Submodular partition functions and duality treewidth/bramble

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Min-Max Theorem for several width parameters

Our goal

Duality treewidth/bramble [Seymour and Thomas 93]
New proof of the min-max theorem for treewidth

Our tool

Submodular partition functions

Generalization

Interpretation of several width-parameters (treewidth, pathwidth, branchwidth, rankwidth, treewidth of matroid) in terms of submodular partition functions.

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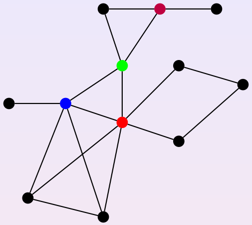
Tree decomposition and treewidth



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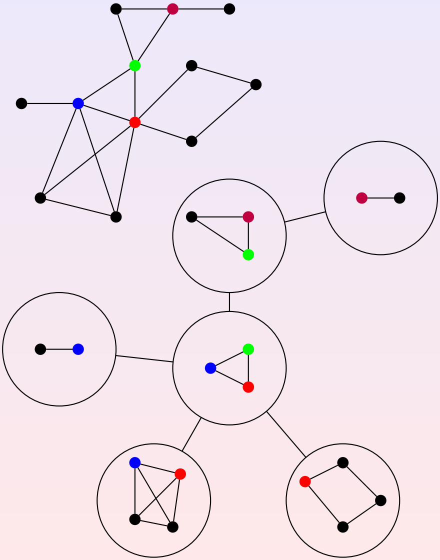

a tree T and bags $(X_t)_{t \in V(T)}$

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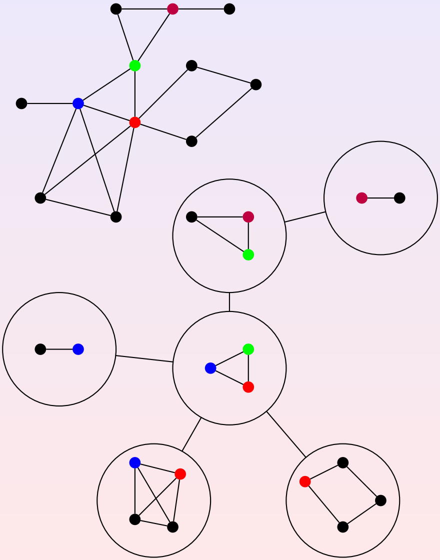

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- every vertex of G is in at least one bag;

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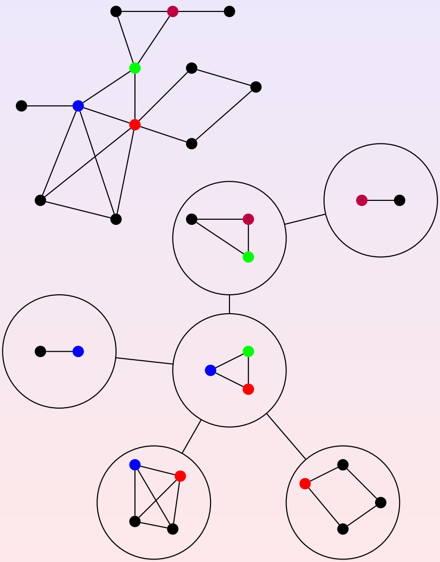

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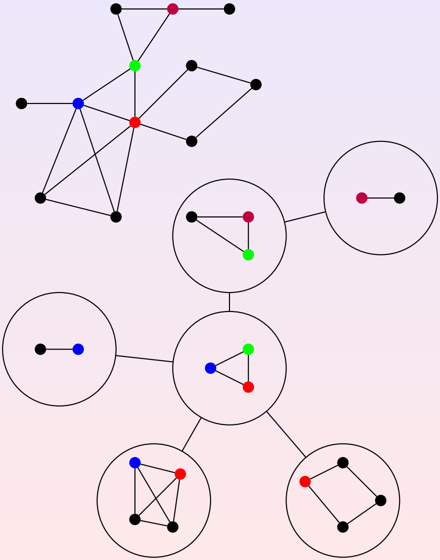

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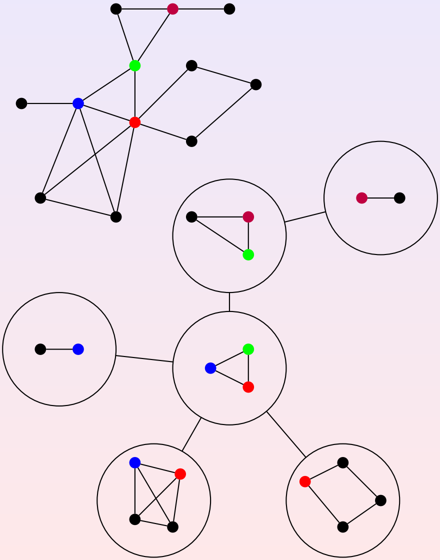

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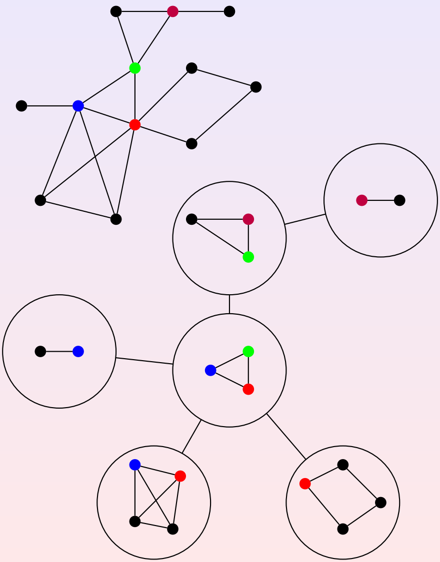

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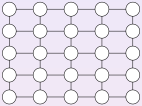
treewidth of G

tw(G), minimum width among all tree-decompositions.

Example of the Grid G_{k*k}



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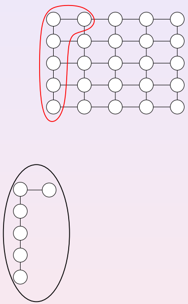

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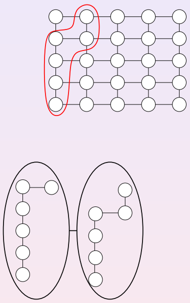

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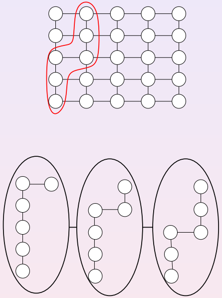

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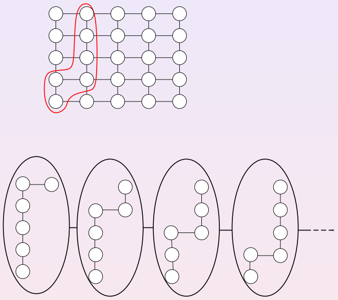

It is easy to find a tree-decomposition, $\mathbf{tw}(G_{k*k}) \leq k$

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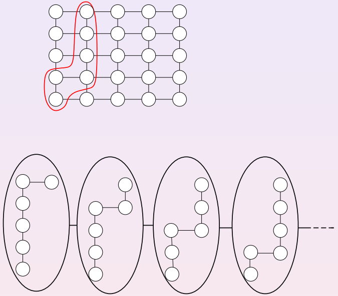

It is easy to find a tree-decomposition, $\mathbf{tw}(G_{k*k}) \leq k$
How to prove that it is an optimal tree-decomposition?

Bramble and bramble-number

Definition

Bramble $\mathcal{B}$: set of connected subsets of $V(G)$, pairwise touching.

- for any $B \in \mathcal{B}$, $B \subseteq V(G)$;
- for any $B_i, B_j \in \mathcal{B}$, $B_i \cup B_j$ connected.

A transversal is a subset $\mathcal{T} \subseteq V(G)$ such that:

For all $B_i \in \mathcal{B}$, $B_i \cap \mathcal{T} \neq \emptyset$

Order of a bramble

$Order(\mathcal{B})$: Minimum size of a transversal of $\mathcal{B}$.

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Bramble-number $\mathbf{bn}(G)$

$\mathbf{bn}(G)$: maximum order among all brambles of G.

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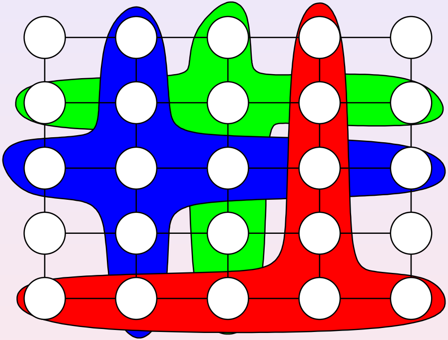

$\mathcal{B}_1$ set of all crosses (one row + one column)

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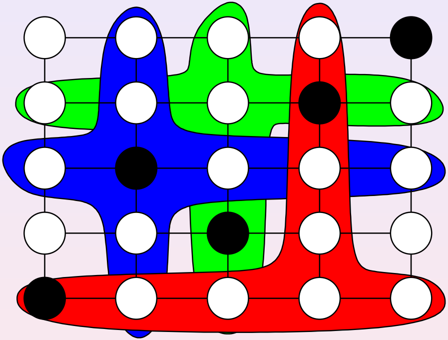

$\mathcal{B}_1$ set of all crosses (one row + one column)
$Order(\mathcal{B}_1) = k$, therefore $\mathbf{bn}(G_{k*k}) \geq k$

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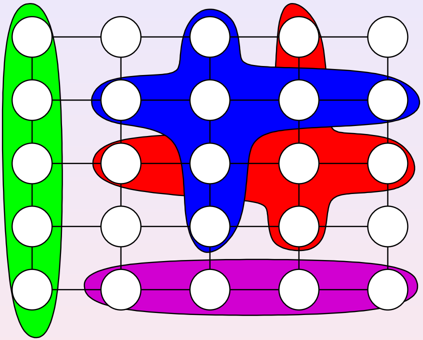

$\mathcal{B}_2$ first column + last row minus its first vertex + set of all crosses of $G_{(k-1)*(k-1)}$

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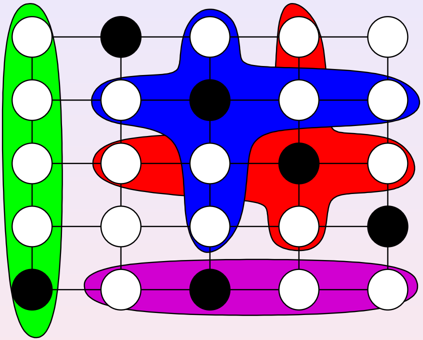

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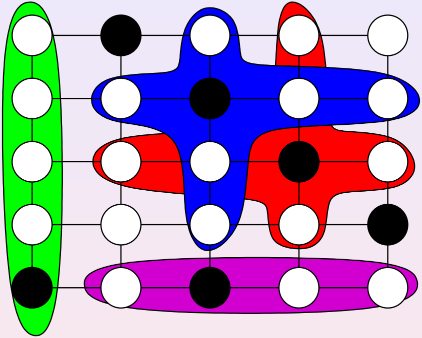

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How to prove that it is a maximal bramble?

Min-Max Theorem

For any graph G, $\mathbf{tw}(G)+1=\mathbf{bn}(G)$

Seymour and Thomas, J. of Comb. Th., 1993.
Graph searching and a min-max theorem for tree-width

$$\min_{(T,X)\ tree-dec.\ of\ G}\ \max_{t\in V(T)} |X_t| = \max_{\mathcal{B} bramble\ of\ G}\ \min_{Y\ transv.\ of\ \mathcal{B}} |Y|$$

Example of the grid

$\mathbf{tw}(G_{k*k})+1=\mathbf{bn}(G_{k*k})=k+1$

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In terms of graph searching

- *Bramble of order* $k+1$ = winning strategy for a visible fugitive against k searchers.
- *Tree-decomposition of width* k = winning strategy for $k+1$ searchers against any visible fugitive.

Partitioning-tree

Definition

A set E. A partitioning-tree T on E

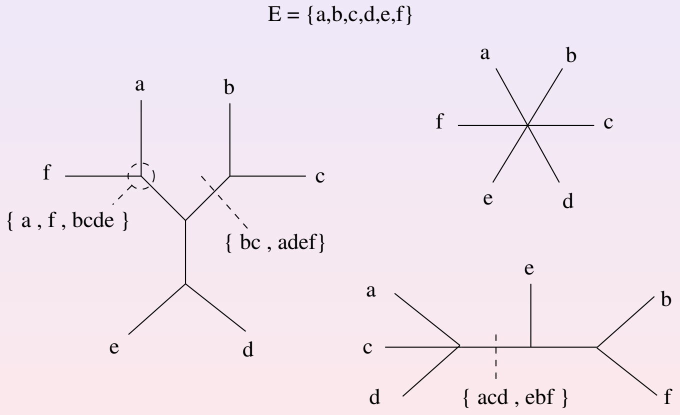
- T a tree;
- a bijection between E and the set of leaves of T.

T-partitions

T defines a set of partitions of E.

- any edge $e \in E(T) \Rightarrow$ a bipartition T_e of E;
- any vertex $v \in V(T) \Rightarrow$ a partition T_v of E.

Partitioning-tree



General Problem

Let $\mathcal{F}$ be a set of partitions of a set E
(but the trivial one $\{E\}$)
$\mathcal{F}$ is a set of admissible partitions of E.

Remarks: The partitions we consider may be degenerated, and the order of the elements of a partition is irrelevant.

Question

Is there an admissible partitioning-tree for $\mathcal{F}$,
i.e. a partitioning-tree T such that $\{T partitions\} \subseteq \mathcal{F}$?

Bramble and principal bramble

Let E be a set, and $\mathcal{F}$ be a set of admissible partitions of E.

$\mathcal{F}$-bramble

A $\mathcal{F}$-bramble is a set $\mathcal{B}$ of subsets of E
$\mathcal{B} = \{X_i \mid X_i \subseteq E\}$

- for any $X_i, X_j \in \mathcal{B}$, $X_i \cap X_j \neq \emptyset$;
- for any $\{E_1, \cdots, E_k\} \in \mathcal{F}$, there is $E_i \in \mathcal{B}$.

principal $\mathcal{F}$-bramble

$\mathcal{B}$ is principal if $\bigcap_{X_i \in \mathcal{B}} X_i \neq \emptyset$.

It is easy to compute a principal $\mathcal{F}$-bramble $\mathcal{B}$:

- pick an element $e \in E$;
- for any partition Y of $\mathcal{F}$, put in $\mathcal{B}$ the element of Y that contains e.

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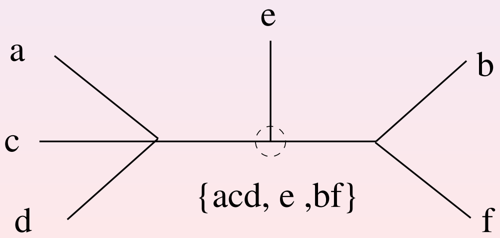
An obstruction to the existence of a partitioning-tree

Lemma

If there is a non-principal $\mathcal{F}$-bramble, then there is no admissible partitioning-tree for $\mathcal{F}$.

$\mathcal{B}$ non-principal $\mathcal{F}$-bramble, T admissible partitioning-tree.

- for any internal vertex u of T, $T_u \in \mathcal{F}$;



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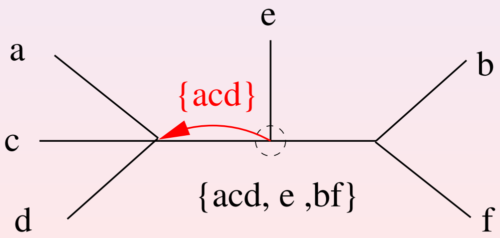
- there is an element X of T_u, $X \in \mathcal{B}$;



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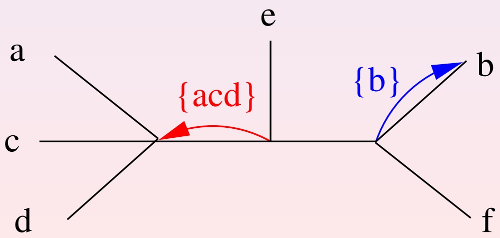
- since $\mathcal{B}$ is not principal, no edge is oriented toward a leaf;



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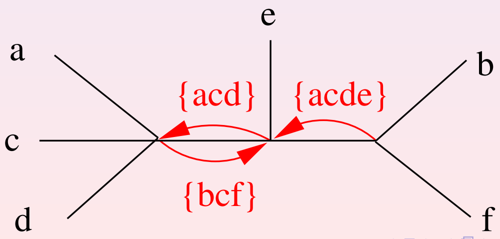
- thus, an edge gets two orientations, $\{acd\} \cap \{bcf\} = \emptyset$, a contradiction.



A new question

We know:
non-principal $\mathcal{F}$-bramble $\Rightarrow$ no admissible partitioning-tree

How to characterize the families $\mathcal{F}$ of partitions of E, s.t. it is an equivalence?

Good family $\mathcal{F}$ of admissible partitions

- either there is a non-principal $\mathcal{F}$-bramble,
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Partition Functions

Partition functions

$\Phi : \{partitions\ of\ E\} \to N$ such that, for any partition $\mathcal{X} = \{X_1, X_2, \cdots, X_n\}$, and $1 \leq i \leq n$, $\Phi(\mathcal{X}) \geq \Phi(\{X_i, X_i^c\})$.

Let Φ be a partition function and let $k \geq 1$.
Let $\mathcal{F}_{\Phi,k}$ be the family of the partitions P, with $\Phi(P) \leq k$.

A k-partitioning-tree T for Φ is an admissible partitioning-tree for $\mathcal{F}_{\Phi,k}$.

A k-bramble for Φ is a $\mathcal{F}_{\Phi,k}$-bramble.

Our Problem

How to characterize the partition functions Φ such that, for any k, $\mathcal{F}_{\Phi,k} = \{partition\ P \mid \Phi(P) \leq k\}$ is a family of good admissible partitions?

In other words

How to characterize the partition functions Φ such that, for any k,

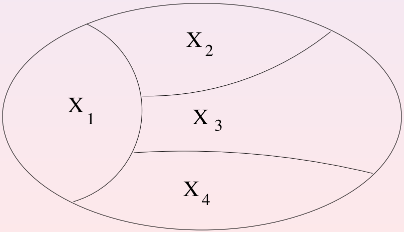
- either there is a non-principal k-bramble for Φ,
- or there is a k-partitioning-tree for Φ?

An operation on partitions

Let $\mathcal{X} = \{X_1, X_2, \cdots, X_n\}$ be a partition of E, and $Y \subset E$ such that $X_1 \cap Y = \emptyset$.

To *push* a partition

By pushing X_1 to Y in $\mathcal{X}$, we get the new partition: $\mathcal{X}_{X_1 \to Y} = \{Y^c, X_2, \cap Y, \cdots, X_n \cap Y\}$



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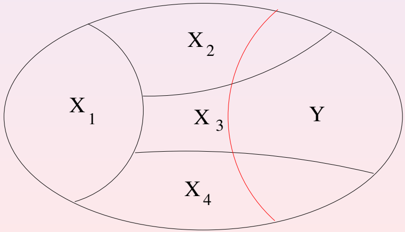

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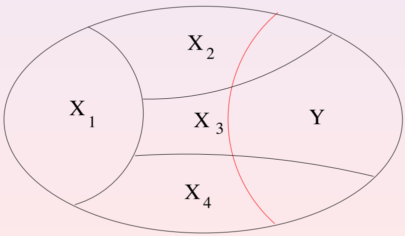

Submodular partition functions

Definition

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$$\Phi(\mathcal{X}) + \Phi(\mathcal{Y}) \geq \Phi(\mathcal{X}_{X_i \to Y_j}) + \Phi(\mathcal{Y}_{Y_j \to X_i})$$

Weakly submodular partition functions

A partition function Φ is weakly submodular if, for any partition $\mathcal{X} = \{X_1, X_2, \cdots, X_n\}$, $\mathcal{Y} = \{Y_1, Y_2 \cdots, Y_m\}$ s.t. $X_i \cap Y_j = \emptyset$.

- either there exists F with $X_i \subseteq F \subseteq Y_j^c$ such that $\Phi(\mathcal{X}) \geq \Phi(\mathcal{X}_{X_i \to F^c})$
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Main Theorem

Theorem: Duality partitioning-tree/bramble

Let Φ be a weakly submodular partition function on a set E, and let $k \geq 1$.

- either there is a non-principal k-bramble for Φ,
- or there is a k-partitioning-tree for Φ.

$\mathcal{F}_{\Phi,k}$ is a good family of admissible partitions

Duality Treewidth / Bramble

Let $G = (V, E)$ be a graph, and $\mathcal{X} = \{X_1, X_2, \cdots, X_n\}$ a partition of E. The border function δ is defined by:
$\delta(\mathcal{X})$ is the set of vertices incident to an edge in X_i and in X_j.

Lemma

$|\delta|$ is a submodular partition function.

Duality treewidth/bramble

If T is a k-partitioning-tree for $|\delta|$, then $(T, (\delta(T_t))_{t \in V(T)})$ is a tree-decomposition of width at most $k - 1$.
We can compute a bramble (in usual sense) of order at least k from any non-principal k-bramble for $|\delta|$.

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To build new weakly submodular functions (1)

Let Φ be a weakly submodular partition function

weakly submodular partition function Φ_p

Let $p \geq 2$. Let Φ_p be defined such that $\Phi_p(\mathcal{X}) = \Phi(\mathcal{X})$ if $\mathcal{X}$ consists of at most p non empty parts, and ∞ otherwise.
Then, Φ_p is weakly submodular.

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The main theorem holds for such a function

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To build new weakly submodular functions (2)

Let f be a submodular function on E, i.e., that satisfies $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$

submodular partition function $\sum_f$

Let $\mathcal{X} = \{X_1, X_2, \cdots, X_n\}$ be a partition. Let us define:
$\sum_f(\mathcal{X}) = \sum_{i \leq n} f(X_i)$.
Then, $\sum_f$ is submodular.

Let f be a symmetric submodular function on E, i.e., satisfying moreover $f(X) = f(X^c)$.

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Other width parameters (1)

Let $G = (V, E)$ be a graph,

Duality branchwidth / tangle [Graph Minors X]

A k-partitioning-tree for the partition function $(\max_{|\delta|})_3$ is a branch decomposition of width at most k.
We can compute a tangle of order at least k from any non-principal k-bramble for $(\max_{|\delta|})_3$.

Duality pathwidth / blockage [Graph Minors I]

A k-partitioning-tree for the partition function $(|\delta|)'_2$ is a path decomposition of width at most $k-1$.
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Let $G = (V, E)$ be a graph,

Rankwidth [Oum et Seymour 06]

- consider the set V;
- based on the symmetric submodular function rk;
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Treewidth of matroid [Hlileny et Whittle 06]

- M a matroid on groud set E with rank function r;
- based on the submodular function r^c such that $r^c(F) = r(F^c)$;
- $\mathcal{X} = \{X_1, \cdots, X_\ell\}$, $\Phi(\mathcal{X}) = \sum_{r^c}(\mathcal{X}) - (\ell - 1)r(E)$.

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