Fractional Combinatorial Games

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Cops & robber games [Nowakowski and Winkler; Quilliot, 83]

Initialization:
1. $C$ places the cops;
2. $R$ places the robber.

Step-by-step:
- each cop traverses at most 1 edge;
- the robber traverses at most 1 edge.

Robber captured:
A cop at same node as robber.

Goal:
Cop-number = minimum number of cops
Cops & robber games [Nowakowski and Winkler; Quilliot, 83]

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An Observer must ensure that a Surfer never reaches a dangerous node
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Surveillance game

Turn by turn: Observer marks $k = 2$ nodes
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then Surfer may move on a adjacent node
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then Surfer may move on an adjacent node
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**then** Surfer may move on a adjacent node
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Turn by turn: Observer marks $k = 2$ nodes

*then* Surfer may move on a adjacent node
Turn by turn: Observer marks $k = 2$ nodes

**then** Surfer may move on a adjacent node
Turn by turn: Observer marks $k = 2$ nodes

**then** Surfer may move on a adjacent node
Turn by turn: Observer marks $k = 2$ nodes

then Surfer may move on a adjacent node
In this example, all nodes are marked

Victory of the Observer using 2 marks per turn
Model: another Two players game

- a *Surfer* starts from safe homebase $v_0$
- in $G$, a dangerous graph

- a *Guard* with some amount $k$ of bullets

**Turn by turn:**

1. the guard secures $\leq k$ nodes;
2. then, the Surfer may move to an adjacent node.

**Defeat:** Surfer in unsafe node  
**Victory:** $G$ safe

Minimize amount of bullets to win for any Surfer’s trajectory

 Surveillance number of $G$ (connected) from $v_0$: $sn(G, v_0)$
Two players play a game on a graph.

Game is played turn-by-turn.

Players play by moving and/or adding tokens on vertices of the graph.

Optimization problem: minimizing number of tokens to achieve some goal.
All these games are hard

- Cops and Robber: $k$ cops are enough?
  - PSPACE-complete in general graphs [Mamino, 2012].
- Surveillance Game: $k$ marks per turn are enough?
  - $k = 2$ NP-complete for Chordal/Bipartite Graphs [Fomin et al, 2012].
  - $k = 4$ PSPACE-complete for DAGS [Fomin et al, 2012].
- Angels and Devils: Does an Angel of power $k$ wins?
  - 1-Angel loses in (infinite) grids [Conway, 1982].
  - 2-Angel wins in (infinite) grids [Máthé, 2007].
- Eternal Dominating set.
- Eternal Vertex Cover.
  - NP-hard [Fomin et al, 2010].
New tools/approaches are required

Several questions remain open

- Meyniel conjecture: \( cn(G) = O(\sqrt{n}) \) in any \( n \)-node graphs?
- Polynomial-time Approximation algorithms?

less difficult but still intriguing

- number of cops to capture fast robber in grids?
- cost of connectivity in surveillance game?
- etc.

Here, we present preliminary results of our new approach
Fractional Combinatorial Game

- Fractional games:
  both players can use “fractions” of tokens.

- Semi-Fractional games:
  only one player (Player C) can use fractions of tokens.

- Integral games: classical games, token are unsplittable
Example: Fractional Cops and Robber

integral game:
cop-number = 2
Example: Fractional Cops and Robber

**Integral game:**
cop-number = 2

**Semi fractional:**
cop-number ≤ 3/2
Example: Fractional Cops and Robber

integral game:
cop-number = 2

semi fractional:
cop-number ≤ 3/2
Example: Fractional Cops and Robber

integral game:
cop-number $= 2$

semi fractional:
cop-number $\leq 3/2$
Example: Fractional Cops and Robber

integral game:
cop-number = 2

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cop-number ≤ 3/2
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Remark:
by definition:
semi-fractional ≤ integral
gap?
relationship with fractional?
### Preliminary results

**Fractional games**

*general framework: fractional relaxation of turn-by-turn games*

**important property:** “convexity” of winning states

### Semi-fractional = fractional (properties of robber’s moves)

*solutions of fractional games provides lower bounds for integral games*

### Algorithm $\mathcal{A}$ to decide which player wins

**tools:** linear programming techniques.

**Bad news:** one step of $\mathcal{A}$ is exponential (exponent: length of the game)

**Hope:** use specificities of games to reduce time-complexity

### Integrality gap

**Bad news:** fractional cop-number $\leq 1 + \epsilon$ for any graph and any $\epsilon > 0$

**Hope:** surveillance game: fractional game gives a probabilistic $\log n$-approximation
States of the Game

In $n$-node graph

- $c \in \mathbb{R}_+^n$ represents the tokens of Player C.
- $r \in \mathbb{R}_+^n$ represents the tokens of Player R.
- $(c, r) \in \mathbb{R}_{2n}^+$ represents the state of the game.

$c = (0.7, 0.2, 0.1)$ and $r = (0, 0.5, 0.5)$;

set of states = polytope

Examples:

- cops and robber: $\sum_{i \leq n} r_i = 1$ and $\sum_{i \leq n} c_i = k$ (# of cops)
- surveillance game: $\sum_{i \leq n} r_i = 1$
Winning states and moves

**Winning states** = convex subset of states

cops and robber: \( \{ (c, r) \mid c_i \geq r_i, i = 1 \cdots n \} \)
surveillance game: \( \{ (c, r) \mid c_i \geq 1, i = 1 \cdots n \} \)

**Moves**

**Slide tokens** along edges = multiplication by stochastic matrix in

\[
\begin{cases}
[\alpha_{i,j}]_{1\leq i,j\leq n} & \forall 1 \leq i, j \leq n, \alpha_{i,j} \geq 0, \text{ and } \\
\forall j \leq n, \sum_{1\leq i\leq n} \alpha_{i,j} = 1, \text{ and } \\
\text{if } \{i, j\} \notin E(G) \text{ then } \alpha_{i,j} = 0
\end{cases}
\]

if \( ij \in E \), an amount \( \alpha_{i,j} \) of the token in \( v_j \) goes to \( v_i \)

**Mark nodes** = add to \( c \) a vector in

\[\{(m_1, \cdots, m_n) \mid \sum_{i \leq n} m_i \leq k\}\]
$\mathcal{R}_{i-1}$: states from which Player C always wins in at most $i - 1$ rounds, when Player R is the first to play.

$C_i = \{ (c, r) \mid \exists \text{move}, (\text{move}(c), r) \in \mathcal{R}_{i-1} \}$
Main Idea of Algorithm 1/2

$\mathcal{R}_{i-1}$: states from which Player C always wins in at most $i - 1$ rounds, when Player R is the first to play.

$\mathcal{C}_i = \{(c, r) \mid \exists \text{move}, (\text{move}(c), r) \in \mathcal{R}_{i-1}\}$

$\mathcal{C}_i$: states from which Player C always wins in at most $i$ rounds when playing first

$\mathcal{C}_i$ is a polytope, computable from $\mathcal{R}_{i-1}$, polynomial-size but in higher dimension

**Problem:** projection
$C_i$: states from which Player C always wins in at most $i$ rounds when playing first.

$R_i = \{(c, r) \mid \forall \text{move}, (c, \text{move}(r)) \in C_i\}$
Main Idea of Algorithm 2/2

$C_i$: states from which Player C always wins in at most $i$ rounds when playing first

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$R_i$: states from which Player C always wins in at most $i$ rounds, when Player R is the first to play.

$R_i = \text{polytope, computable from } C_i$, polynomial-size.

**Trick:** “just” have to reenforce each constraint
Bad news and Good news

Fractional cop-number is one :(

**Strategy:** place $f_1 = 1/n$ cop per node
At step $i$,

1. $h_i = \sum_{j \leq i} f_j$ cop “follows” the robber. 
   $1 - h_i$ cop remains
2. place $f_{i+1} = \frac{1 - h_i}{n}$ cop per node.

$h_i \rightarrow i \rightarrow \infty \ 1$

Meyniel conjecture seems safe...

approximation for surveillance number? :)

Surveillance game: inequality defining the polytopes are similar to set cover
Proof based on approximation of set cover
for the moment: only probabilistic strategy
Conclusion and Future Work

Promising framework (we hope)

Lot of work remains:
- polynomial algorithm? (it is if length of game is bounded)
- approximation?
- fast robber?
- careful analysis of polytopes in each game
- etc.
Thank you