Probabilistically Checkable Proofs (PCP) and Hardness of Approximations

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Outline

1. Introduction on Approximation Algorithms
2. PCP & PCP Theorem ($NP = PCP(O(\log n), O(1))$)
3. PCP Theorem ⇔ MAX-3SAT is hard to Approximate
4. From Label-Cover to MAX-Clique

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Approximation Algorithms

Π a maximization Problem

c-Approximation for Π $1 < c$ constant or depends on input length

- deterministic polynomial-time algorithm $\mathcal{A}$
- for any input I, $\mathcal{A}$ returns a solution with value at least $OPT(I)/c$.

Example: MAX-3SAT
input: a 3CNF formula Φ
output: maximum (over all assignments A) number of clauses of Φ satisfied by A

remark: MAX-3SAT is NP-hard

Algorithm: Random assignment $\Rightarrow$ $\frac{8}{7}$-approximation for MAX-3SAT

each clause is satisfied with probability $\frac{7}{8}$

Application of PCP Theorem : above algorithm (derandomized) is optimal unless $P = NP$.

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Hierarchy of NP-hard Problems

Fully Polynomial Time Approximation Scheme (FPTAS)

For any ϵ, $\exists$ $(1+\epsilon)$-approximation algorithm, polynomial in both n and $1/\epsilon$
e.g., Knapsack

Polynomial Time Approximation Scheme (PTAS)

For any ϵ, $\exists$ $(1+\epsilon)$-approximation algorithm, polynomial in n (typically $n^{f(\epsilon)}$)
but no FPTAS (unless P=NP) e.g., Euclidean TSP

$\Theta(1)$-approximation but no PTAS (unless P=NP)

e.g., Vertex Cover, MAX-3SAT

$O(\log n)$-approximation but no constant approximation (unless P=NP)

e.g., Set Cover

only $n^{\Theta(1)}$-approximation, no $n^{1-\epsilon}$-approximation for any $\epsilon > 0$ (unless P=NP)

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Reminder: Definition of NP

$L \in NP$ iff $\exists$ a deterministic Verifier V s.t.

for any input (word) w

- V polynomial-time in $|w|$
- and
  - if $w \in L$ then $\exists$ proof (certificate) x with $V(w,x) = 1$
  - if $w \notin L$ then $\forall$ proof x, $V(w,x) = 0$

examples:
3-SAT: proof= assignment of variables
3-Coloriage: proof= Labelling of vertices

Verifier checks (1) that certificate is consistent (2) that the answer is correct

Definition of the class PCP(r,q)

$L \in PCP(r,q)$ iff $\exists$ a randomized Verifier V s.t.

for any input w

- V polynomial-time in $|w|$, uses a string ρ of $O(r(|w|))$ bits of randomness and only checks $O(q(|w|))$ bits of the proof, and
  - if $w \in L$ then $\exists$ proof x with $Pr_\rho(V(w,x,\rho)=1)=1$ (**completeness**).
  - if $w \notin L$ then $\forall$ proof x, $Pr_\rho(V(w,x,\rho)=1) \leq 1/2$ (**soundness**).

proof x has size $\leq q(|w|) \cdot 2^{r(|w|)}$ and only $O(q(|w|))$ bits of x are checked.

If the proof x is correct, then w is accepted with proba 1 (**completeness**).
Otherwise (if proof x is a fake), w is rejected with probability at least $1/2$ (**soundness**).

tradeoff between randomness (r) and number of bits (q) checked

Relationships between PCP and NP

$NP \subseteq PCP(0, poly(n))$

idea: NP-proof has polynomial size and the PCP-Verifier can check it completely

$PCP(O(\log n), O(1)) \subseteq NP$

idea: NP-Verifier: apply PCP-Verifier for each of the $2^{O(\log n)}$ possible random strings
i.e., "check the full proof" in polynomial-time

$NP \subseteq PCP(n^{O(1)}, O(1))$

idea: Increase the size of NP-proof s.t. if it is a fake, the mistakes are "everywhere"

PCP Theorem: $NP \subseteq PCP(O(\log n), O(1))$ Corollary: $NP = PCP(O(\log n), O(1))$

Initial proof: [Arora,Safra FOCS'92] [Arora,Lund,Motwani,Sudan,Szegedy FOCS'92]
[Arora FOCS'95]

improvements: [Raz STOC'95] [Hastad JACM'01]
"easier" proof: [Dinur STOC'06, JACM'07]

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PCP Theorem

$NP = PCP(O(\log n), O(1))$ [Arora *et al.*'92]

Consequence: for any $L \in NP$, $\exists$ a Verifier V s.t.

for any input w of size n

- V polynomial-time in $|w|$, uses a string ρ of $O(\log n)$ bits of randomness and only checks $O(1)$ bits of the proof, and
  - if $w \in L$ then $\exists$ proof x with $Pr_\rho(V(w,x,\rho) = 1) = 1$ (**completeness**).
  - if $w \notin L$ then $\forall$ proof x, $Pr_\rho(V(w,x,\rho) = 1) \leq 1/2$ (**soundness**).

Remark 1: $PCP(O(\log n), O(1))$ is stable by polynomial-time reductions.

i.e., if $L' \prec_{poly} L$ and $L \in PCP(O(\log n), O(1))$ then $L' \in PCP(O(\log n), O(1))$

$L' \prec_{poly} L$ means $\exists$ a poly-time reduction from L to L' (i.e., L not "easier" than L').

Remark 2: V may have extra/restricted properties [Arora *et al.*'92, Raz'95,Hastadt'01]

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MAX-3SAT is hard to Approximate (1/2)

Goal: prove that $\exists c_0$ s.t. $\forall \epsilon$, there is no $(c_0 - \epsilon)$-approx of MAX-3SAT unless $P = NP$

MAX-3SAT

input: a 3CNF formula Φ
output: maximum (over all assignments A) number of clauses of Φ satisfied by A

GAP-3SAT$_s$ $0 < s < 1$

input: a 3CNF formula Φ with m clauses
output:

- YES if Φ is satisfiable
- NO if at most $s \cdot m$ clauses are satisfiable (for all assignments)
- anything otherwise

Lemma

If $\exists s < 1$ s.t. GAP-3SAT$_s$ is NP-hard
then there is no s-approx of MAX-3SAT unless $P = NP$

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MAX-3SAT is hard to Approximate (2/2)

PCP ⇔ Hardness of Approximation

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PCP ⇔ Hardness of Approximation

$NP \subseteq PCP(O(\log n), O(1)) \Leftrightarrow \exists s < 1$ s.t. GAP-3SAT$_s$ is NP-hard

$\Leftarrow$ roughly, prove that GAP-3SAT$_s \in PCP(O(\log n), O(1))$

For any $L \in NP$ there is Φ polynomial-time s.t.

- for any word w, $\Phi(w)$ is an instance of GAP-3SAT$_s$
- if $w \in L$ then $\Phi(w)$ satisfiable; else at most a fraction s of the clauses can be satisfied

Verifier:

1. Compute $\Phi(w)$, m its number of clauses
2. use $\lceil \log m \rceil$ random bits to choose a clause
3. read the value of the litterals in this clause
4. answer 1 if the clause is satisfied, 0 otherwise

If $w \notin L$ (i.e., at most $s.m$ clauses can be satisfied), it is detected with proba $\geq 1 - s$. Repeat constant number of times to achieve the desired probability

MAX-3SAT is hard to Approximate (2/2)

PCP ⇔ Hardness of Approximation

$NP \subseteq PCP(O(\log n), O(1)) \Leftrightarrow \exists s < 1$ s.t. GAP-3SAT$_s$ is NP-hard

$\Rightarrow$ reduce $L \in NP$ to GAP-kSAT$_{1-\frac{1}{2^{k+1}}}$

L has a Verifier V using $O(\log n)$ random bits and read $k = O(1)$ bits in the proof.
given a word w and a random string ρ, $V(w, \rho, x) \in \{0, 1\}$ reads k bits of the proof x

$f_{w,\rho} : x \to V(w, \rho, x)$ expressed as k-SAT formula with $\leq 2^k$ clauses.

$$Let\ \Phi_w = \wedge_{\rho \in \{0,1\}^{O(\log n)}} f_{w,\rho}.$$

if $w \in L$ then Φ_w satisfied by x
otherwise at most a fraction $1 - \frac{1}{2^{k+1}}$ of clauses can be satisfied.

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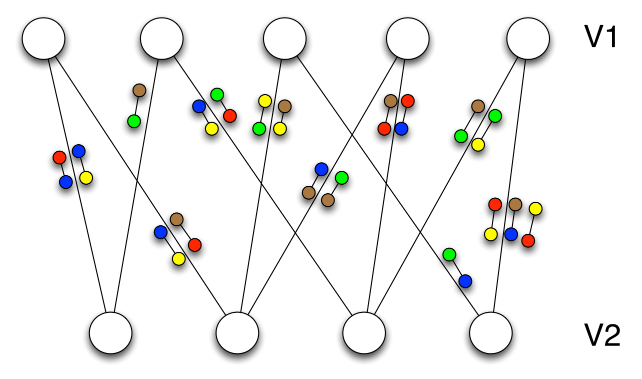
LABEL COVER (Maximization version)

regular bipartite graph $G = (V_1 \cup V_2, E)$ and, $\forall e \in E$, partial function $\Pi_e : [1, N] \to [1, N]$



Let $c : V \to [1, N]$ be a coloring of V.
an edge $e = \{u, v\} \in E$, $u \in V_1$, $v \in V_2$, is covered if $\Pi_e(c(u)) = c(v)$.

Goal: Find a coloring maximizing the fraction of covered edges.

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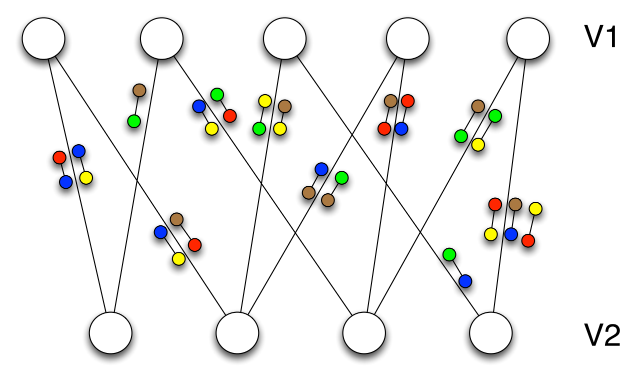

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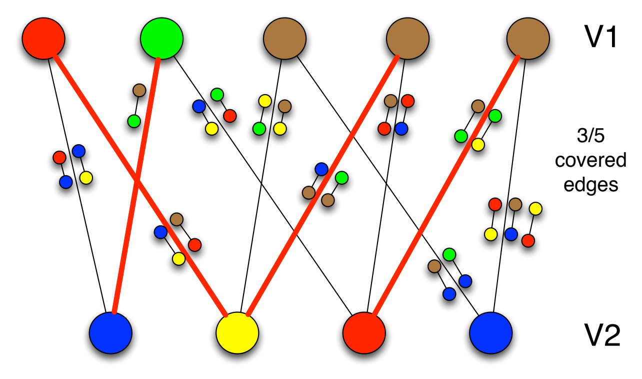

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LABEL COVER hard to approximate

$\exists$ reduction f from 3SAT to LABEL COVER s.t.

for any instance I of 3SAT and instance $f(I)$ of LABEL COVER

if X clauses of I can be satisfied, $\geq X$ edges of $f(I)$ can be covered
if Y edges of $f(I)$ can be covered, $\geq Y/3$ clauses of I can be satisfied

Gap-Preserving Reduction

Corollary

$\exists \epsilon$ s.t. finding an $(1+\epsilon)$-approximation to Label Cover is NP-hard.

LABEL COVER very hard to approximate

Theorem

$\forall 0 < \epsilon < 1/2$ and $c(n) \geq 2^{\log^{0,5-\epsilon} n}$, and there is a reduction f in time $n^{poly(\log n)}$, from any NP problem Π to Label Cover such that

- for any $w \in \Pi$, $f(w)$ has value 1
- for any $w \notin \Pi$, $f(w)$ has value at most $1/c(n)$.

Corollary: if $\exists 0 < \epsilon < 1/2$ and a $2^{\log^{0,5-\epsilon} n}$-approximation algorithm for Label Cover, then any problem in NP can be solved in time $n^{poly(\log n)}$.

Conjecture: Exponential Time Hypothesis (ETH)

There is no sub-exponential-time algorithm to solve 3-SAT unless $P = NP$

Corollary 1: Assuming ETH

$\forall \epsilon$, there is no $2^{\log^{0,5-\epsilon} n}$-approximation algorithm for Label Cover unless $P = NP$

Corollary 2: Assuming ETH

$\forall \epsilon$, there is no $2^{\log^{0,5-\epsilon} n}$-approximation algorithm for Max Clique unless $P = NP$

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Proof of Cor. 2: “Max Clique hard to approx.”

Gap preserving reduction from Label Cover
Reduce any instance of Label Cover to instance of Max Clique s.t. both optima coincide

Let $G = (V_1 \cup V_2, E)$ and, $\forall e \in E$, $\Pi_e : [1, N] \rightarrow [1, N]$

- for any $e \in E$ and any $a, b \in [1, N]$ s.t., $\Pi_e(a) = b$
  add a vertex (e, a, b)
- 2 nodes are adjacent if they are consistent
  e.g., (uv, a, b) and (uw, a', b') are adjacent iff $a = a'$

Proof of Theorem

$L \in RPCP(r, s, p)$ iff $\exists$ a restricted Verifier V s.t.

- V polynomial-time in $|w|$, uses a string ρ of $O(r(|w|))$ bits of randomness
- the proof uses an alphabet with $2^{O(s(|w|))}$ symbols
- the proof consists of 2 Tables T_1, T_2
- V chooses 1 location (chosen uniformly at random) in each table, and read only the corresponding 2 symbols a_1 and a_2
- V confirms that T_1 coherent with T_2: for any a_1 at most one symbol a_2 makes V accepting
  - if $w \in L$ then $\exists$ proof x with $Pr_\rho(V(w, x, \rho) = 1) = 1$ (**completeness**).
  - if $w \notin L$ then $\forall$ proof x, $Pr_\rho(V(w, x, \rho) = 1) \leq 2^{-p(|w|)}$ (**soundness**).

Theorem [Feige,Lovász'92] improved by Raz'95

For any integer $k \geq 2$,

$$NP \subseteq RPCP(\log^{2k+2} n, \log^{k+2} n, \log^{k} n)$$

For any $L \in NP$, use a restricted PCP-Verifier of it to build an instance of Label Cover.

References

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lecture of Nicolas Schabanel on proof PCP theorem (video online)

lecture of Luca Trevisan on PCP (video online)