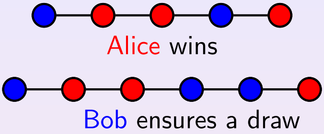
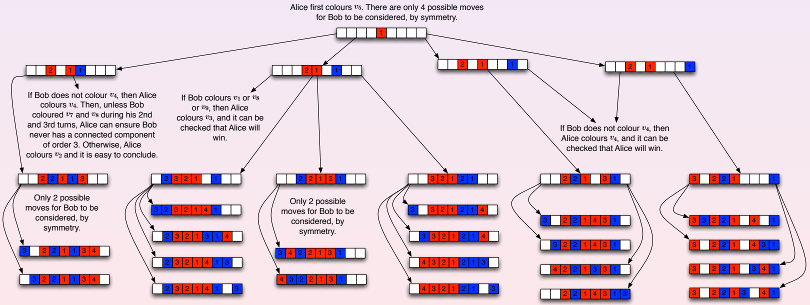


LCSG in simple graph classes

[Bensmail, Fioravantes, Mc Inerney, N., 2021]

Alice wins

Bob ensures a draw



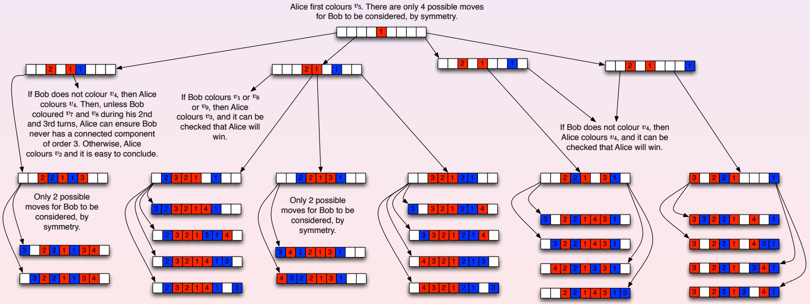
LCSG in simple graph classes

[Bensmail, Fioravantes, Mc Inerney, N., 2021]

Theorem

For all $n \geq 1$, Alice wins in the path P_n if and only if $n \in \{1, 3, 5, 7, 9\}$.

- If $n \in \{1, 3, 5, 7, 9\}$, then case analysis.
- If n is even, then Bob can draw (reflection graph).
- For odd $n \geq 11$, non trivial induction with pre-coloured vertices.



LCSG in simple graph classes

[Bensmail, Fioravantes, Mc Inerney, N., 2021]

Theorem

For all $n \geq 1$, Alice wins in the cycle C_n if and only if n is odd.

LCSG in simple graph classes

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For all $n \geq 1$, Alice wins in the cycle C_n if and only if n is odd.

***A*-perfect graph**

Where Alice can ensure a single red component (of size $\lceil \frac{n}{2} \rceil$).

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The outcome of the game in any cograph G is computable in linear time.

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Theorem

The outcome of the game in any cograph G is computable in linear time.

Proof: Induction on $|V|$: return the outcome and whether G is A-perfect or not.

Join $G = G_1 \oplus G_2$ (with $|V_1| \leq |V_2|$): G is A-perfect. Moreover Bob cannot ensure a single component iff $|V_1| = 1$ and G_2 not A-perfect.

Disjoint union of joins $G = \bigcup_i G_i$ (with all G_i are cographs resulting from a join). Case analysis based on:

- whether or not G_i is a join with a singleton,
- parity of $|V(G_i)|$,
- whether or not G_i is A-perfect.

Maker-Breaker LCSG

[Bensmail, Fioravantes, Mc Inerney, N., Oijid, 2023]

Input: A graph $G = (V, E)$ and $k \in \mathbb{N}$.

Game $(V, \mathcal{F})$ with $\mathcal{F}$ all subsets of at least k vertices inducing a connected subgraph.

Alice wins iff she gets a red connected subgraph with at least k vertices.

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New graph invariant

$c_g(G)$: maximum integer k such that Alice wins in G.

Rmk: For every graph G, $\lfloor\frac{\Delta(G)}{2}\rfloor + 1 \leq c_g(G) \leq \lceil\frac{|V|}{2}\rceil$

$$c_g(G) = \lceil\frac{|V|}{2}\rceil \text{ iff } G \text{ is } A\text{-perfect.}$$

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c_g of simple graph classes

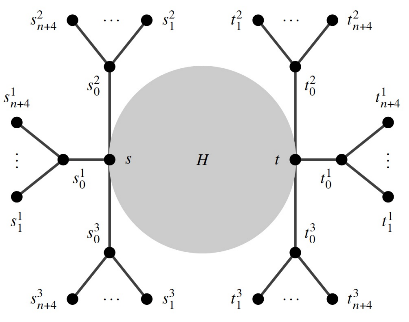
- $c_g(P_n) = c_g(C_n) = 2$ for every $n \geq 3$.
- $c_g(G)$ can be computed in linear time in the class of $(q, q-4)$ graphs.

M-B LCSG / Complexity

[Bensmail, Fioravantes, Mc Inerney, N., Oijid, 2023]

Theorem

Given a graph G and an integer $k \geq 1$, it is PSPACE-complete to decide whether $c_g(G) \geq k$, even if G is a bipartite, split or planar graph.



Planar case: $c_g(G) \geq |V(H)| + 5$ iff Alice wins the planar extended Hex game in H.

M-B LCSG / A-perfect

[Bensmail, Fioravantes, Mc Inerney, N., Oijid, 2023]

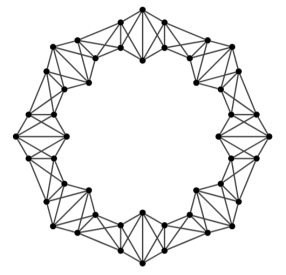
Some sufficient conditions to be A-perfect

If $\Delta(G) + \delta(G) \geq n$ or $|E(G)| - 3 \geq \frac{(n-2)(n-3)}{2}$, then G is A-perfect.

Lemma

There exist arbitrarily large d-regular A-perfect graphs iff $d \geq 4$.

(if G cubic and A-perfect, then $|V(G)| \leq 16$)



Further work on Largest Con. Subgraph game

About A-perfect graphs

- Complexity of deciding if a graph is A-perfect?
- Other necessary and/or sufficient conditions?

Other graph classes

- Complexity of deciding if Alice wins the LCSG in trees?
- Complexity of deciding if $c_g(T) \geq k$ in a tree T?
- Value or bounds for $c_g(Grid)$? $c_g(Hexagonal\ Grid) = 6$. $c_g(P_n \square P_m) \leq 2\min\{n, m\}$.
- Interval graphs...?

Other variants

- Connected variant: Alice can only colour a vertex adjacent to a red vertex.

H-game

joint work with E. Duchêne, V. Gledel, F. Mc Inerney, N. Oijid, A. Parreau and M. Stojaković .

Maker-Breaker game on edge-set of graphs

Given a graph $G = (V, E)$, Alice and Bob select elements of E.

On complete graphs $G = K_n$ [Chvátal, Erdös, 78]

- Connectivity game — *Alice* wants the edges of a spanning tree
- Hamiltonian game
- Perfect matching game
- H-game — *Alice* wants the edges of a copy of a fixed graph H.

Goal: find the threshold bias — e.g., [Krivelevich 11]

Also played in Erdös-Rényi random graph model G

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Here, complexity of the outcome in any graph G.

Complexity

[Duchêne, Gledel, Mc Inerney, N., Oijid, Parreau, Stojaković]

Arboricity-k game: Alice wins if red graph has arboricity at least k

Theorem: Arboricity-k game is in P ($k = 2$: cycle game)

Alice wins the arboricity-k game in G iff $k \leq \lceil arboricity(G)/2 \rceil$.

Theorem: Perfect matching game is PSPACE-complete

Reduction from Uniform POS CNF 6.

***H*-game:** H is a fixed graph. Given a graph G as input, can Alice select edges to create an induced copy of H?

Theorem: H-game is PSPACE-complete. More precisely

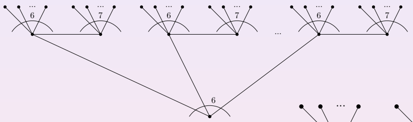
- $\exists$ a tree T, s.t. T-game PSPACE-c in graphs with diameter ≤ 6.
- $\exists$ a graph H' of order 51 (size 57), s.t. H'-game PSPACE-c.

Complexity *Tree*-game

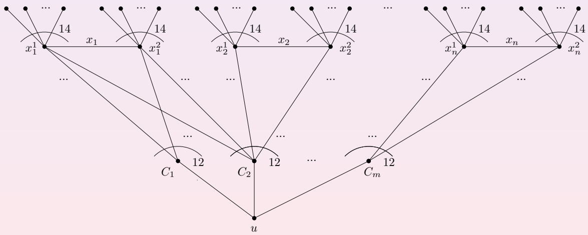
[Duchêne, Gledel, Mc Inerney, N., Oijid, Parreau, Stojaković]

Theorem: $\exists$ a tree T, s.t. T-game PSPACE-c.

Reduction from Uniform POS CNF 6 where falsifier starts.



The tree T.



Scheme of the reduction.

Alice wins $\Leftrightarrow$ Falsifier wins

T-game: some poly. cases [Duchêne, Gledel, Mc Inerney, N., Oijid, Parreau, Stojaković]

Theorem: Bob wins the P_4-game in G iff.

1. G is bipartite and all the vertices of degree at least 3 are in the same part; or
2. G is an odd cycle; or
3. G is a subgraph of the bull, K_4, or a C_5 with a leaf attached to one vertex.

Theorem: Star-game in trees

Let $\ell \geq 1$, the outcome of the $K_{1,\ell}$-game can be decided in linear time in a trees.

Theorem: Parameterized complexity

- The $K_{1,\ell}$-game is FPT by the length t of the game.
  (small (w.r.t. ℓ) max. degree + bounded (in t) diameter)
- The H-game in trees is FPT by the length of the game.

Further work on the H-game

Complexity in general graphs

- Smallest graph H such that H-game is PSPACE-complete?
- Dichotomy of graphs H: H-game PSPACE-complete *vs.* Polynomial?
- Is the H-game $W[1]$-hard parameterized by the length of the game?

H-game in trees

- Complexity?
- Cost of connectedness?

(if Alice's edges must induce a connected component)

H-game in other graph classes?

Other conventions: Avoider-Enforcer, Waiter-Client...?

Conclusion: many other games

Impartial games: allowable moves depend only on the position, not on the current player. [Sprague 36, Grundy 39, Berlekamp, Conway, Guy 82]

Impartial games in graphs

Alice and Bob alternately "select" vertices, according to some common rule.

The first player who cannot "select" a vertex loses.

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Kayles game / Clique forming game: selected vertices must induce a stable set (resp., a clique) [Schaefer 78]

Coloring game: Alice and Bob colour vertices with k colours in a proper way.

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Harmonious Coloring game: see the next talk of Nicolas Martins :)

[Linhares, Martins, N., Sampaio 24+]

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Obrigado!