Cops and robber games in graphs

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COATI

Pursuit-Evasion Games

2-Player games

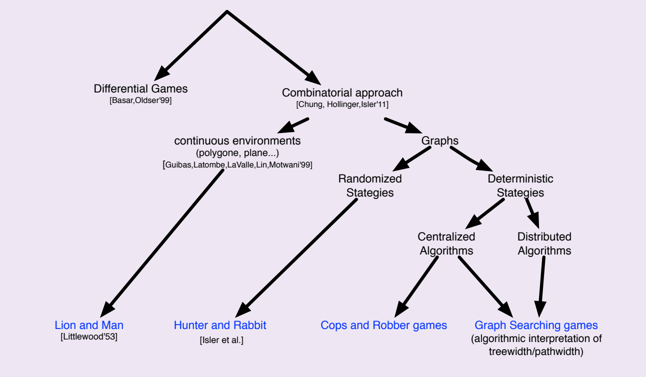
A team of mobile entities (Cops) track down another mobile entity (Robber)

Always one winner

- **Combinatorial Problem:**
  Minimizing some resource for some Player to win
  e.g., minimize number of Cops to capture the Robber.
- **Algorithmic Problem:**
  Computing winning strategy (sequence of moves) for some Player
  e.g., compute strategy for Cops to capture Robber/Robber to avoid the capture

natural applications: coordination of mobile autonomous agents
(Robotic, Network Security, Information Seeking...)
but also: Graph Theory, Models of Computation, Logic, Routing...

Pursuit-Evasion: Over-simplified Classification



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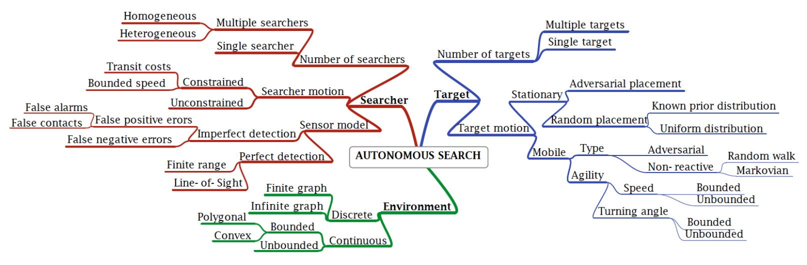
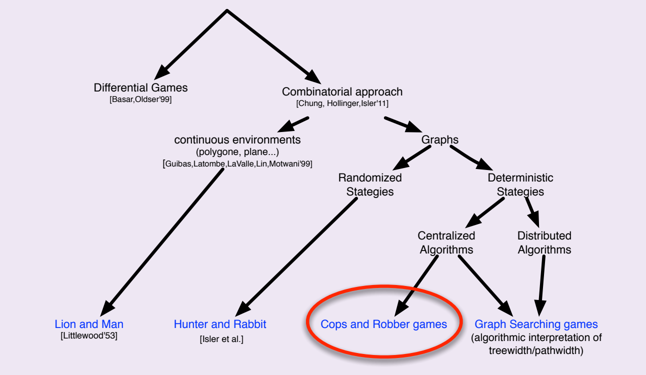


[Chung,Hollinger,Isler'11]

Pursuit-Evasion: Over-simplified Classification



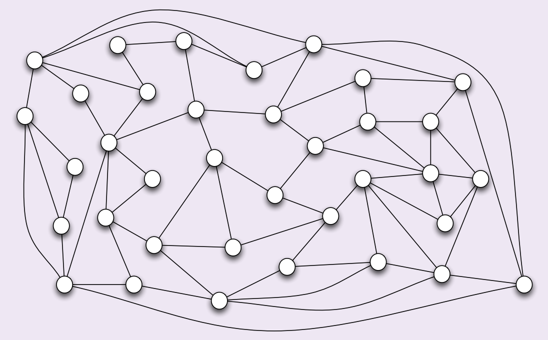
Today: focus on Cops and Robber games

Goal of this talk: illustrate that studying Pursuit-Evasion games helps

- Offer new approaches for several structural graph properties
- Models for studying several practical problems
- Fun and intriguing questions

Cops & Robber Games [Nowakowski and Winkler; Quilliot, 1983]

Rules of the $\mathcal{C}\&\mathcal{R}$ game



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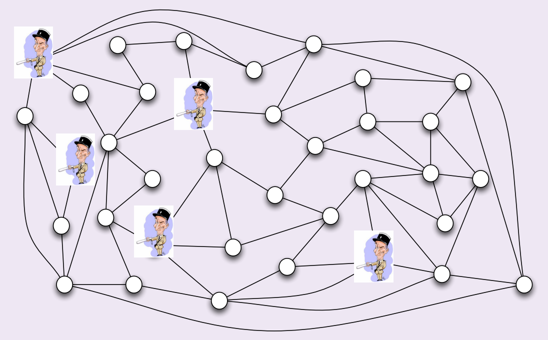

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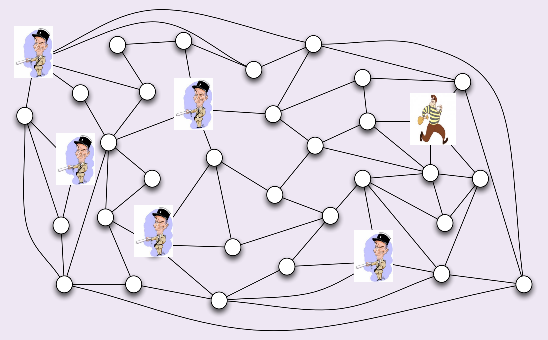

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1. Place $k \geq 1$ Cops $\mathcal{C}$ on nodes
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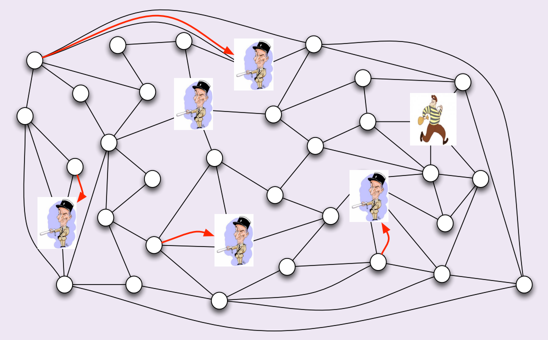
3. Turn by turn
   (1) each $\mathcal{C}$ slides along ≤ 1 edge



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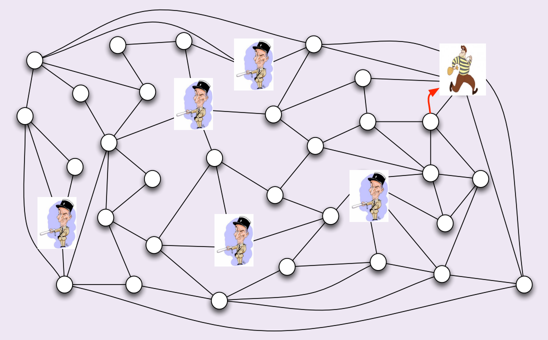

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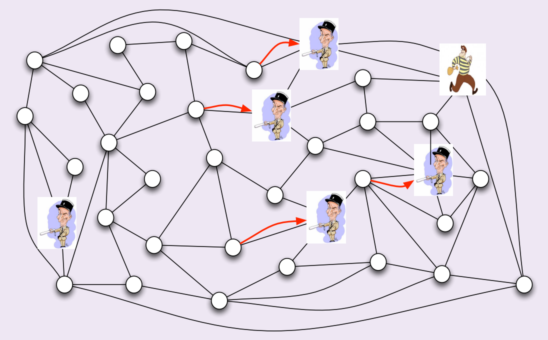

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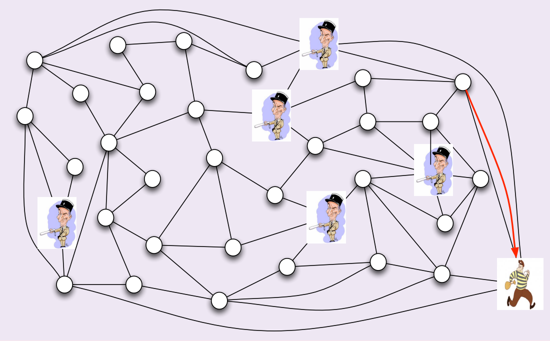
Goal of the $\mathcal{C}\&\mathcal{R}$ game

- Robber must avoid the Cops



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Goal of the $\mathcal{C}\&\mathcal{R}$ game

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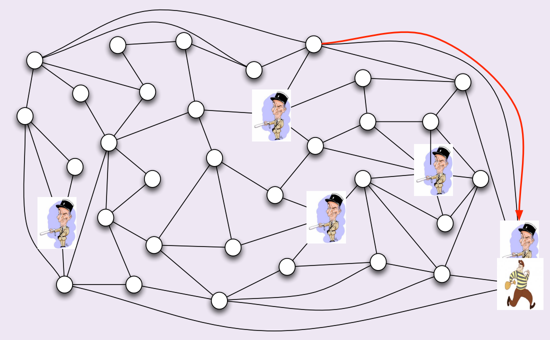
- Cops must capture Robber (i.e., occupy the same node)



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Goal of the $\mathcal{C}\&\mathcal{R}$ game

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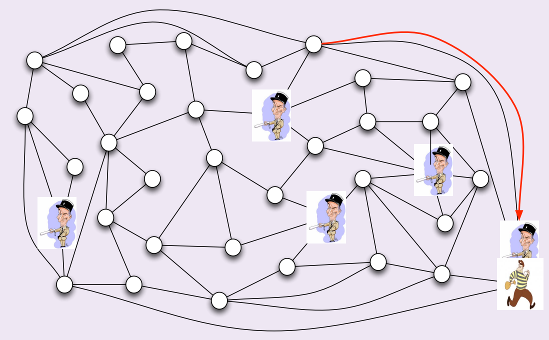
Cop Number of a graph G

$cn(G)$: min # Cops to win in G



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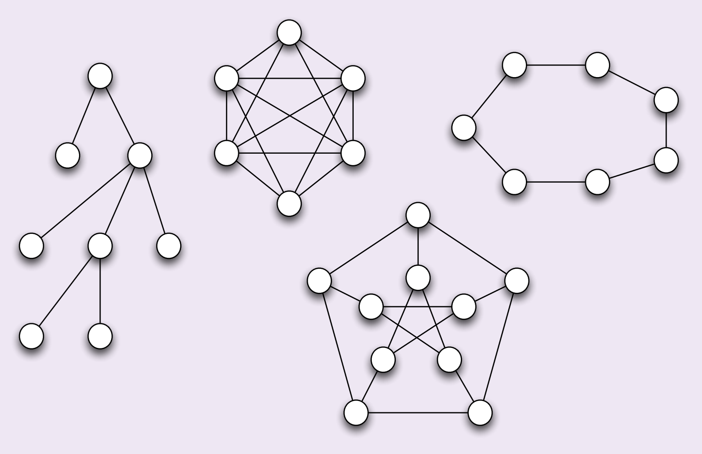

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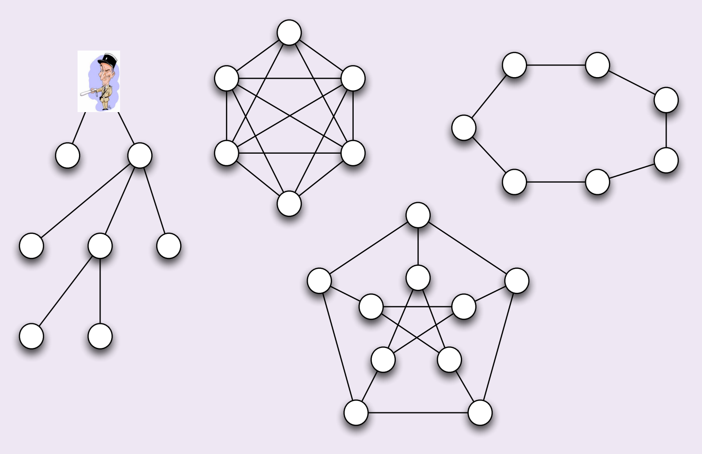

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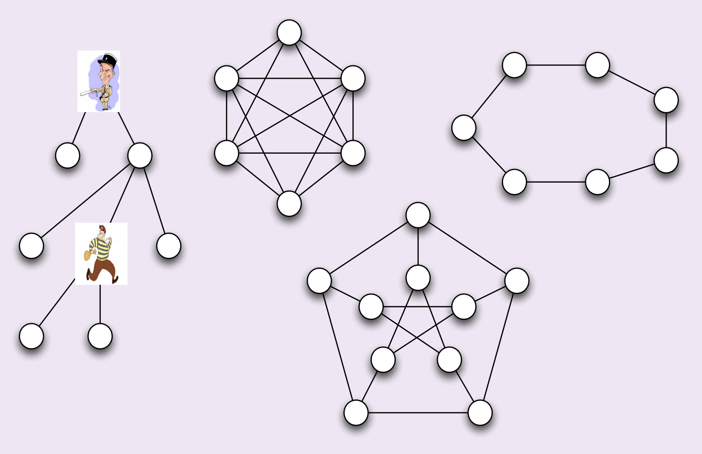

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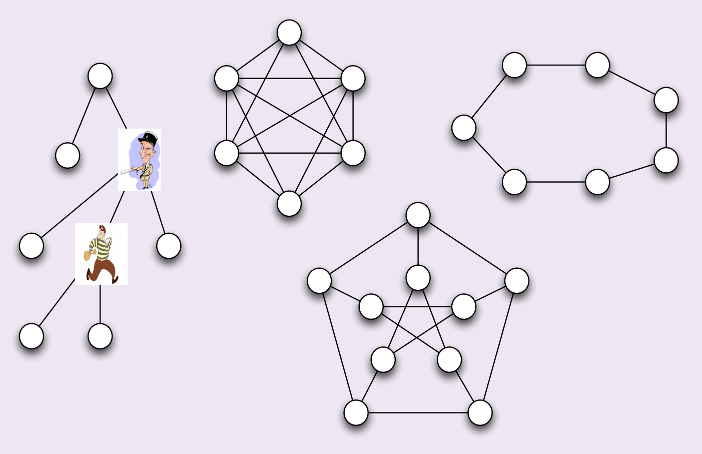

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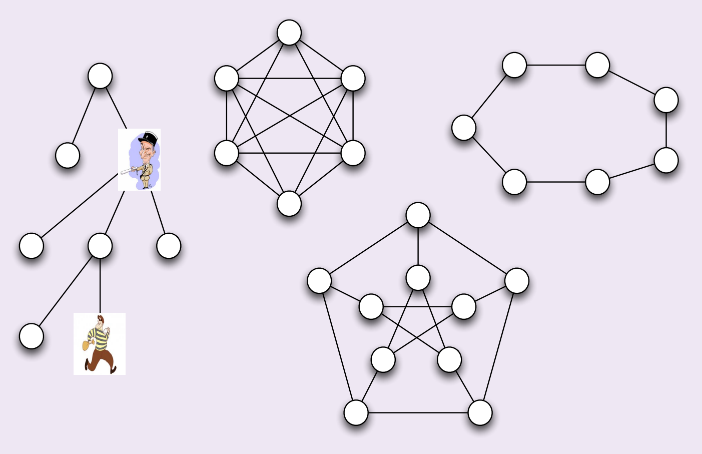

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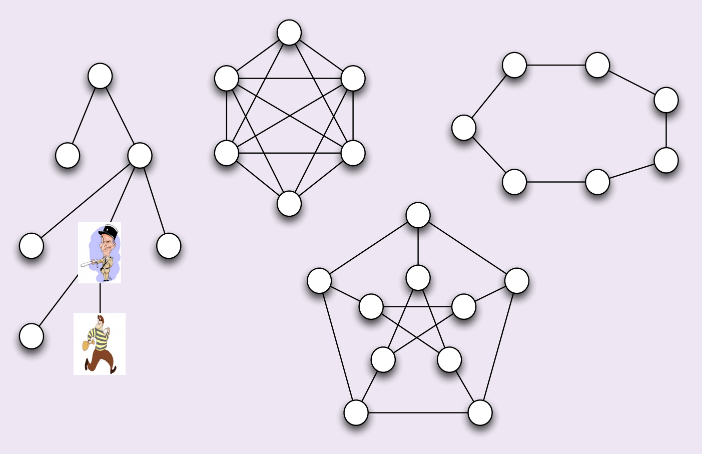

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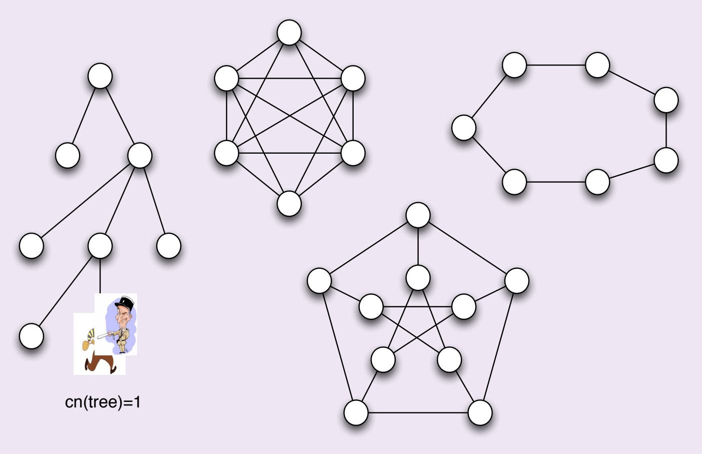

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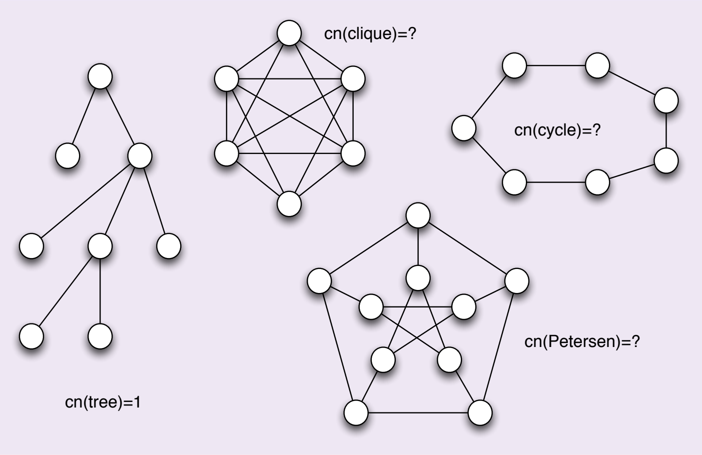

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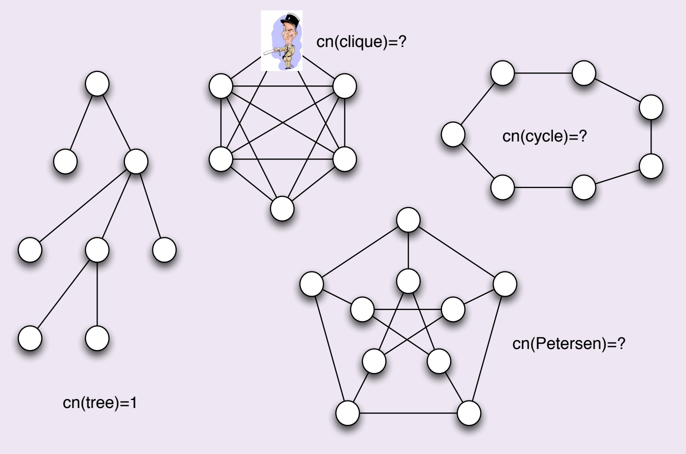

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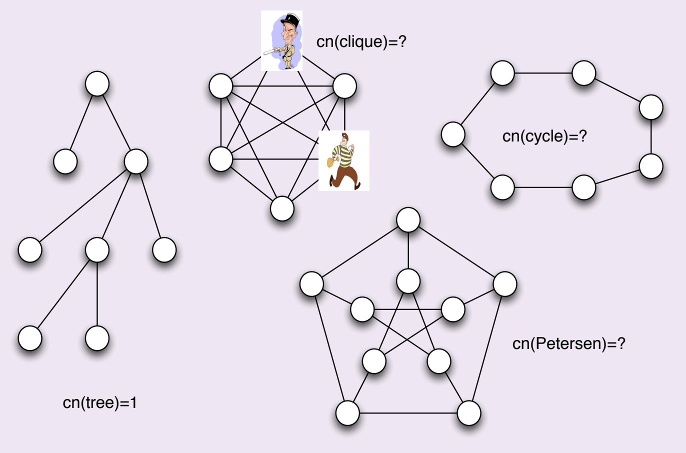

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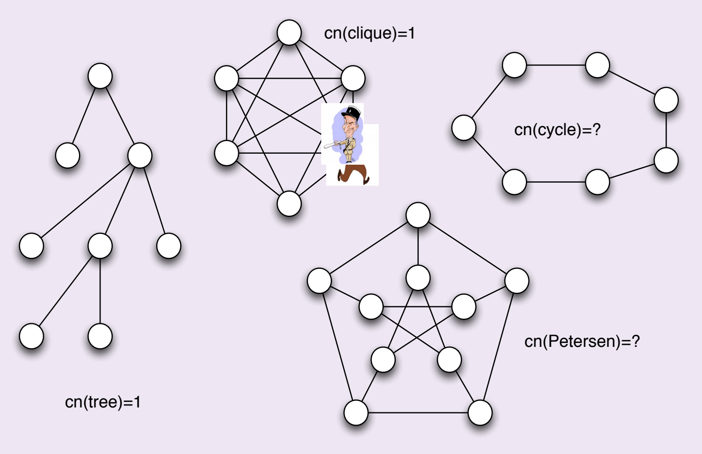

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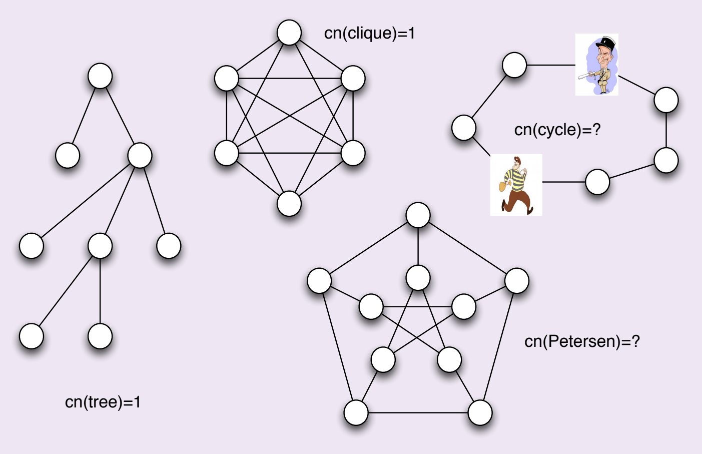

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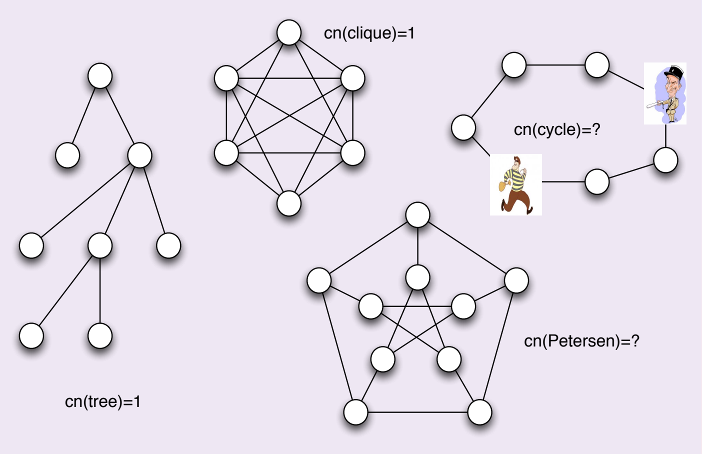

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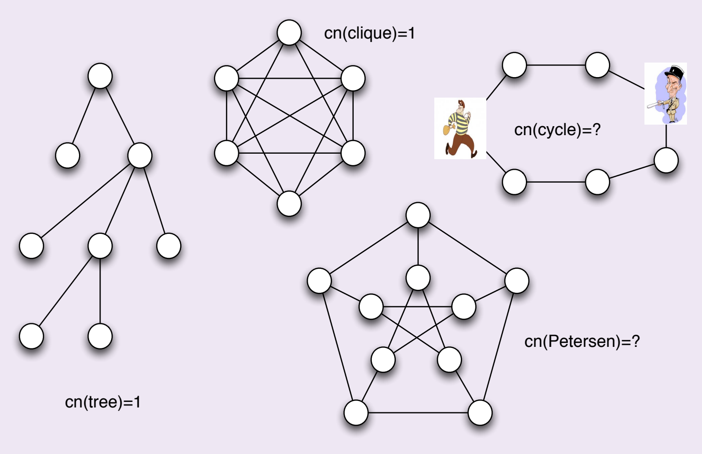

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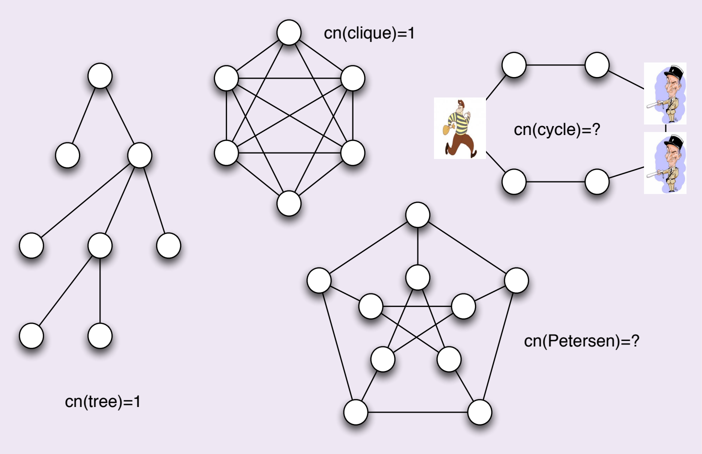

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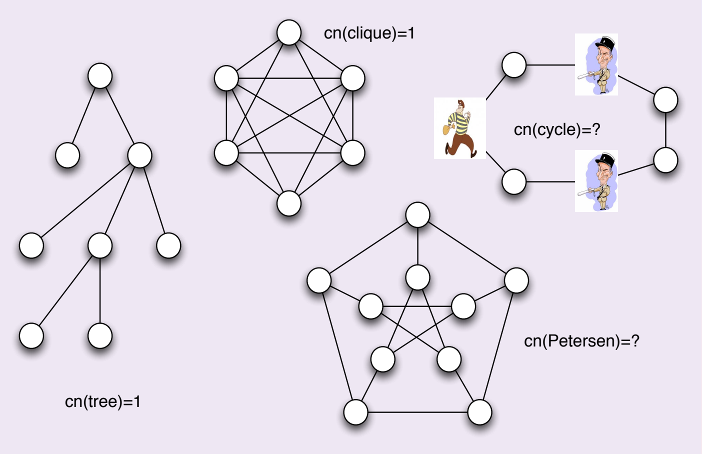

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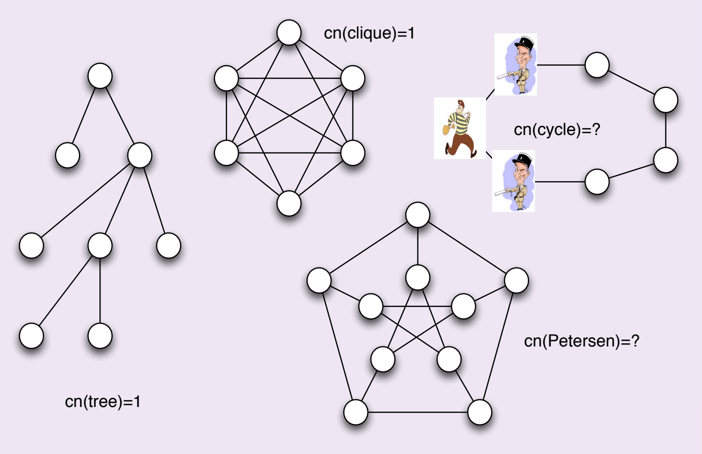

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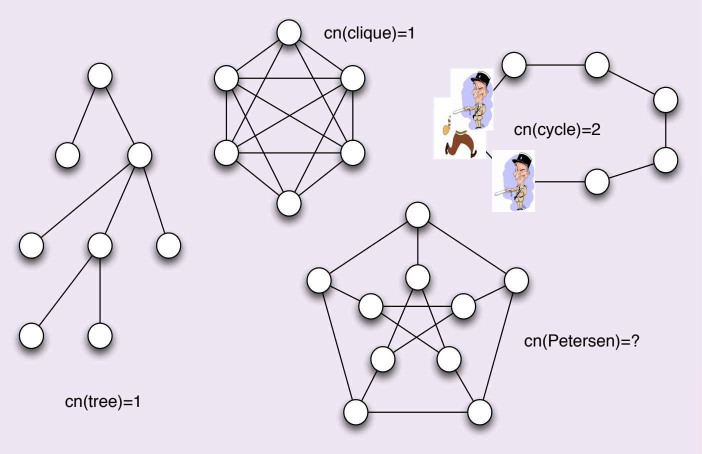

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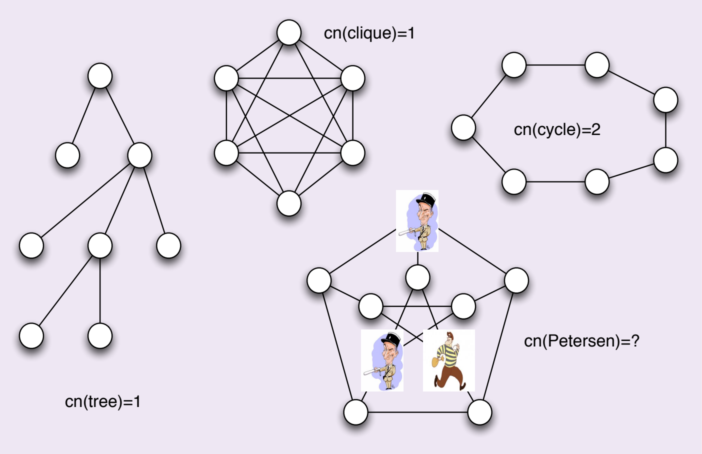

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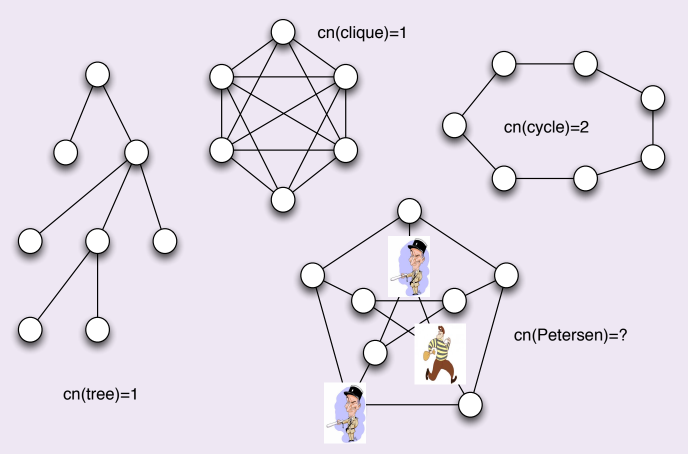

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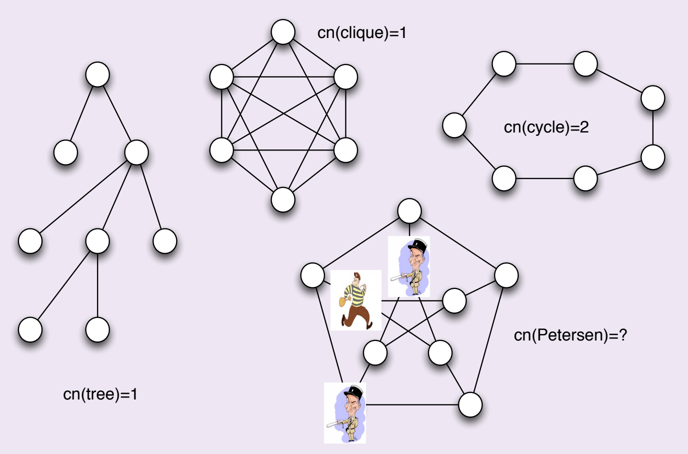

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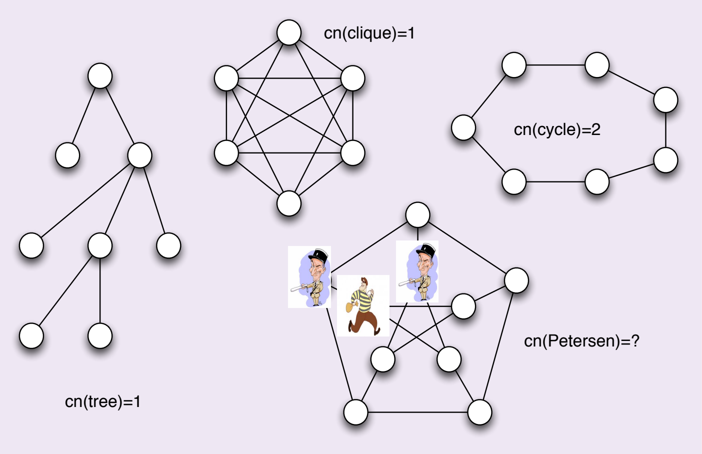

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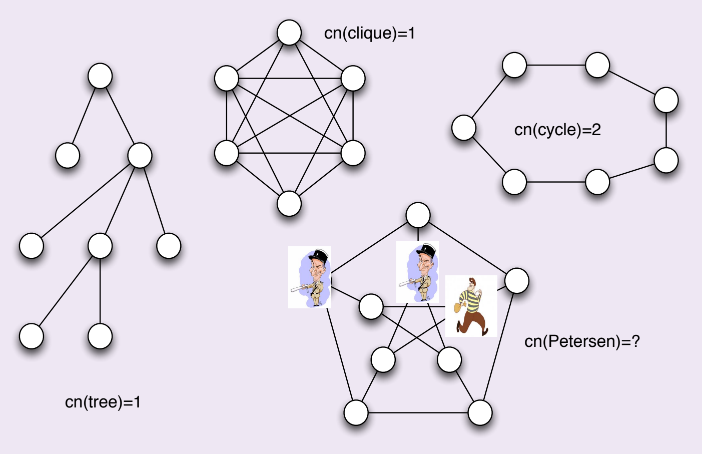

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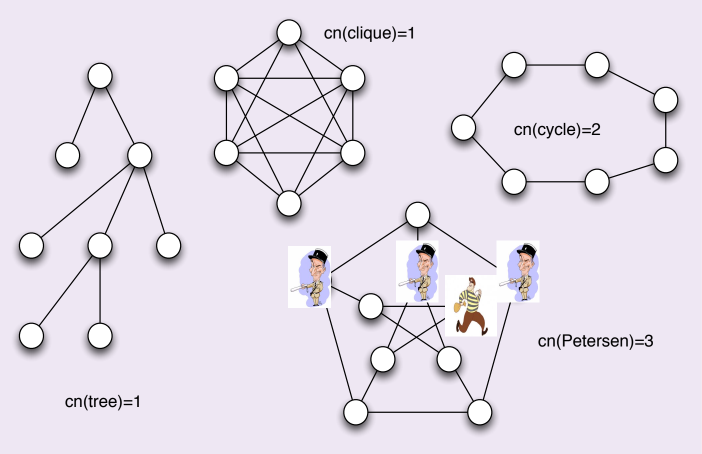

Easy remark: For any graph G, $cn(G) \leq \gamma(G)$ the size of a min dominating set of G.

Complexity: a graph G, $cn(G) \leq k$?

Seminal paper: $k = 1$ [Nowakowski and Winkler; Quilliot, 1983]

$cn(G) = 1$ iff $V = \{v_1, \cdots, v_n\}$ and, $\forall i < n$, $\exists j > i$ s.t., $N(v_i) \cap \{v_i, \cdots, v_n\} \subseteq N[v_j]$.
(dismantable graphs) can be checked in time $O(n^3)$

Generalization to any k [Berarducci, Intrigila'93] [Hahn, MacGillivray'06] [Clarke, MacGillivray'12]

$cn(G) \leq k$? can be checked in time $n^{O(k)}$ $\in EXPTIME$

EXPTIME-complete in directed graphs [Goldstein and Reingold, 1995]

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(i.e., no algorithm in time $f(k)n^{O(1)}$ expected)

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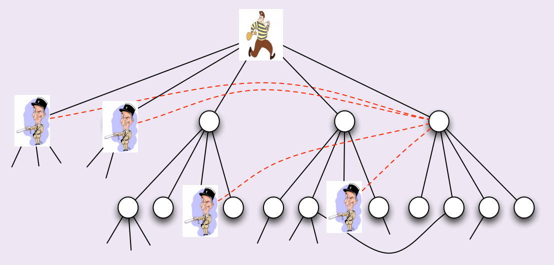
Large girth (smallest cycle) AND large min degree ⇒ large cop-number

G with min-degree d and girth $> 4 \Rightarrow cn(G) \geq d$. [Aigner and Fromme 84]



- for any k, d, there are d-regular graphs G with $cn(G) \geq k$ [Aigner and Fromme 84]
- $cn(G) \geq d^t$ in any graph with min-degree d and girth $> 8t-3$ [Frankl 87]
- for any k, there is G with diameter 2 and $cn(G) \geq k$ (e.g., Kneser graph $KG_{3k,k}$)

Graphs with high cop-number

Large girth (smallest cycle) AND large min degree $\Rightarrow$ large cop-number

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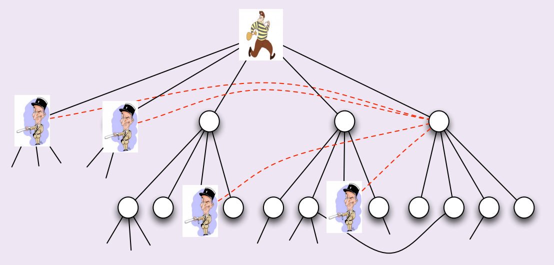

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Meyniel Conjecture

$\exists$ n-node graphs with degree $\Theta(\sqrt{n})$ and girth > 4

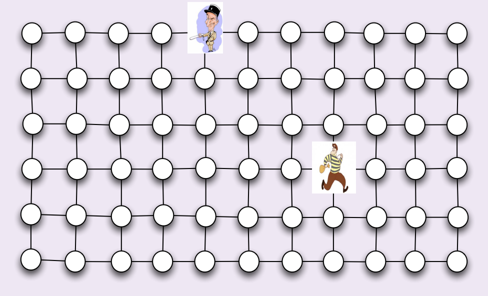
$\Rightarrow \exists$ n-node graphs G with $cn(G) = \Omega(\sqrt{n})$
(e.g., projective plan, random $\sqrt{n}$-regular graphs)

Meyniel Conjecture

Conjecture: For any n-node connected graph G, $cn(G) = O(\sqrt{n})$ [Meyniel 85]

Link with Graph Structural Properties

Reminder: For any graph G, $cn(G) \leq \gamma(G)$ the dominating number of G.



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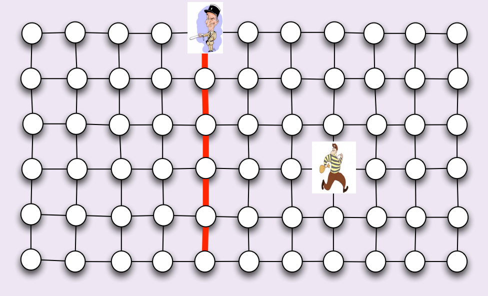

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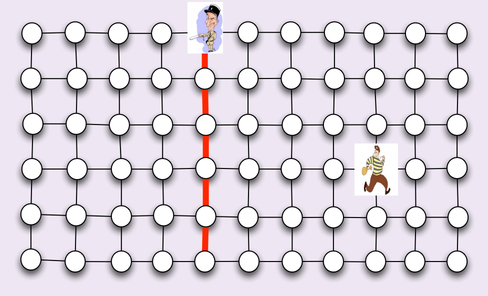
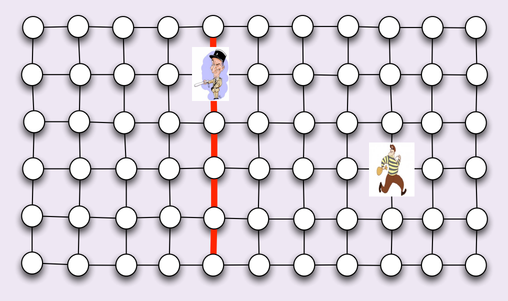


Link with Graph Structural Properties

Reminder: For any graph G, $cn(G) \leq \gamma(G)$ the dominating number of G.



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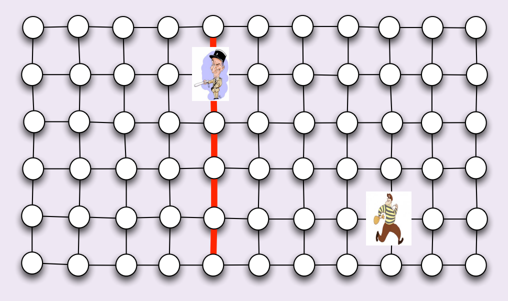

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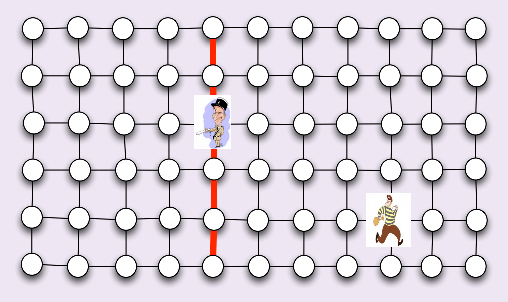

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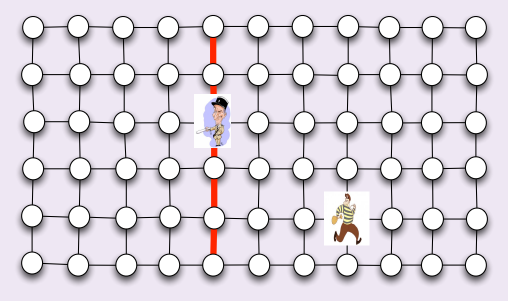

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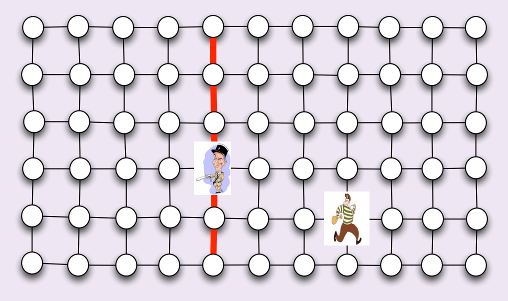

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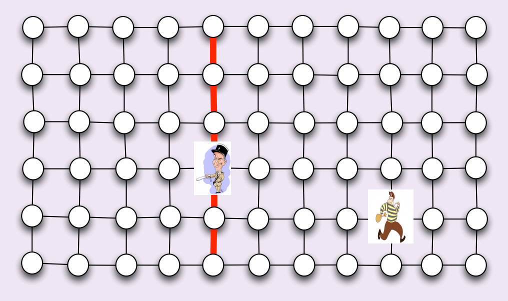

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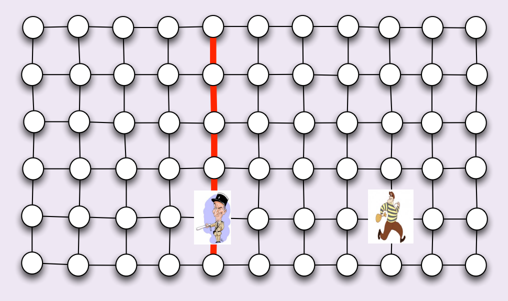

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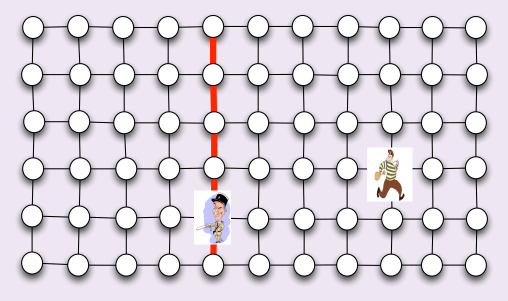

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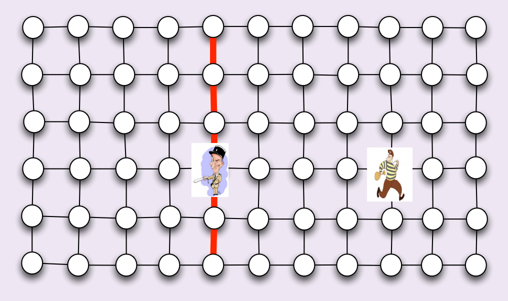

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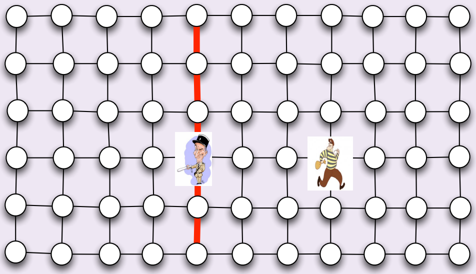

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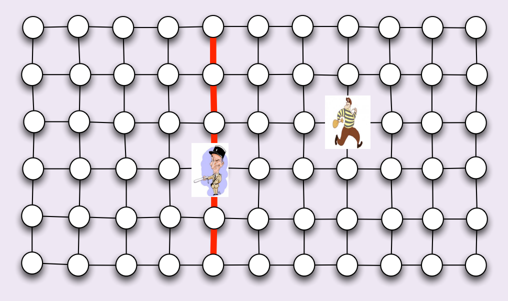

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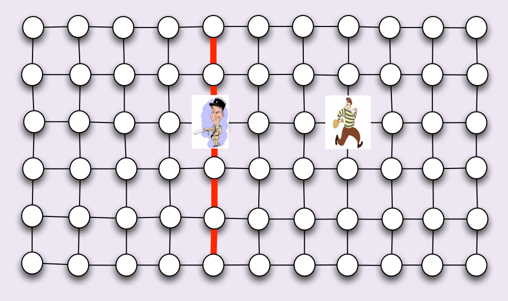

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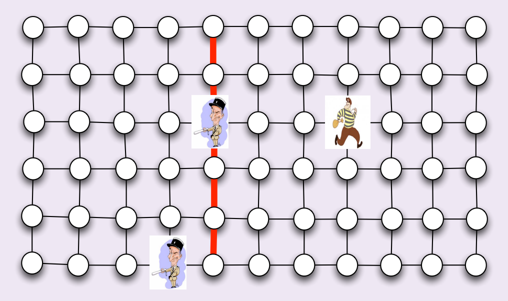

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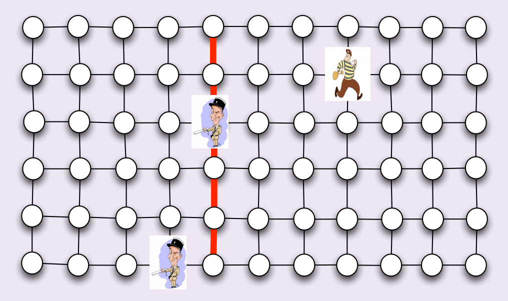

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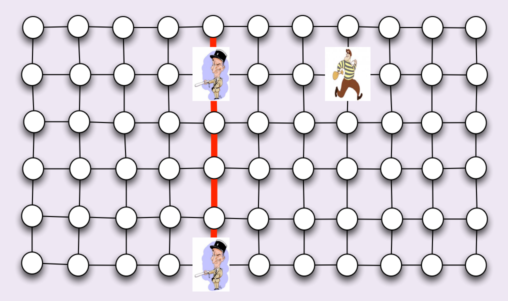

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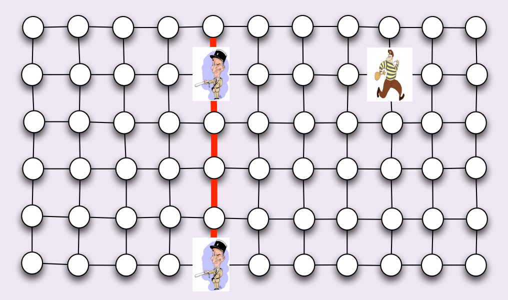

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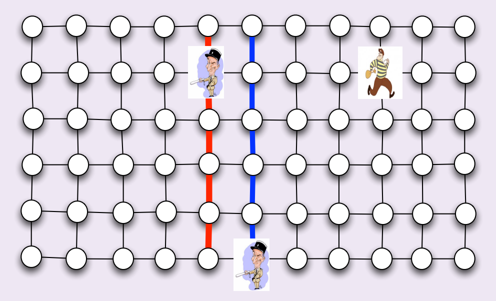

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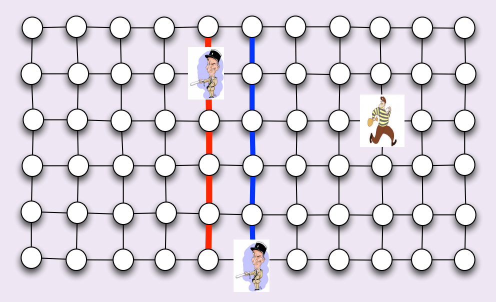

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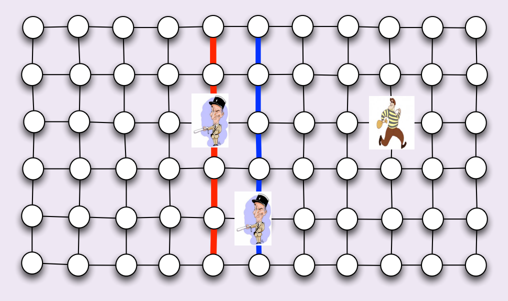

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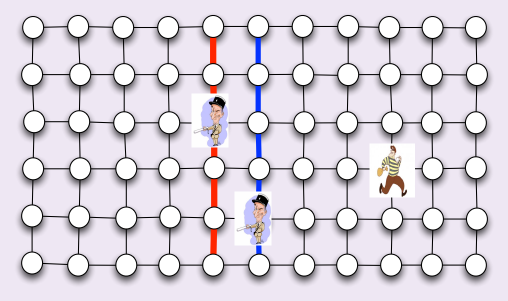

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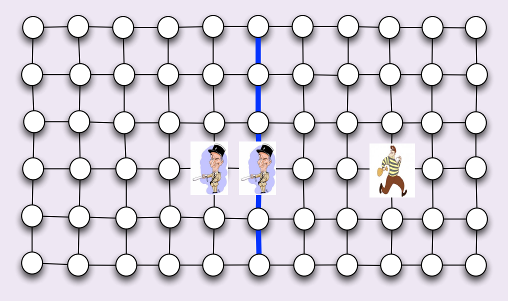

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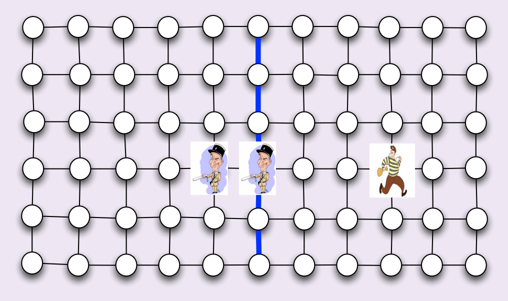

Lemma [Aigner, Fromme 1984]

1 Cop is sufficient to "protect" a shortest path P in any graph.
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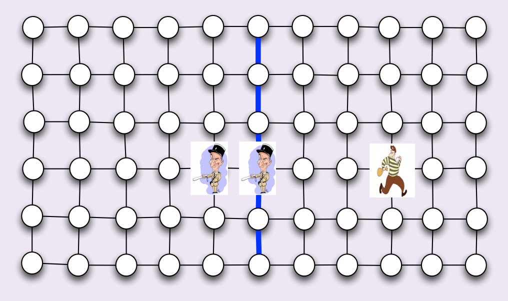

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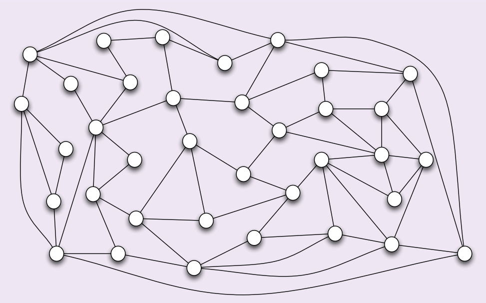
$\Rightarrow$ Cop-number related to both structural and metric properties

1 Cop can protect 1 shortest path: applications (1)



Cop-number vs. graph structure a surprising (?) example

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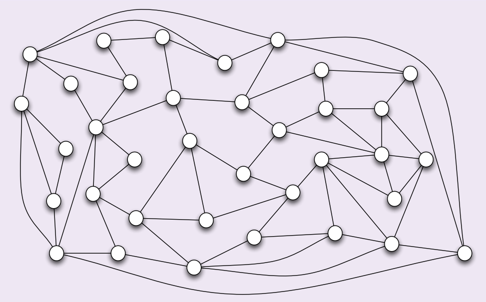

For any planar graph G (there is a drawing of G on the plane without crossing edges), there exists separators consisting of ≤ 3 shortest paths

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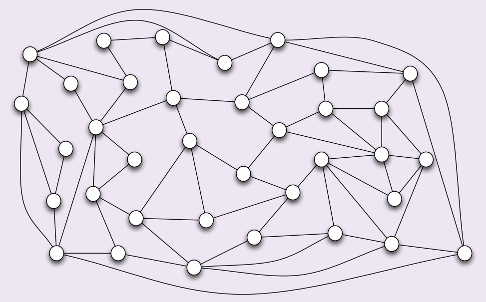

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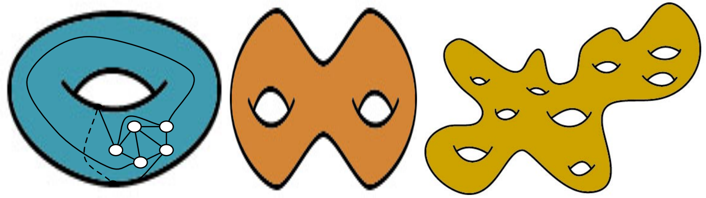
Cop-number vs. graph structure — a surprising (?) example

$cn(G) \leq 3$ for any planar graph G [Aigner and Fromme 84]

1 Cop can protect 1 shortest path: applications (2)

G with genus $\leq g$: can be drawn on a surface with $\leq g$ "handles".



Cop-number vs. graph structure — let's go further

$cn(G) \leq \lfloor \frac{3g}{2} \rfloor + 3$ for any graph G with genus $\leq g$ [Schröder, 01]

Conjectures: $cn(G) \leq g + 3$? $cn(G) \leq 3$ if G has genus 1?

G is H-minor-free if no graph H as minor — "generalize" bounded genus [Robertson,Seymour 83-04]

$cn(G) < |E(H)|$ [Andreae, 86]

Application [Abraham,Gavoille,Gupta,Neiman,Tawar, STOC 14]

"Any graph excluding K_r as a minor can be partitioned into clusters of diameter at most Δ while removing at most $O(r/\Delta)$ fraction of the edges."

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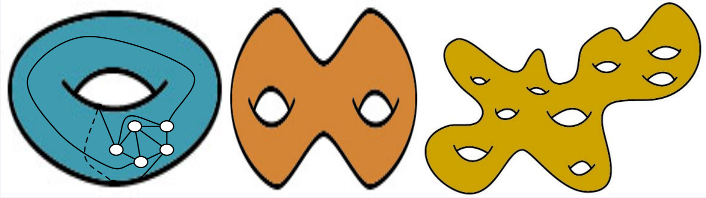

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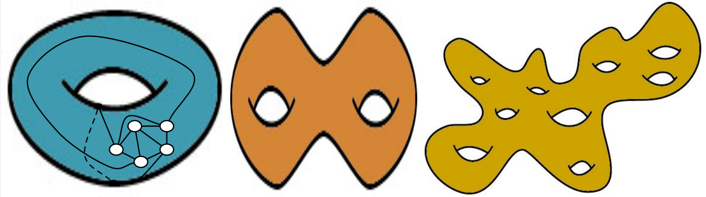

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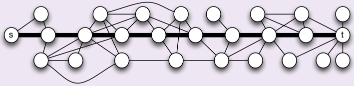

Lemma shortest-path-caterpillar = closed neighborhood of a shortest path [Chiniforooshan 2008]

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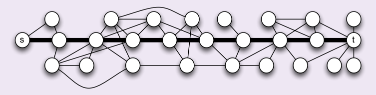

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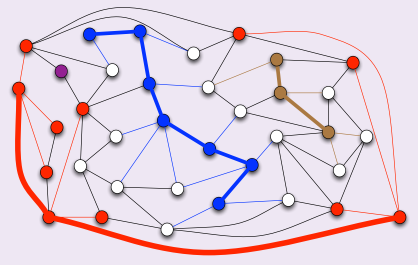

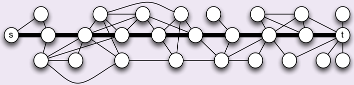

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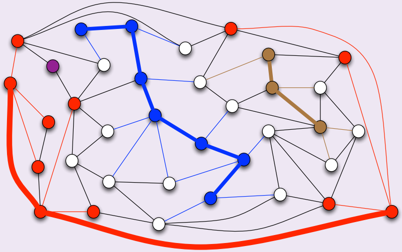
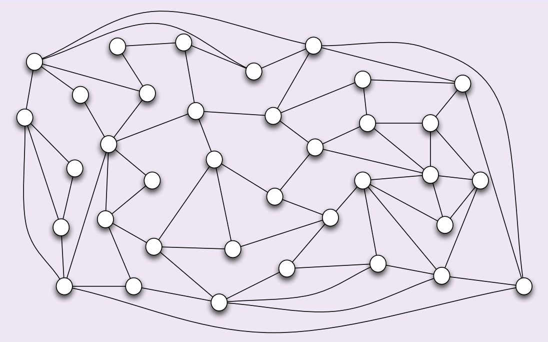


For any graph G, $cn(G) = O(n/\log n)$ [Chiniforooshan 2008]

From a simple universal strategy...



A simple universal strategy (Cops must occupy an **induced** path)

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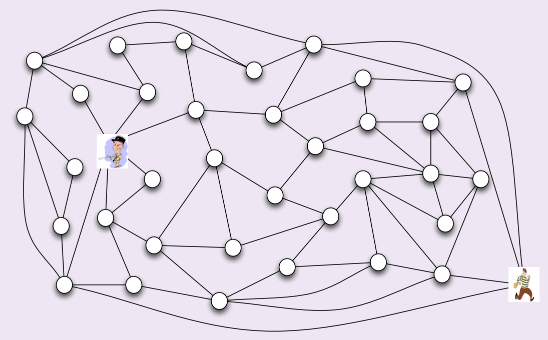

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(1) start in a node

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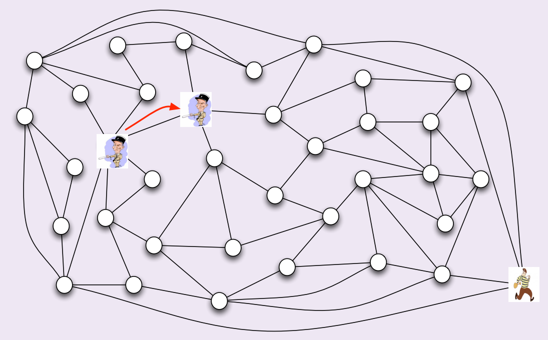

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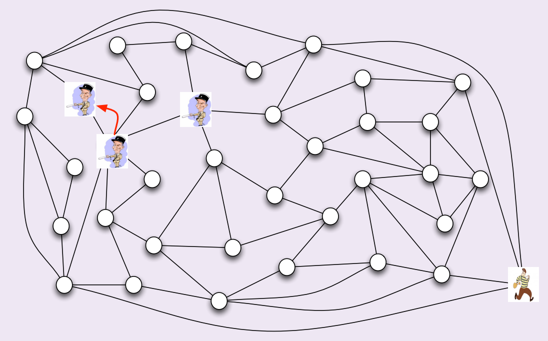

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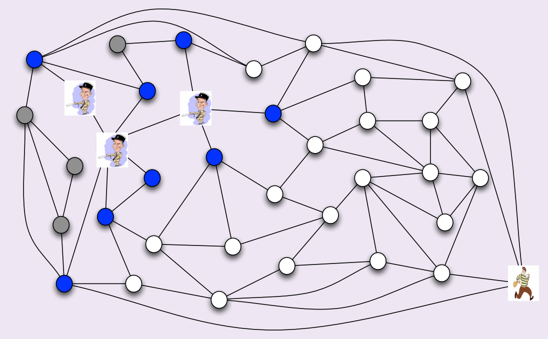

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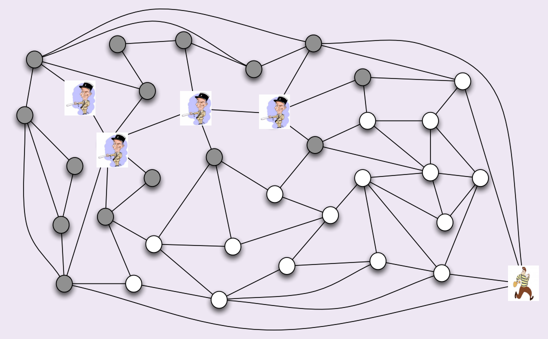

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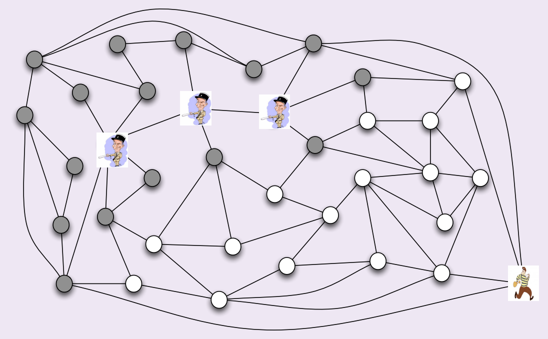

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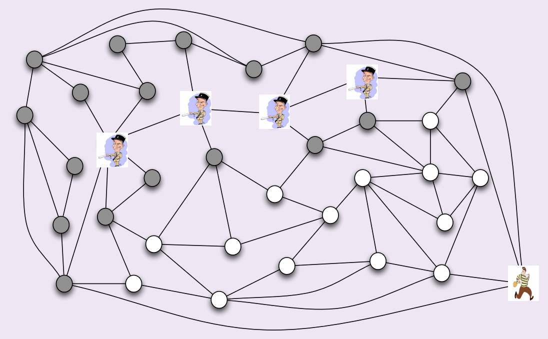

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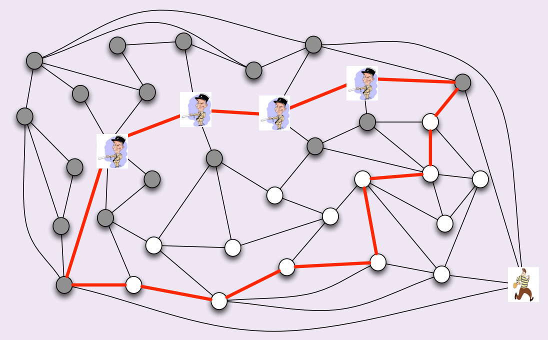

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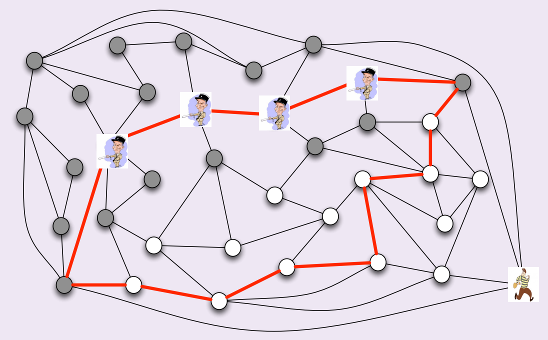

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Theorem [Kosowski,Li,N.,Suchan, ICALP'12, Algorithmica]

$cn(G) \leq k-1$ in any graph G with maximum induced cycle of length k (k-chordal)

...to a Structural Result and Applications to Compact Routing

Recursive decomposition using **separators** with short **dominating induced path**

Theorem [Kosowski,Li,N.,Suchan, ICALP'12, Algorithmica]

There is a $O(m^2)$ algorithm that, for any graph G with m edges and max degree Δ,

- either returns an induced cycle of length $\geq k+1$,
- or compute a tree-decomposition with width $\leq (k-1)(\Delta-1)+2$.

Applications [Kosowski,Li,N.,Suchan, ICALP'12, Algorithmica

$tw(G) = O(\Delta \cdot k)$ if G has no induced cycle of length $> k$.
improve the bound $tw(G) = O(\Delta^{O(k)})$ [Bodlaender,Thilikos 1997]

Application: Compact routing scheme in k-chordal graphs additive stretch: $O(k \log \Delta)$, Routing Tables: $O(k \log n)$ bits.

Progress on Meyniel Conjecture

Meyniel Conjecture [85]: For any n-node connected graph G, $cn(G) = O(\sqrt{n})$

	cn	
dominating set $\leq k$	$\leq k$	[folklore]
treewidth $\leq t$	$\leq t/2+1$	[Joret, Kaminski,Theis 09]
chordality $\leq k$	$< k$	[Kosowski,Li,N.,Suchan 12]
genus $\leq g$	$\leq \lfloor \frac{3g}{2} \rfloor + 3$	*(conjecture $\leq g+3$)* [Schröder, 01]
H**-minor free**	$\leq \lvert E(H) \rvert$	[Andreae, 86]
degeneracy $\leq d$	$\leq d$	[Lu,Peng 12]
diameter 2	$O(\sqrt{n})$	—
bipartite diameter 3	$O(\sqrt{n})$	—
Erdös-Réyni graphs	$O(\sqrt{n})$	[Bollobas *et al.* 08] [Luczak, Pralat 10]
Power law	$O(\sqrt{n})$	*(big component?)* [Bonato,Pralat,Wang 07]

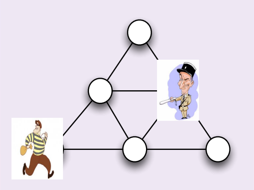
A long story not finished yet...

- $cn(G) = O(\frac{n}{\log \log n})$ [Frankl 1987]
- $cn(G) = O(\frac{n}{\log n})$ [Chiniforooshan 2008]
- $cn(G) = O(\frac{n}{2^{(1-o(1))\sqrt{\log n}}})$ [Scott, Sudakov 11, Lu,Peng 12]

note that $\frac{n}{2^{(1-o(1))\sqrt{\log n}}} \geq n^{1-\epsilon}$ for any $\epsilon > 0$

When Cops and Robber can run

New variant with speed: Players may move along several edges per turn
$cn_{s',s}(G)$: min # of Cops with speed s' to capture Robber with speed s, $s \geq s'$.



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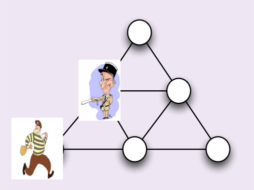

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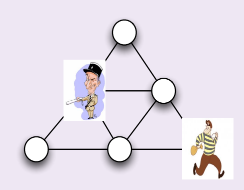

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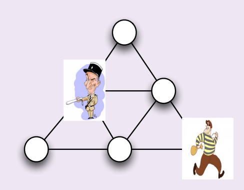

Meyniel Conjecture [Alon, Mehrabian'11] and general upper bound [Frieze,Krivelevich,Loh'12] extend to this variant

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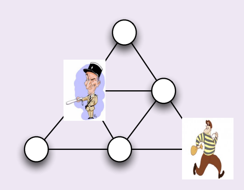

Meyniel Conjecture [Alon, Mehrabian'11] and general upper bound [Frieze,Krivelevich,Loh'12] extend to this variant

... but fundamental differences (recall: planar graphs have $cn_{1,1} \leq 3$)

$cn_{1,2}(G)$ unbounded in grids [Fomin,Golovach,Kratochvil,N.,Suchan TCS'10]

Open question: $\Omega(\sqrt{\log n}) \leq cn_{1,2}(G) \leq O(n)$ in $n \times n$ grid G exact value?

When Cops and Robber can run

G is Cop-win $\Leftrightarrow$ 1 Cop sufficient to capture Robber in G

Structural characterization of Cop-win graphs for any speed s and s'

[Chalopin,Chepoi,N.,Vaxès SIDMA'11]

generalize seminal work of [Nowakowski,Winkler'83]

hyperbolicity δ of G: measures the "proximity" of the metric of G with a tree metric

New characterization and algorithm for hyperbolicity

- bounded hyperbolicity $\Rightarrow$ one Cop can catch Robber almost twice faster

  [Chalopin,Chepoi,N.,Vaxès SIDMA'11]
- one Cop can capture a faster Robber $\Rightarrow$ bounded hyperbolicity

  [Chalopin,Chepoi,Papasoglu,Pecatte SIDMA'14]

- $O(1)$-approx. sub-cubic-time for hyperbolicity [Chalopin,Chepoi,Papasoglu,Pecatte SIDMA'14]
- $\textit{tree-length}(G) \leq \lfloor \frac{\ell}{2} \rfloor \textit{tw}(G)$ for any graph G with max-isometric cycle ℓ

  $\Rightarrow O(\ell)$-approx. for $\textit{tw}$ in bounded genus graphs [Coudert,Ducoffe,N. 14]

Conclusion / Open problems

Meyniel Conjecture [1985]: For any n-node connected graph G, $cn(G) = O(\sqrt{n})$

Conjecture [?]: For any n-node connected graph G with genus g, $cn(G) \leq g + 3$

simpler(?) questions

- $cn(G) \leq 3$ if G has genus ≤ 1?
- how many cops with speed 1 to capture a robber with speed 2 in a grid?
- when Cops can capture at distance?
  [Bonato,Chiniforooshan,Pralat'10] [Chalopin,Chepoi,N.,Vaxès'11]
- Many other variants and questions... (e.g. [Clarke'09] [Bonato, *et a.*'13]...)
- Directed graphs ??

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