Spy-Game on graphs

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Search Games: Theory and Algorithms

Leiden, Netherlands, June 28th, 2016

Many Two-player games in Graphs

Cops and Robber [Nowakowski and Winkler; Quilliot 1983]

A team of *Cops* attempts to capture one *Robber*

Angels and Devils [Conway 1996]

Angel moves on the graph, *Devil* blocks vertices

Eternel Domination [Burger *et al* 2004]

A team of *Defenders* protects nodes from one *Attacker*

....

Spy Game [this work]

A team of *Guards* attempts to stay close to the *Spy*

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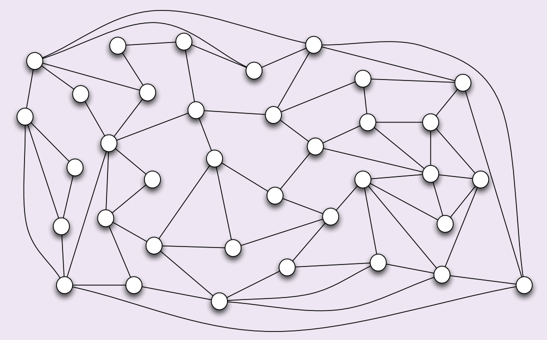
Motivations to define yet another game

Cops & Robber Games [Nowakowski and Winkler; Quilliot, 1983]

Rules of the $\mathcal{C}\&\mathcal{R}$ game



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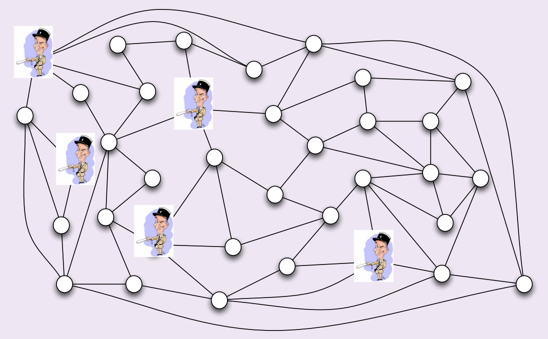

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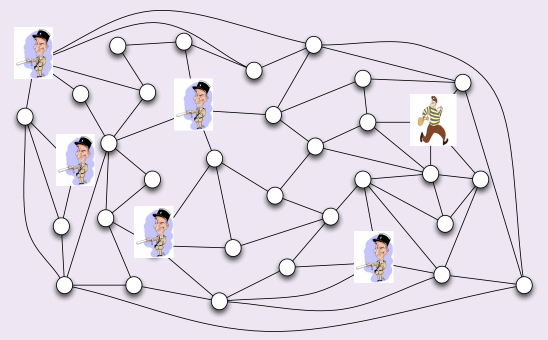

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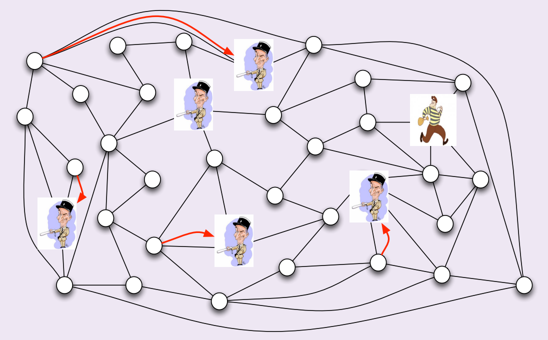
   (1) each $\mathcal{C}$ slides along ≤ 1 edge



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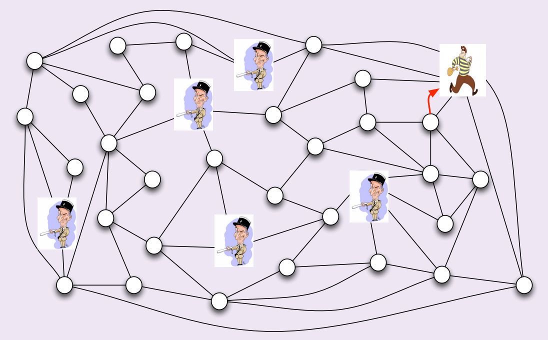

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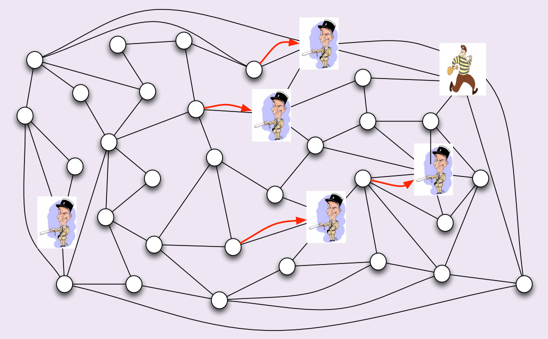

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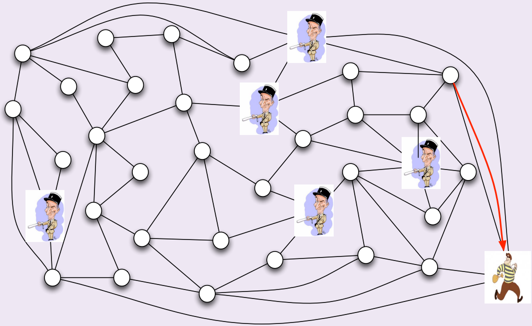

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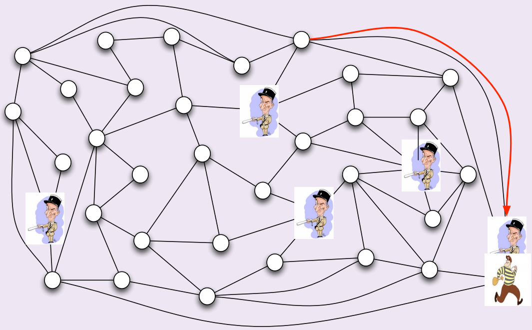

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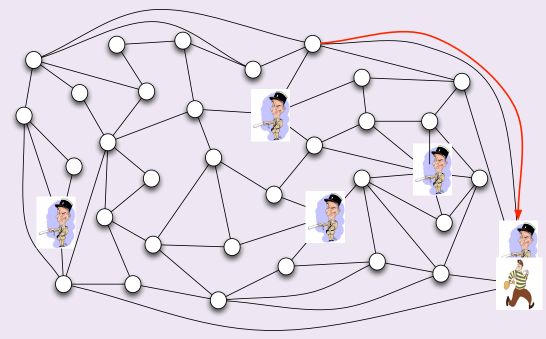
Cop Number of a graph G

$cn(G)$: min # Cops to win in G



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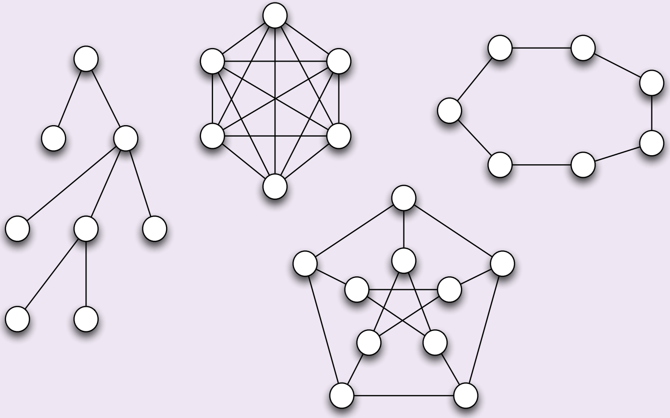

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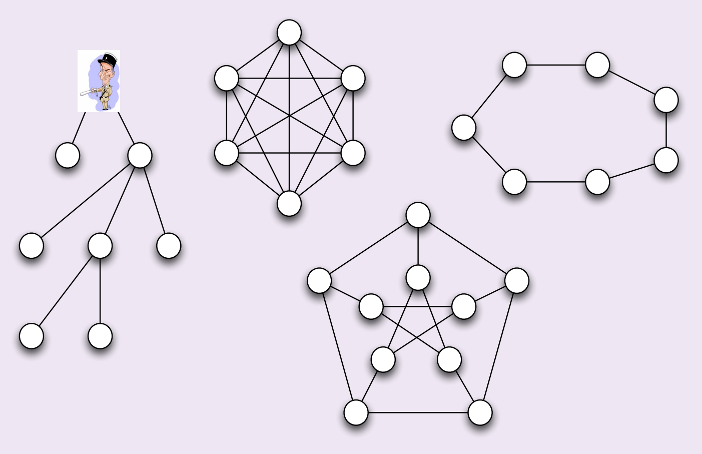

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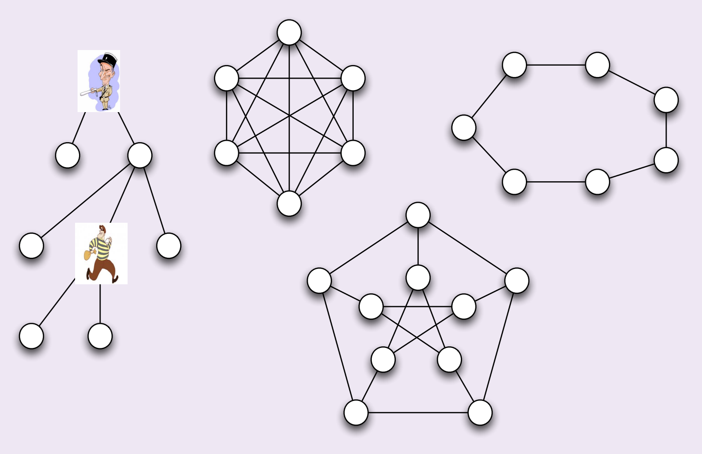

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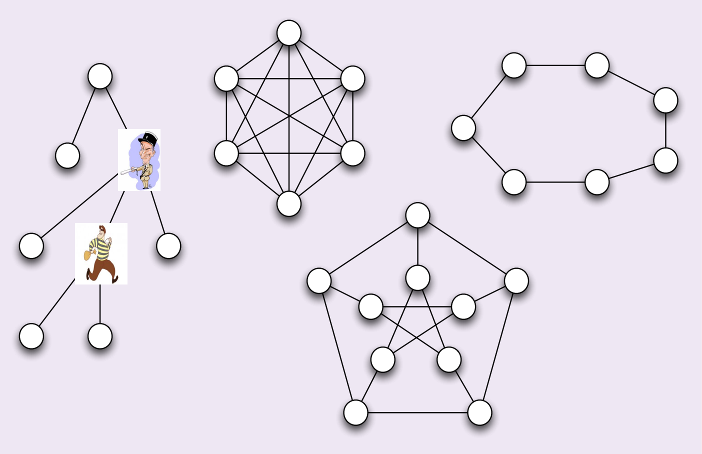

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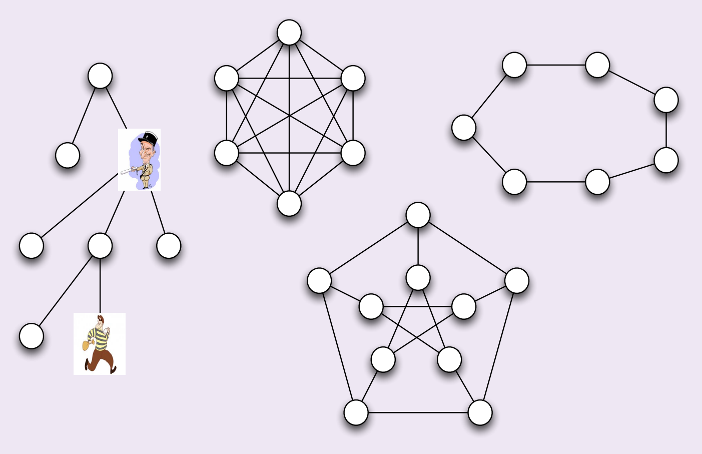

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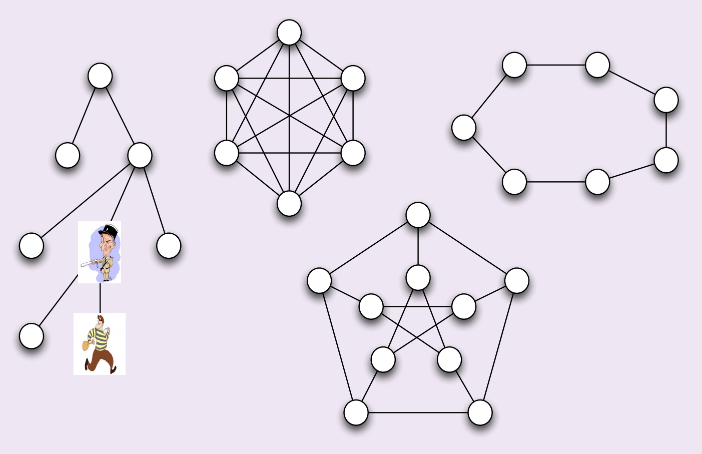

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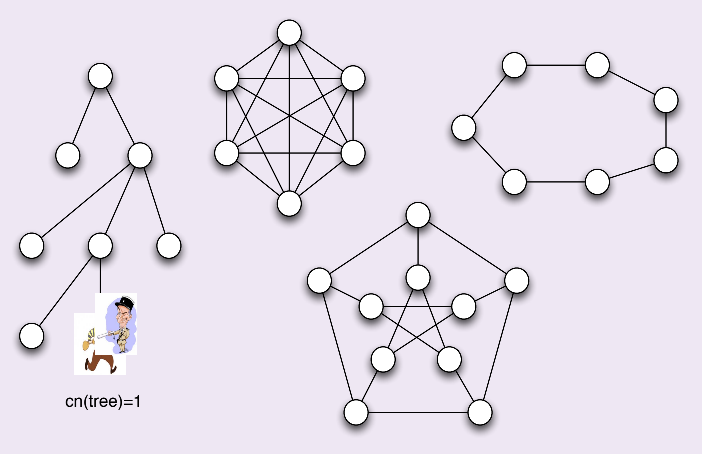

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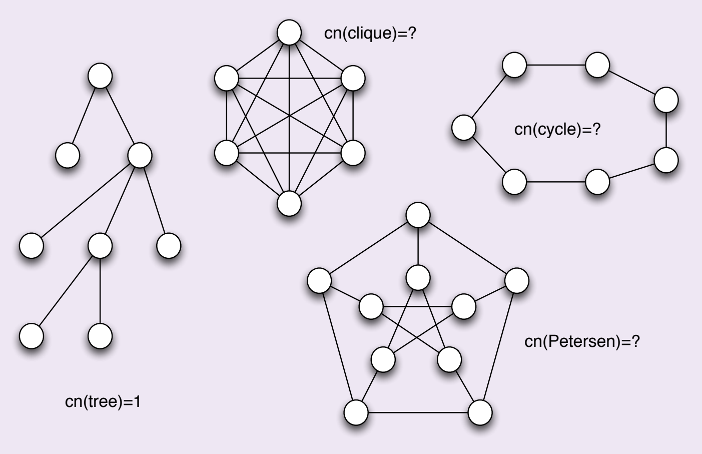

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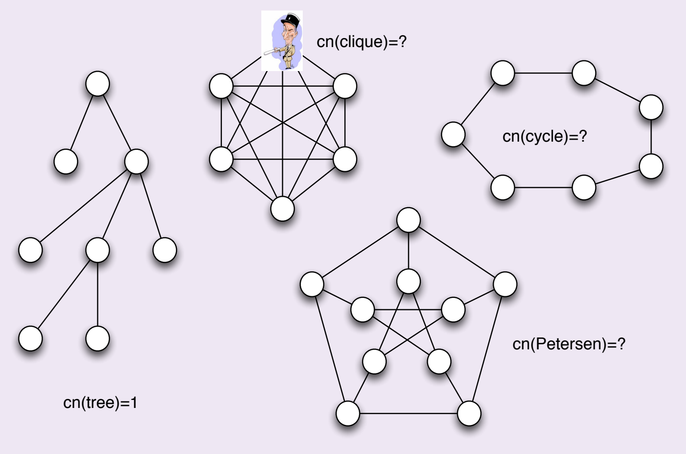

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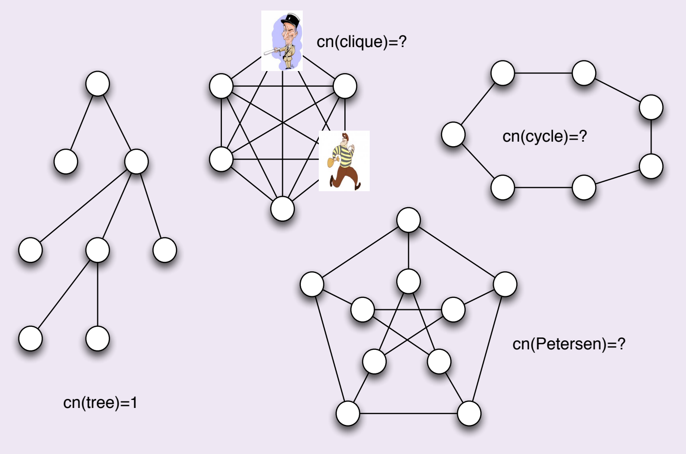

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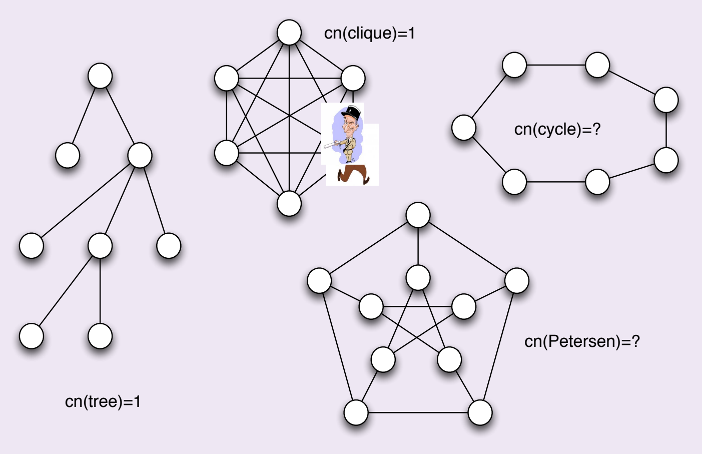

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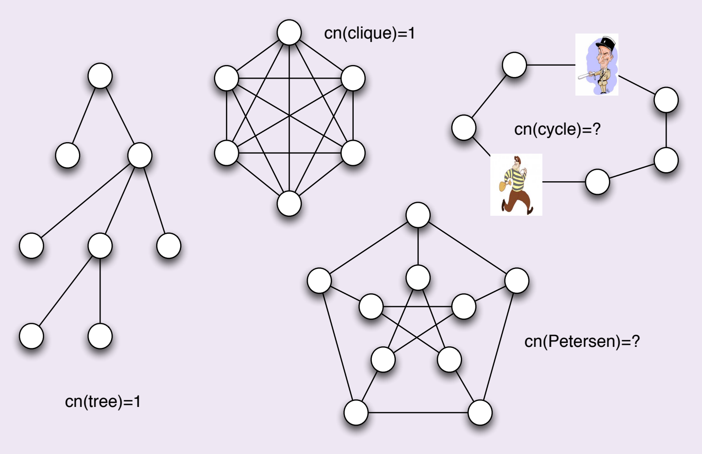

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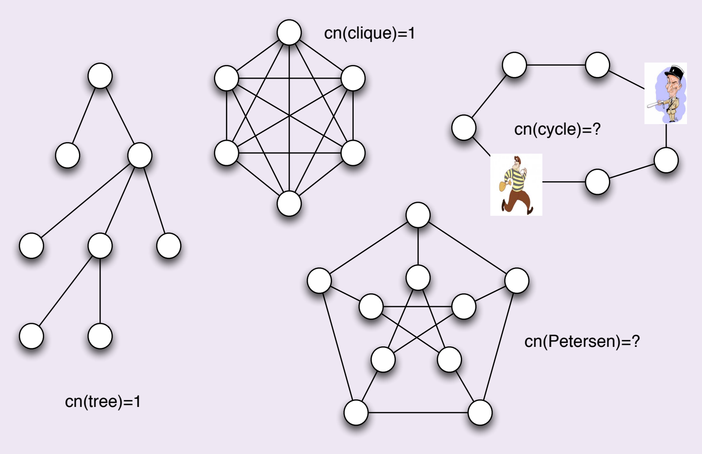

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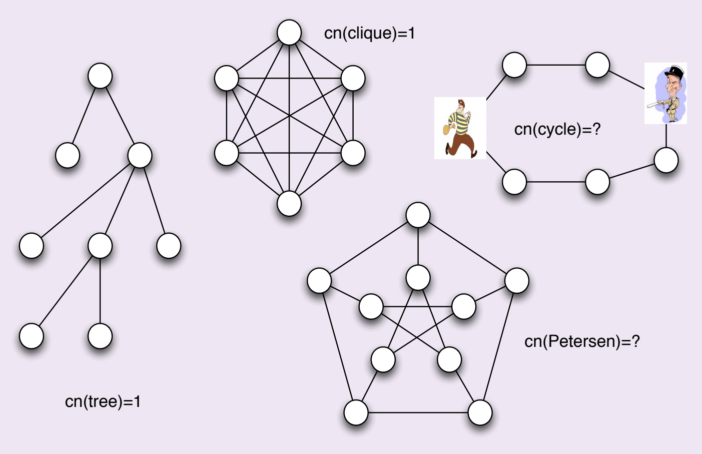

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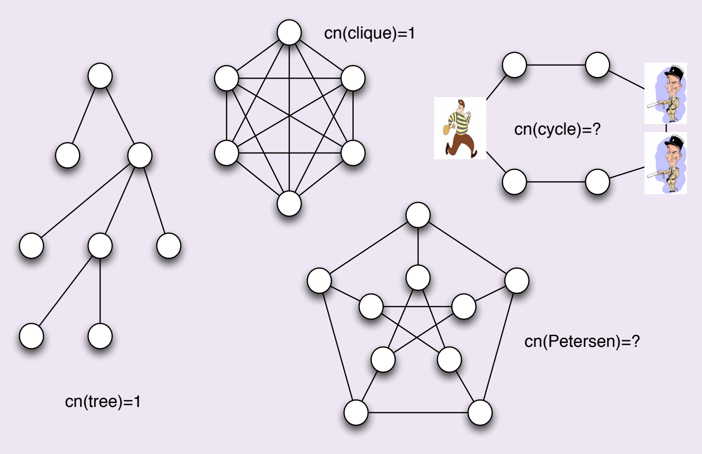

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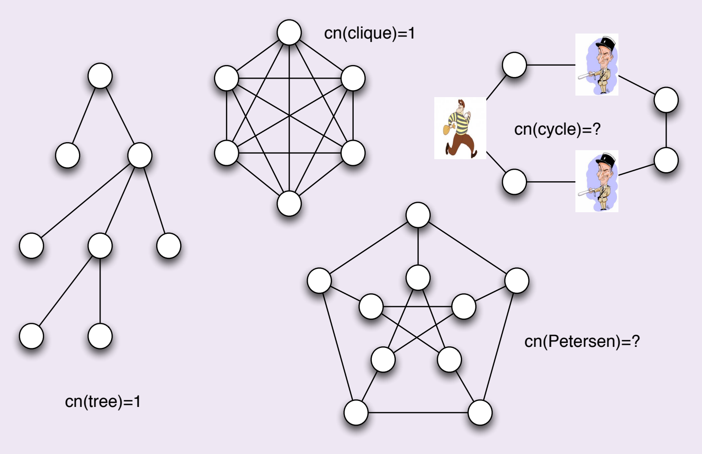

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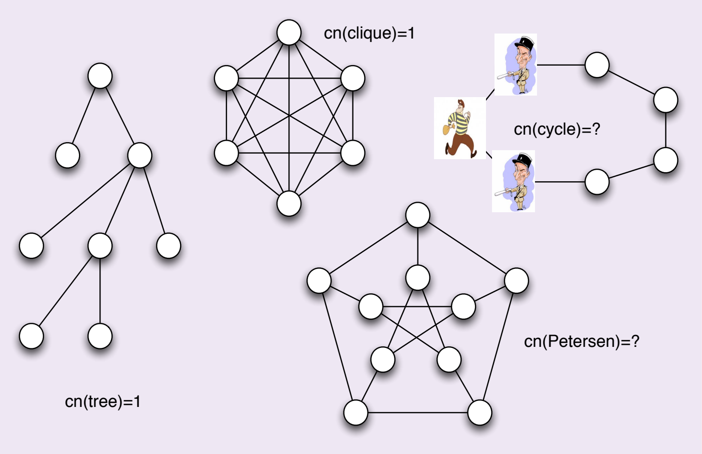

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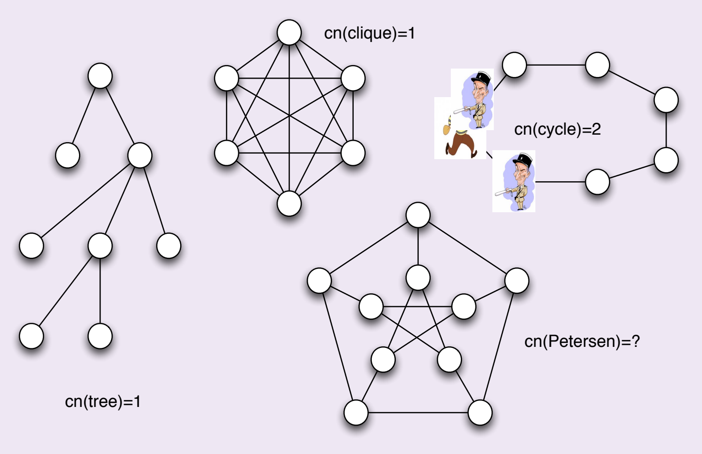

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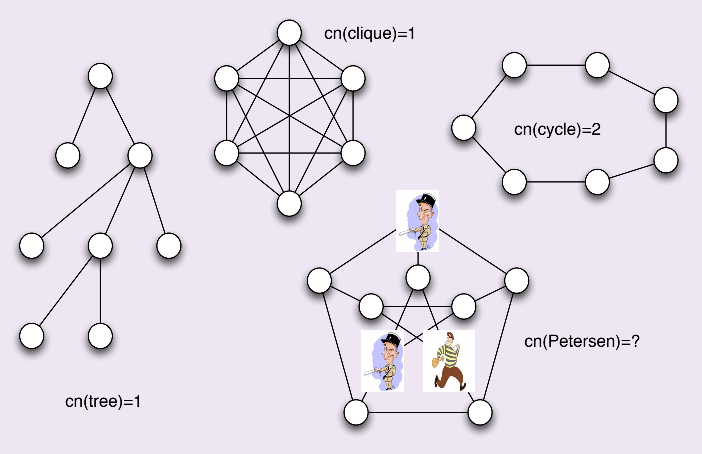

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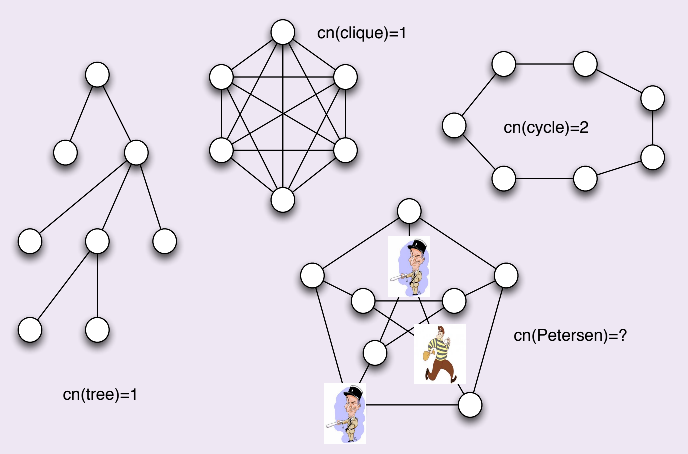

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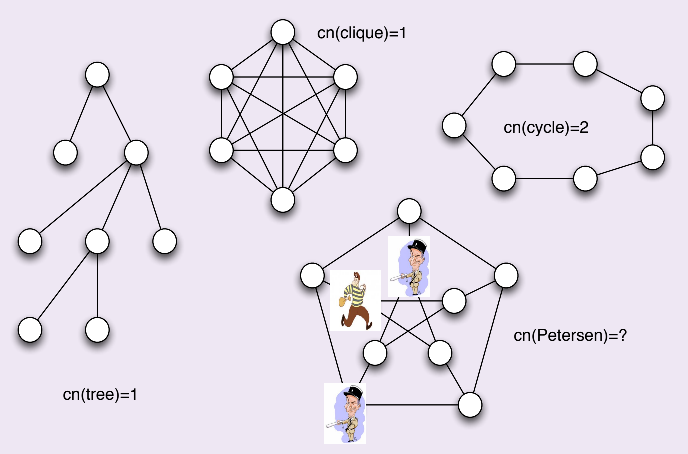

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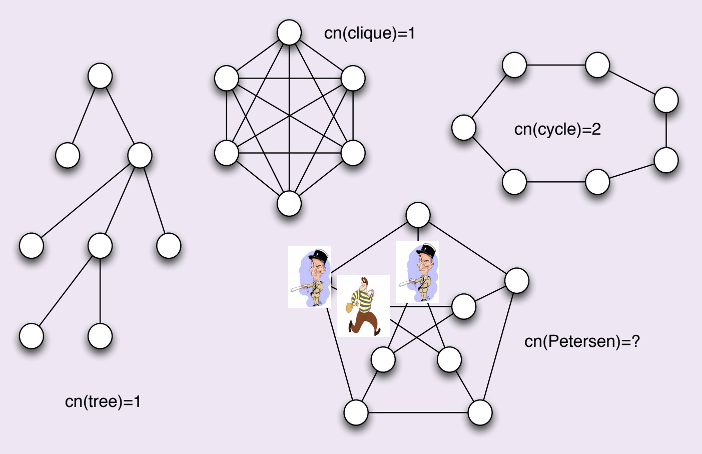

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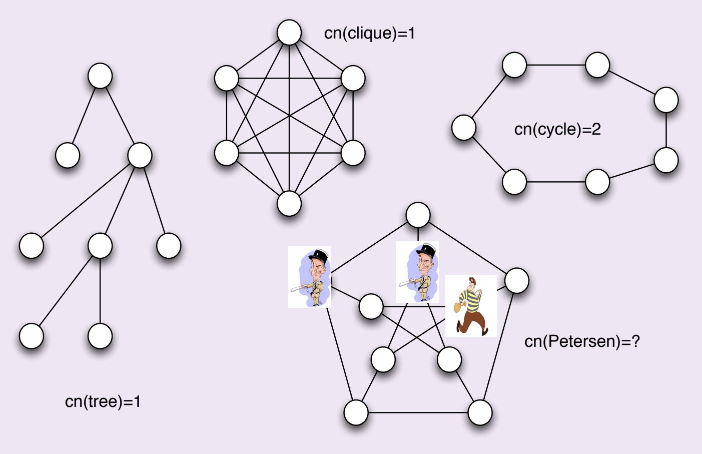

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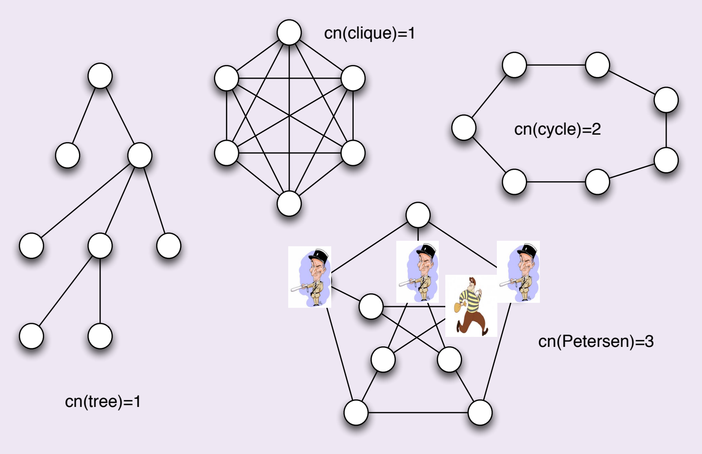

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True in many graph classes	cn	
dominating set $\leq k$	$\leq k$	[folklore]
treewidth $\leq t$	$\leq t/2+1$	[Joret, Kaminski,Theis 09]
chordality $\leq k$	$< k$	[Kosowski,Li,N.,Suchan 12]
genus $\leq g$	$\leq \lfloor \frac{3g}{2} \rfloor + 3$	(Conjecture $\leq g+3$) [Schröder, 01]
planar graphs	≤ 3	[Aigner,Fromme, 84]
grids	$= 2$	[folklore]
H-**minor free**	$\leq \lvert E(H) \rvert$	[Andreae, 86]
degeneracy $\leq d$	$\leq d$	[Lu,Peng 12]
diameter 2	$O(\sqrt{n})$	—
bipartite diameter 3	$O(\sqrt{n})$	—
Erdös-Réyni graphs	$O(\sqrt{n})$	[Bollobas *et al.* 08] [Luczak, Pralat 10]
Power law	$O(\sqrt{n})$	[Bonato,Pralat,Wang 07]

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Currently known best general upper bound...

- $cn(G) = O(\frac{n}{2^{(1-o(1))\sqrt{\log n}}})$ [Scott, Sudakov 11, Lu,Peng 12]

note that $\frac{n}{2^{(1-o(1))\sqrt{\log n}}} \geq n^{1-\epsilon}$ for any $\epsilon > 0$

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To tackle Meyniel's conjecture: new variants have been defined.

When the Robber can run [Fomin,Golovach,Kratochvil,N.,Suchan'10]

New variant with speed: The Robber may move along several edges per turn
$cn_s(G)$: min # of Cops to capture Robber with speed $s \geq 1$.

Meyniel Conjecture [Alon, Mehrabian'11] and general upper bound [Frieze,Krivelevich,Loh'12] extend to this variant

... but fundamental differences (recall: planar graphs have $cn_1 \leq 3$)

$cn_2(G)$ unbounded in grids [Fomin,Golovach,Kratochvil,N.,Suchan'10]

$$\Omega(\sqrt{\log n}) \leq cn_2(G_{n\times n}) \leq O(n) \text{ in } n \times n \text{ grid } G_{n\times n}$$

Open question: exact value of $cn_2(G_{n\times n})$?

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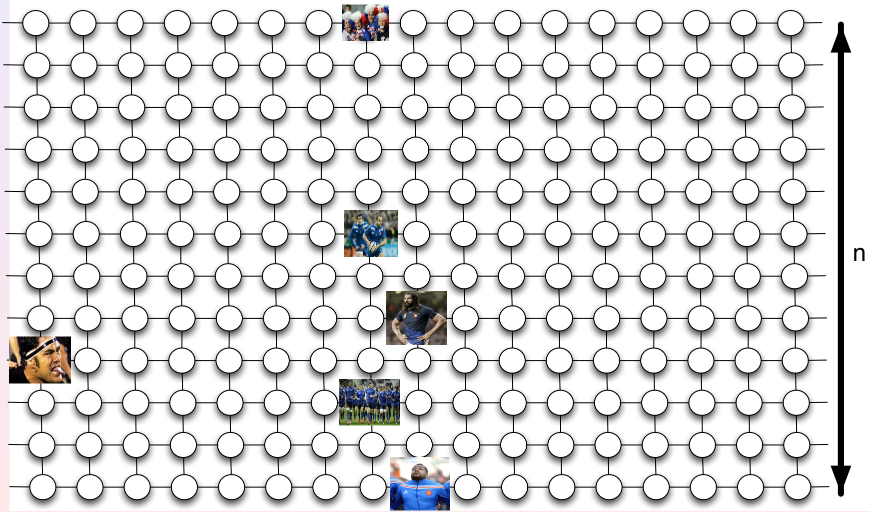

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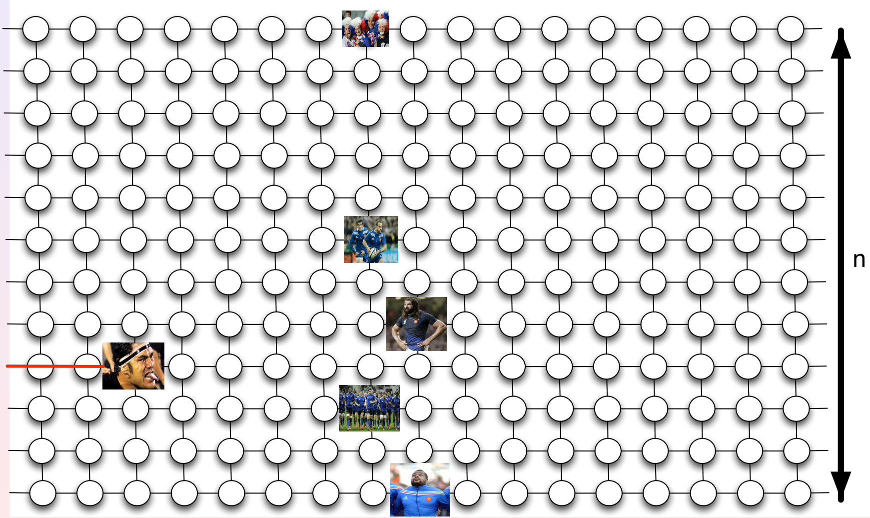

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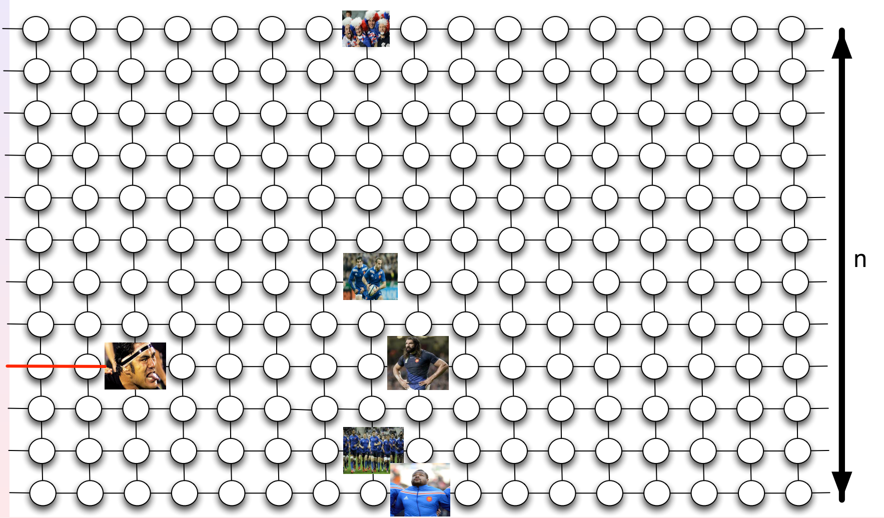

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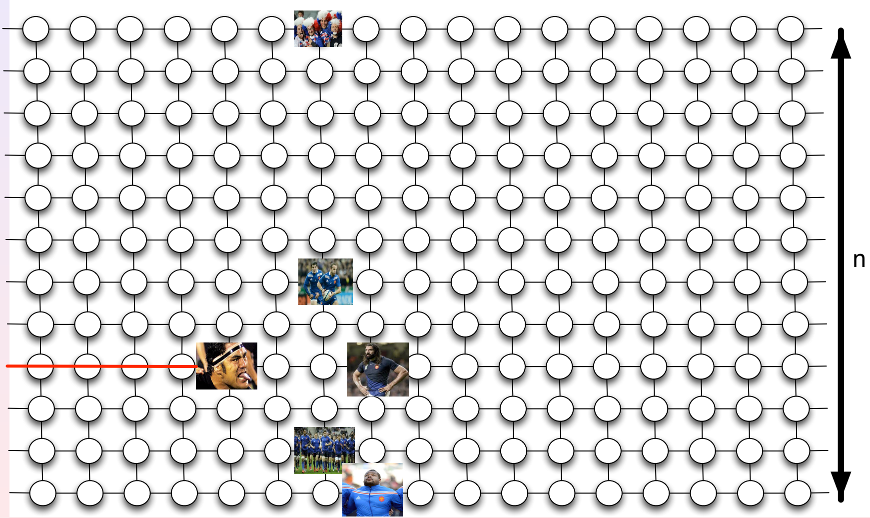

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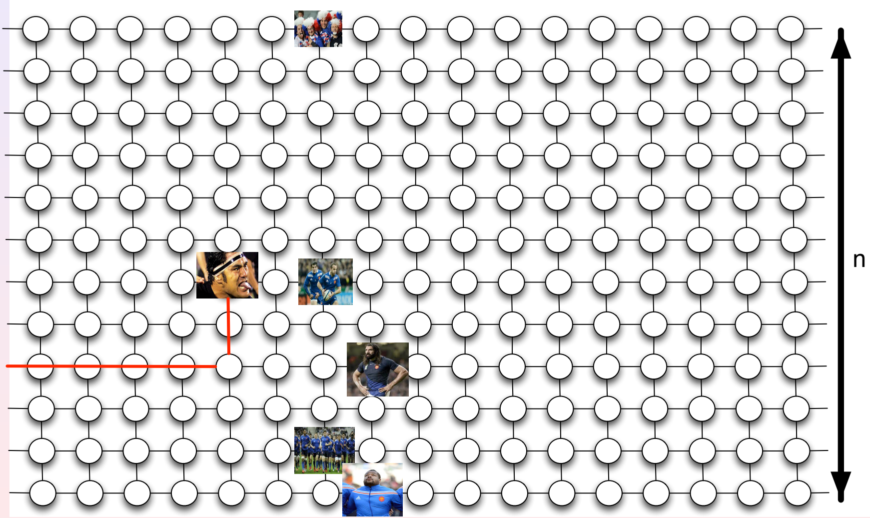

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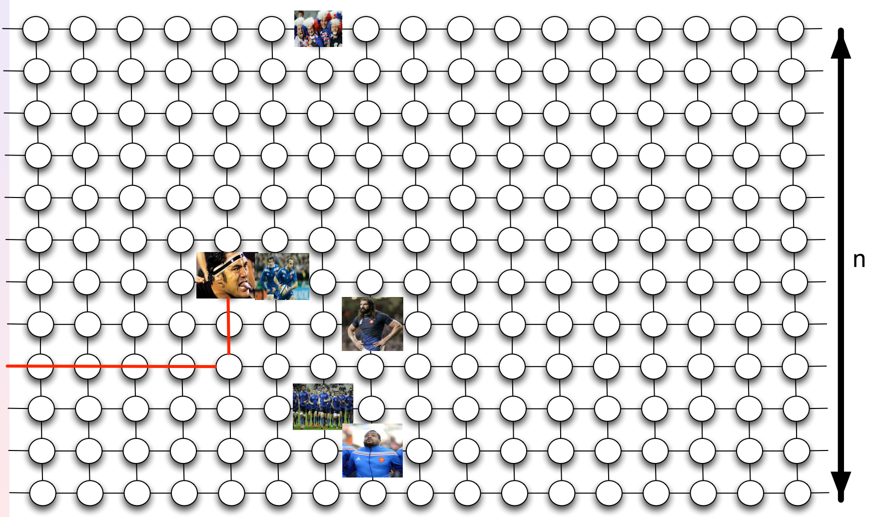

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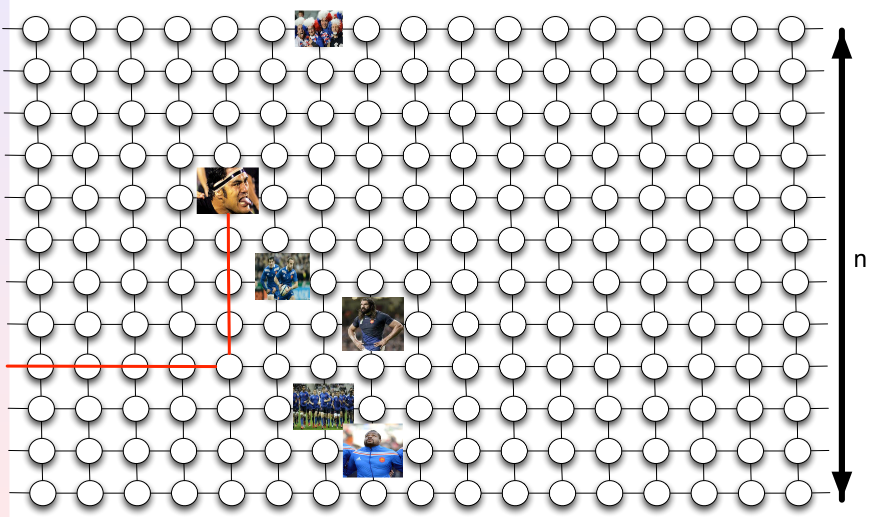

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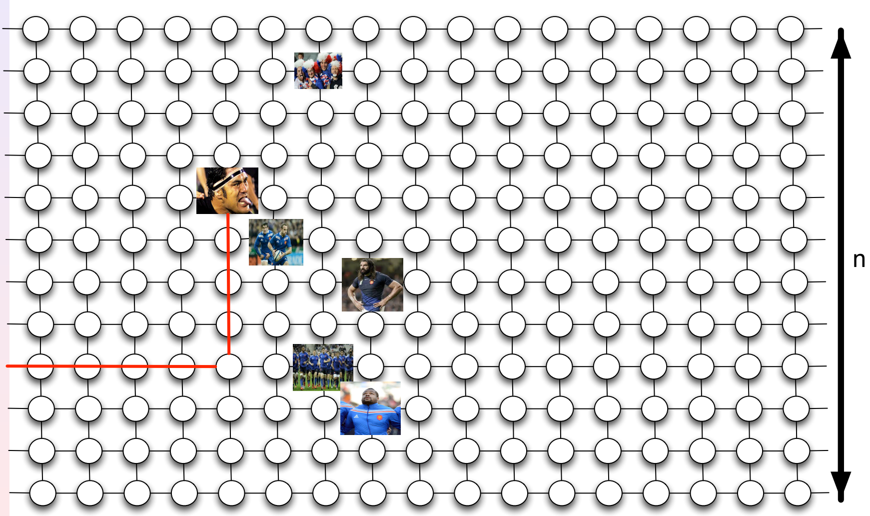

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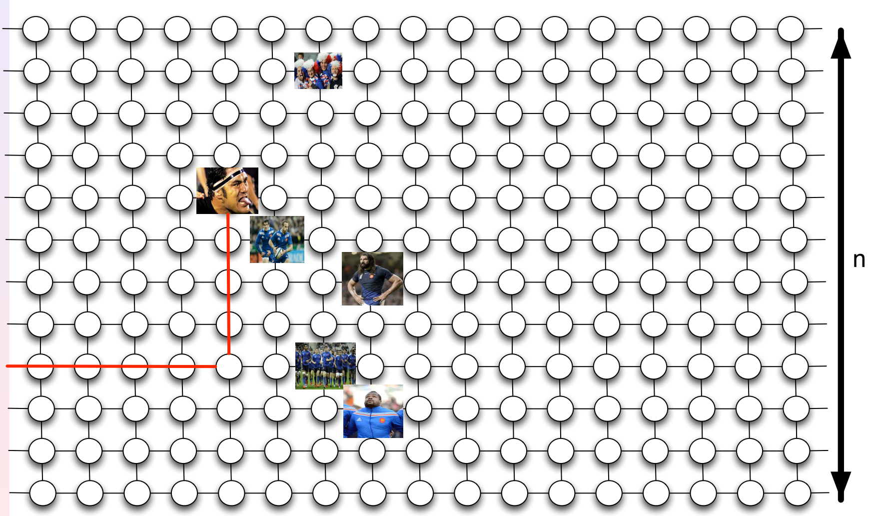

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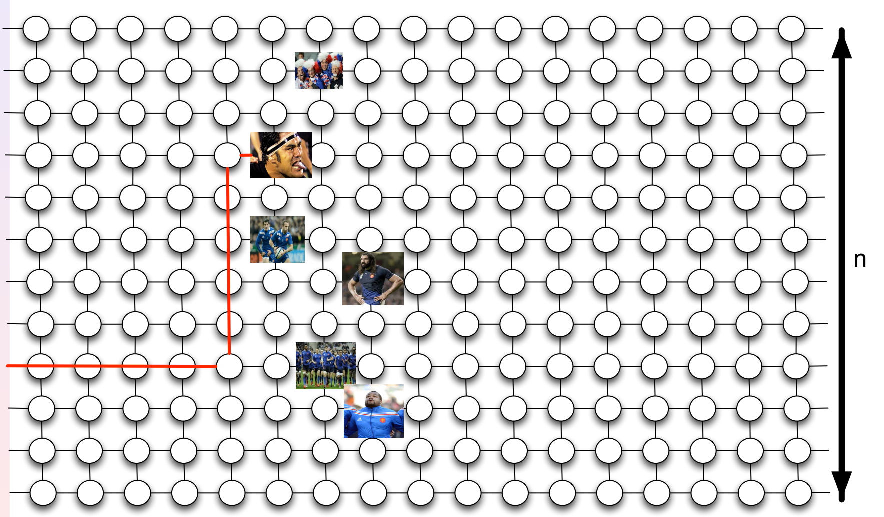

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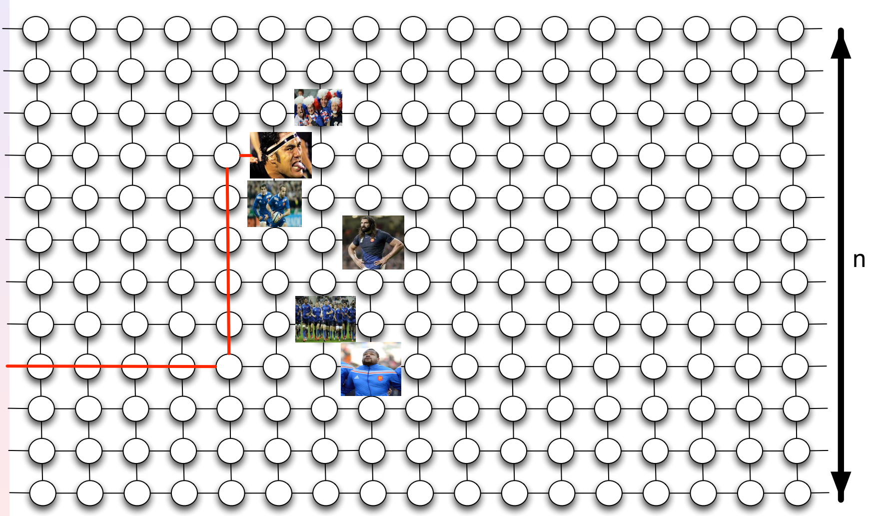

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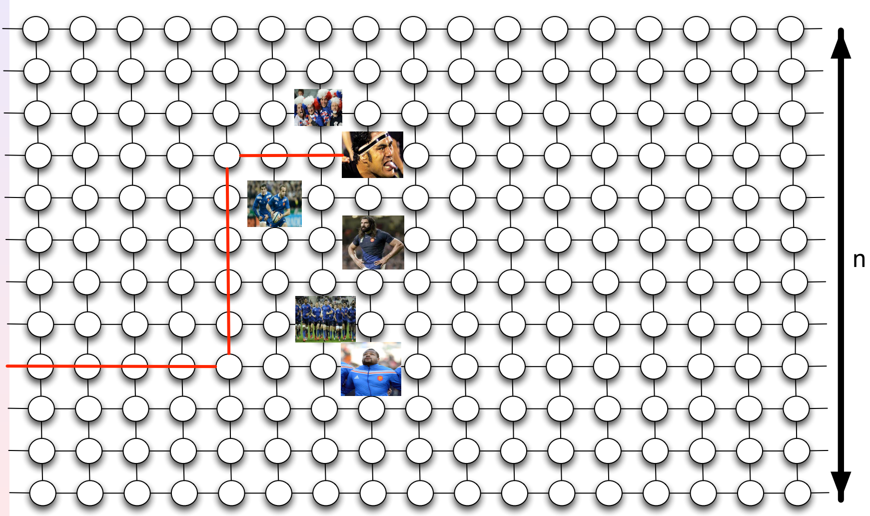

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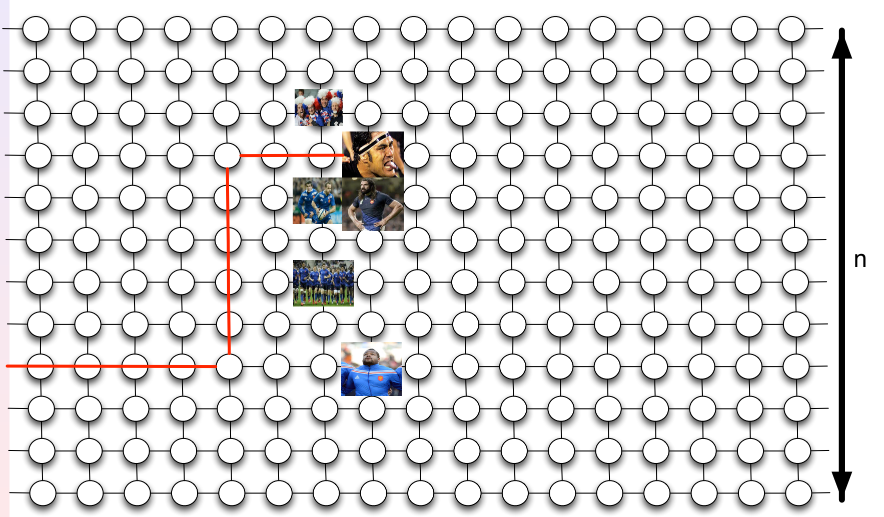

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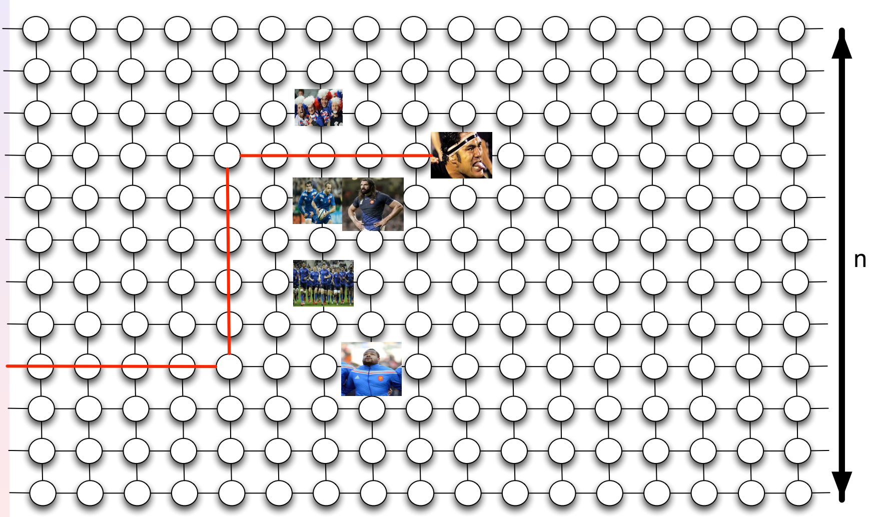

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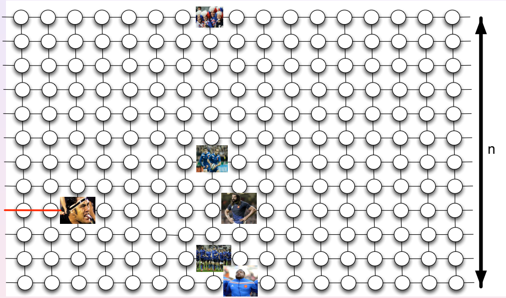
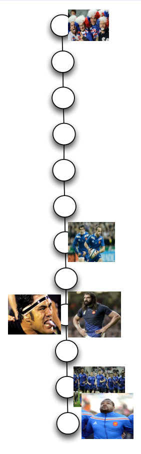


Project the whole game on One column

Goal of Robber: go far to all cops

"Find a hole"



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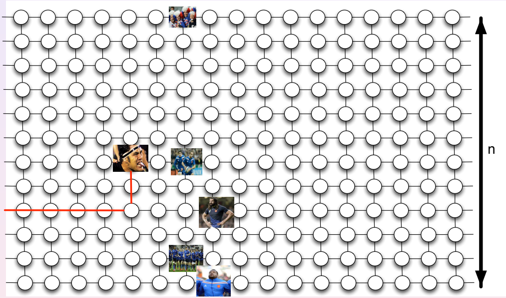

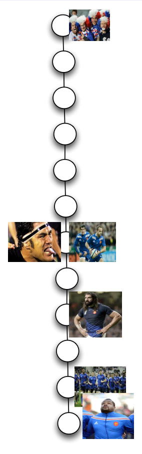

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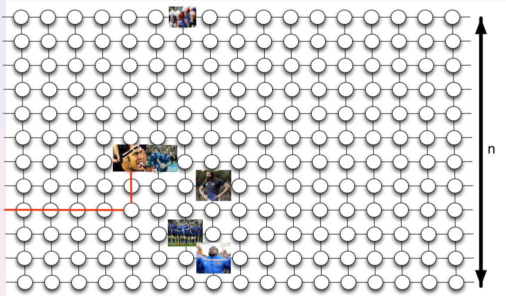

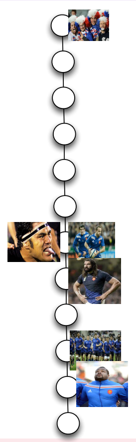

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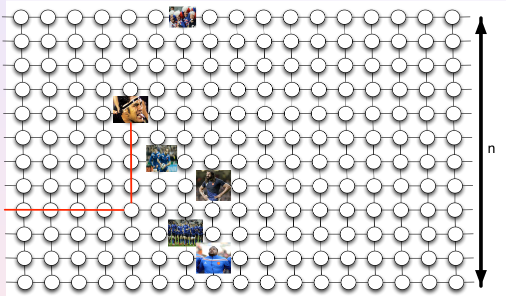

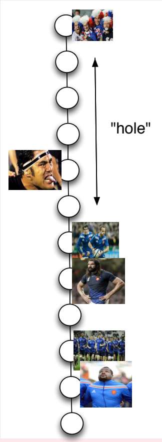

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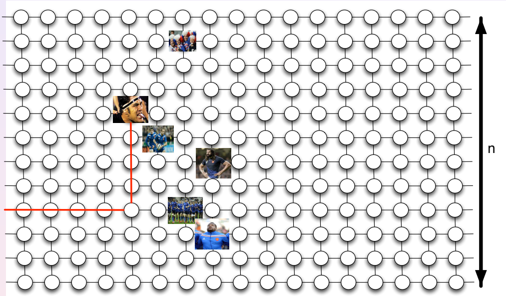

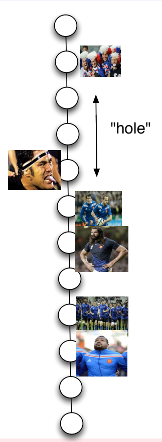

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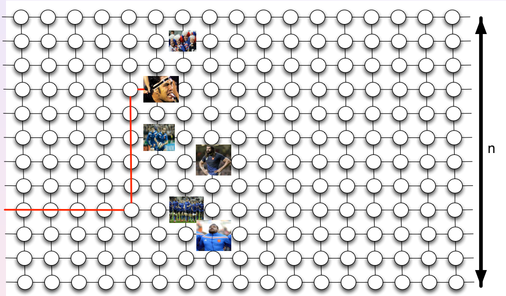

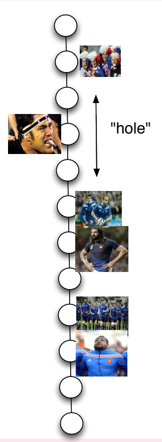

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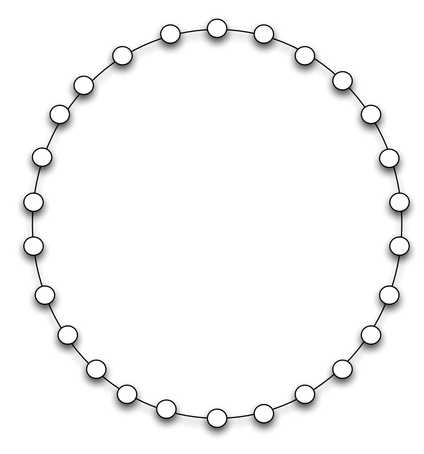
Spy Game in graphs

Rules of the Spy game



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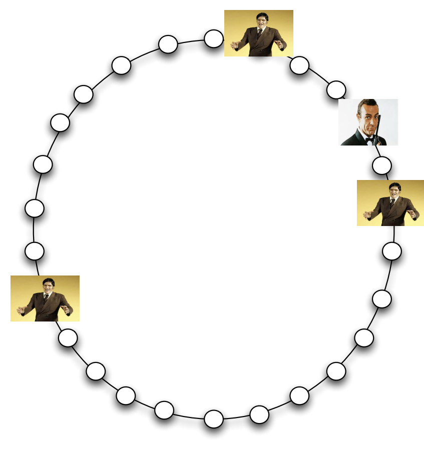

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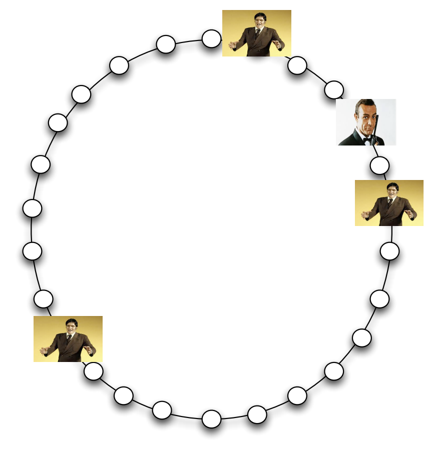

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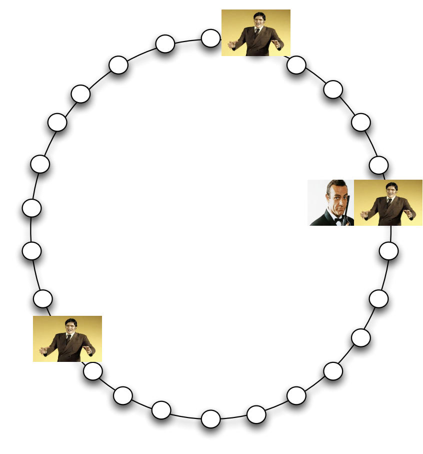

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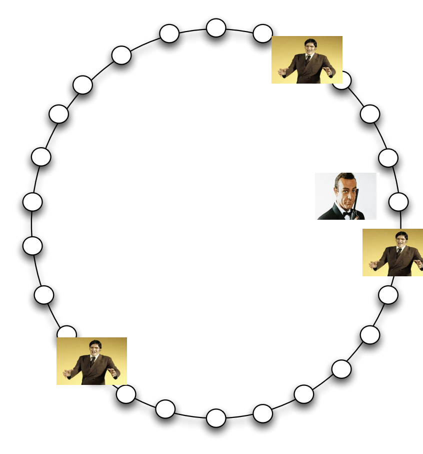

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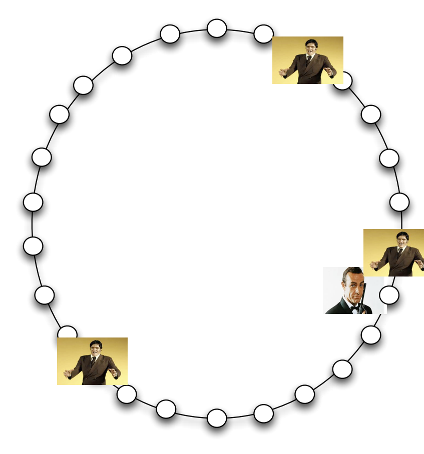

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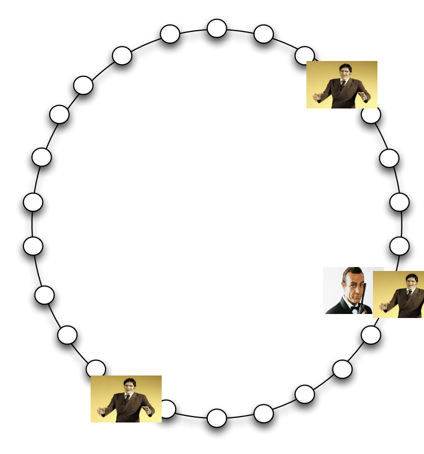

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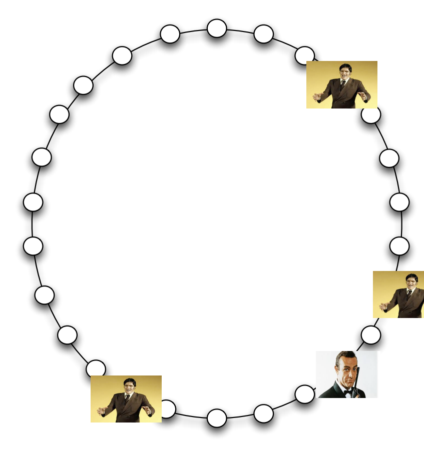

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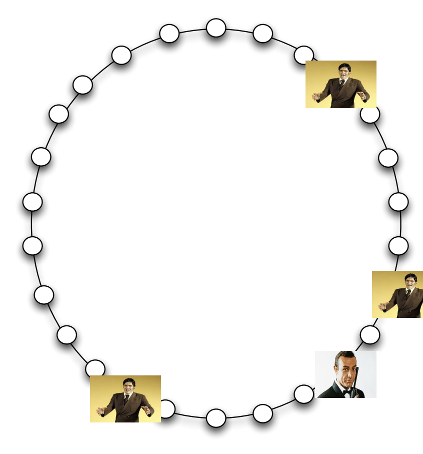

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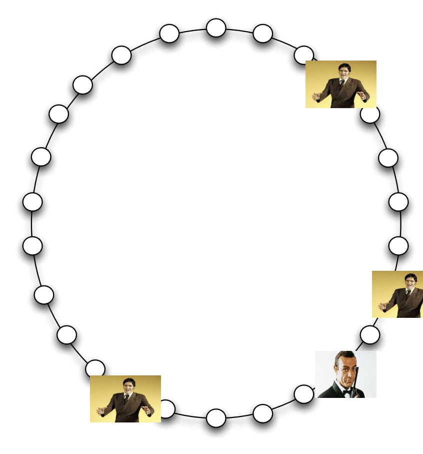
$\Leftrightarrow$ Cops and Robber for $s = 1$, $d = 0$
$\Leftrightarrow$ Eternal Domination for $s = \infty$, $d = 0$



Complexity

- Computing $gn_{s,d}$ is NP-hard (reduction to Set Cover)
- Computing $gn_{s,d}$ is PSPACE-hard in DAGs if Guards are placed first

Case of Paths and Cycles on n vertices

- Paths: $\left\lfloor \frac{n(s-1)}{2ks} \right\rfloor \le d_{s,k}(P_n) \le \left\lceil \frac{(n+1)(s-1)}{2ks} \right\rceil$
- Cycles: $\left\lfloor \frac{(n-1)(s-1)}{k(2s+2)-4} \right\rfloor \le d_{s,k}(C_n) \le \left\lfloor \frac{(n+1)(s-1)}{k(2s+2)-4} \right\rfloor$

Case of grids: # of guards is super-linear in the side n

$\exists \epsilon > 0$ such that $gn_{s,d}(G_{n\times n}) = \Omega(n^{1+\epsilon})$ in any $n \times n$ grid $G_{n\times n}$

Fractional relaxation of the game

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Lower Bound: Spy starts from one end and runs !

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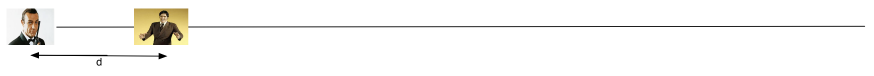

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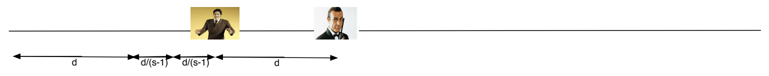
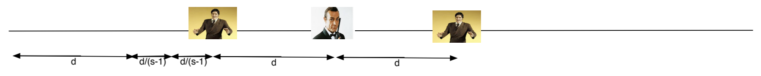


Lower Bound: Spy starts from one end and runs !
One guard is "consumed" after $2d/(s-1)$ steps



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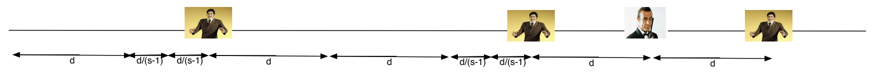
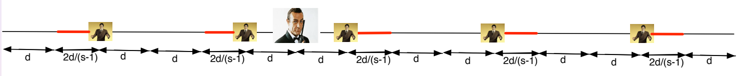
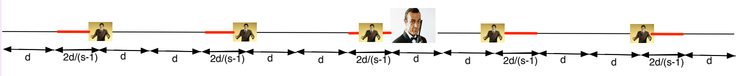


Lower Bound: Spy starts from one end and runs !
Hence, $n \leq k \cdot 2ds/(s-1)$



Upper Bound: Each guard is assigned its own area of length $\leq 2ds/(s-1)$.



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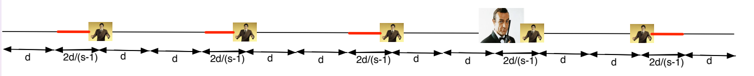

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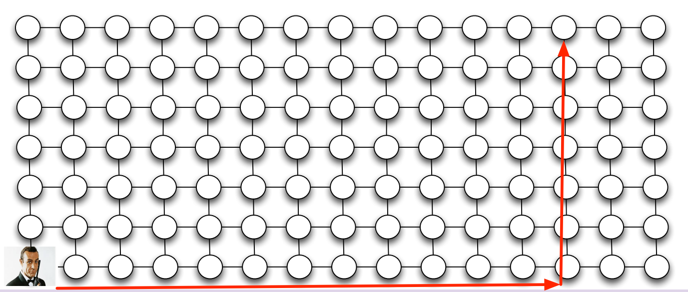
Case of Grids

Lemma 1: $gn_{s,d}(G_{n\times n}) = \Omega(n \log n)$ in $n \times n$ grid

Consider only "L - strategies"



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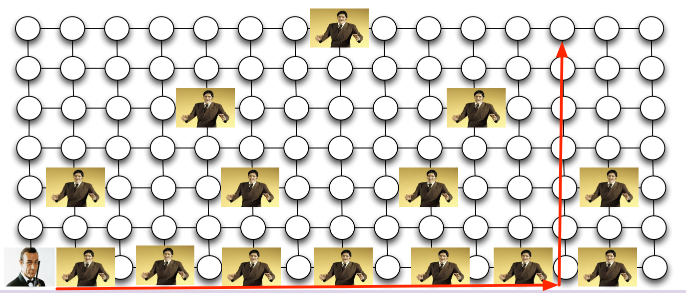

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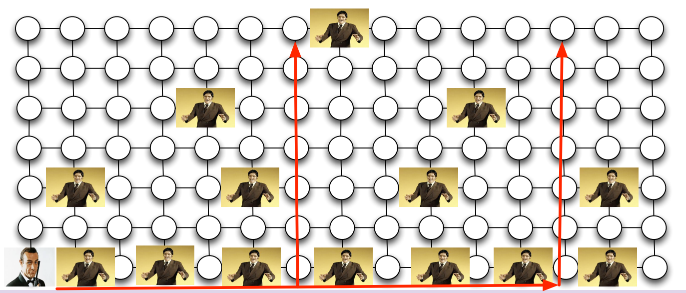

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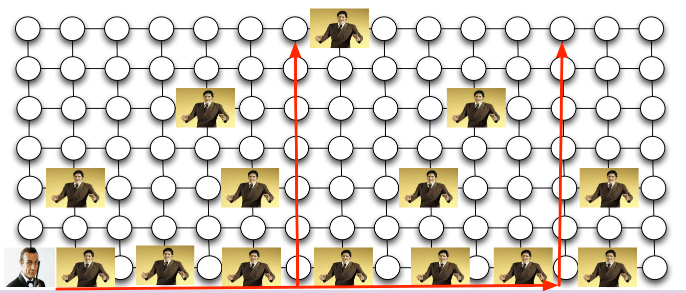

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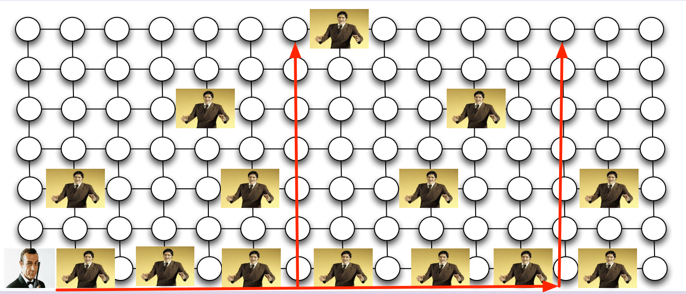

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Not enough for the announced bound:

$$gn_{s,d}(G_{n\times n}) = \Omega(n^{1+\epsilon}) \text{ for some } \epsilon > 0$$

We would like to reduce guards "density" in order to recurse

Fractional Spy-Game

Consider "fractional guards"

$fg(v) \in \mathbb{R}^+$: amount of guard on vertex v

Total amount of guards

$\sum_{v \in V(G)} fg(v)$

Moves of guards

It is a flow!

Winning condition: control the Spy at each step

$\sum_{v \in B(Spy,d)} fg(v) \geq 1$

$B(Spy, d)$: ball of radius d centered on the Spy

$\textit{frac-gn}_{s,d}(G)$: min amount of fractional guards required to win

Theorem: super-linear and sub-quadratic in grids

$\exists \epsilon, \beta$ such that

$$\Omega(n^{1+\epsilon}) \leq \textit{frac-gn}_{s,d}(G_{n\times n}) \leq O(n^{2-\beta}).$$

Clearly, $\textit{frac-gn}_{s,d}(G) \leq gn_{s,d}(G)$ for any graph G

Corollary

$gn_{s,d}(G_{n\times n}) = \Omega(n^{1+\epsilon})$ for some $\epsilon > 0$

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Goal: find such a strategy for the guards, i.e., define

$fg_t(v)$: amount of cops on v at step t

"Trick": Consider very simple strategy

time-independent + decreasing function of the distance

$$fg(v) = \frac{1}{dist(v, Spy)^{\beta}}.$$

main result (using flows and duality)

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In fractional version, the density can be reduced!

for $a \in \mathbb{N}^*$, after at most $2n$ steps against k guards, the amount of guards at distance $\leq 2n/a$ from the spy is $< k(aH(a))^{-1}$.

Theorem: $\exists \alpha > 0$ such that $\Omega(n^{1+\alpha}) \leq frac\text{-}gn_{s,d}(G_{n \times n})$.

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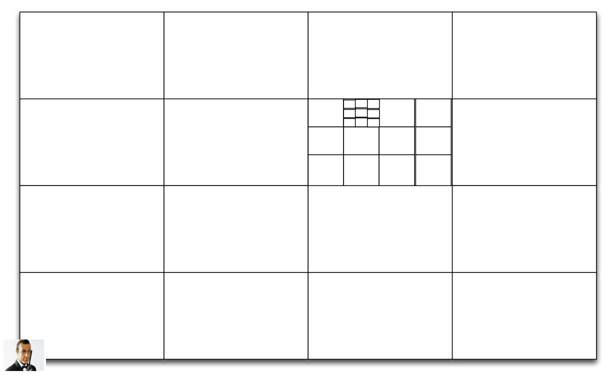

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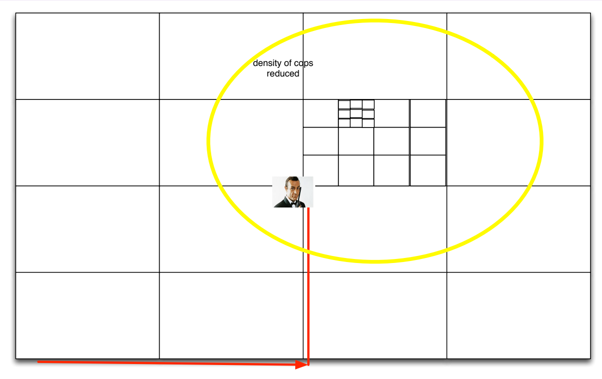

Fractional Spy-Game

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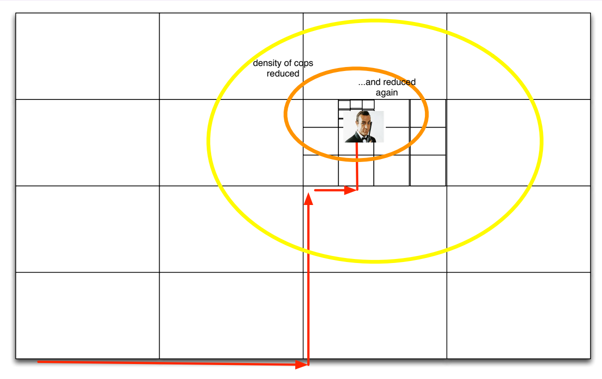

Current Work / Open problems

On Cops and Robber games:

- Meyniel Conjecture [1985]:

  For any n-node connected graph G, $cn(G) = O(\sqrt{n})$

- How many cops with speed 1 to capture a robber with speed 2 in a grid?

On Spy game:

- Complexity of computing $gn_{s,d}$? of computing $frac\text{-}gn_{s,d}$?
- actual value of $gn_{s,d}(G)$ in $n \times n$ grid G?

  We know: $\exists \epsilon, \beta > 0$, $\Omega(n^{1+\epsilon}) \leq frac\text{-}gn \leq gn \leq O(n^2)$.

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- other graph classes: trees, bounded treewidth...?

Thank you !

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