# Fractional Combinatorial Games

F. Giroire[1] N. Nisse[1] S. Pérennes[1] R. P. Soares[1,2]

[1] COATI, Inria, I3S, CNRS, UNS, Sophia Antipolis, France
[2] ParGO Research Group, UFC, Fortaleza, Brazil

EURO 2013, stream Graph Searching

Roma, July 4th, 2013

# Cops & robber games [Nowakowski and Winkler; Quilliot, 83]

**Initialization:**

1. $\mathcal{C}$ places the cops;
2. $\mathcal{R}$ places the robber.

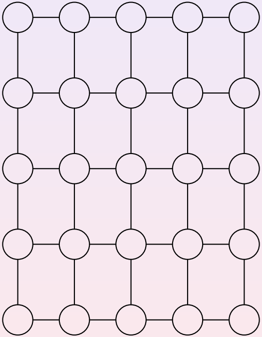
**Step-by-step:**

- each cop traverses at most 1 edge;
- the robber traverses at most 1 edge.

**Robber captured:**
A cop at same node as robber.

**Goal:**

Cop-number=minimum number of cops



**Initialization:**

1. $\mathcal{C}$ places the cops;
2. $\mathcal{R}$ places the robber.

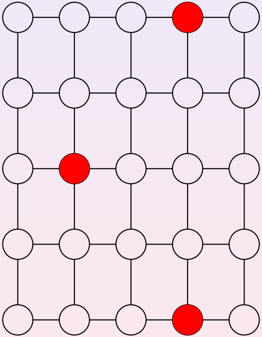
**Step-by-step:**

- each cop traverses at most 1 edge;
- the robber traverses at most 1 edge.

**Robber captured:**
A cop at same node as robber.

**Goal:**

Cop-number=minimum number of cops



# Cops & robber games [Nowakowski and Winkler; Quilliot, 83]

**Initialization:**

1. $\mathcal{C}$ places the cops;
2. $\mathcal{R}$ places the robber.

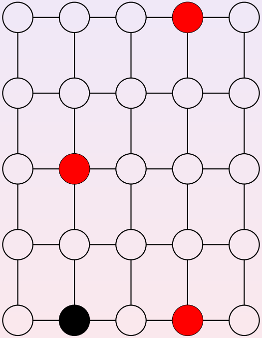
**Step-by-step:**

- each cop traverses at most 1 edge;
- the robber traverses at most 1 edge.

**Robber captured:**
A cop at same node as robber.

**Goal:**

Cop-number=minimum number of cops



# Cops & robber games [Nowakowski and Winkler; Quilliot, 83]

**Initialization:**

1. $\mathcal{C}$ places the cops;
2. $\mathcal{R}$ places the robber.

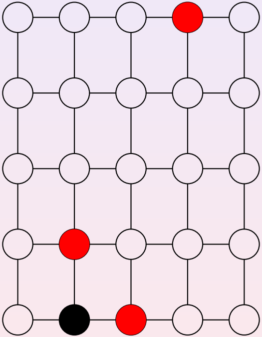
**Step-by-step:**

- each cop traverses at most 1 edge;
- the robber traverses at most 1 edge.

**Robber captured:**
A cop at same node as robber.

**Goal:**

Cop-number=minimum number of cops



# Cops & robber games [Nowakowski and Winkler; Quilliot, 83]

**Initialization:**

1. $\mathcal{C}$ places the cops;
2. $\mathcal{R}$ places the robber.

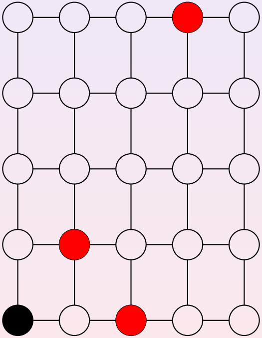
**Step-by-step:**

- each cop traverses at most 1 edge;
- the robber traverses at most 1 edge.

**Robber captured:**
A cop at same node as robber.

**Goal:**

Cop-number=minimum number of cops



# Cops & robber games [Nowakowski and Winkler; Quilliot, 83]

**Initialization:**

1. $\mathcal{C}$ places the cops;
2. $\mathcal{R}$ places the robber.

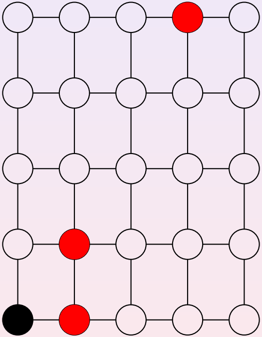
**Step-by-step:**

- each cop traverses at most 1 edge;
- the robber traverses at most 1 edge.

**Robber captured:**
A cop at same node as robber.

**Goal:**

Cop-number=minimum number of cops



# Cops & robber games [Nowakowski and Winkler; Quilliot, 83]

**Initialization:**

1. $\mathcal{C}$ places the cops;
2. $\mathcal{R}$ places the robber.

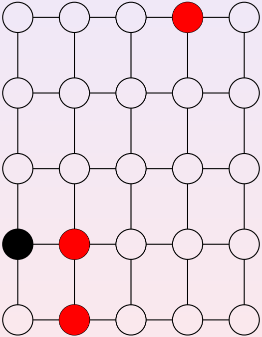
**Step-by-step:**

- each cop traverses at most 1 edge;
- the robber traverses at most 1 edge.

**Robber captured:**
A cop at same node as robber.

**Goal:**

Cop-number=minimum number of cops



**Initialization:**

1. $\mathcal{C}$ places the cops;
2. $\mathcal{R}$ places the robber.

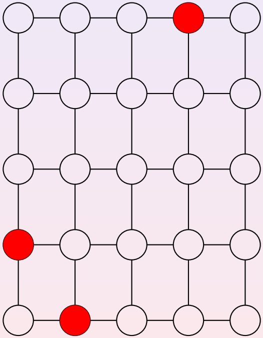
**Step-by-step:**

- each cop traverses at most 1 edge;
- the robber traverses at most 1 edge.

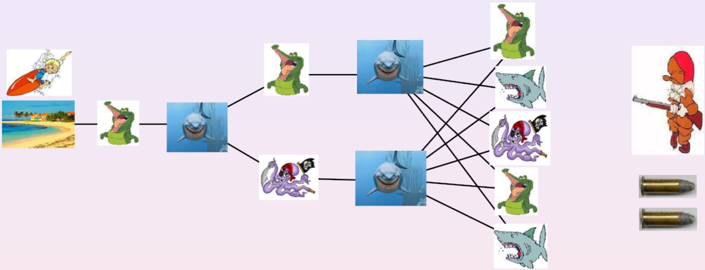
**Robber captured:**
A cop at same node as robber.

**Goal:**

Cop-number=minimum number of cops

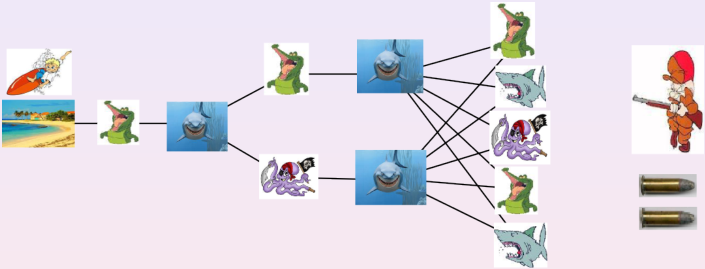


# Surveillance game [Fomin,Giroire,Mazauric,Jean-Marie,Nisse 12]



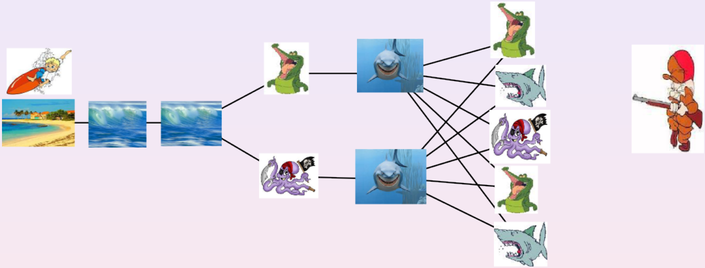
An Observer must ensure that a Surfer never reaches a dangerous node

# Surveillance game [Fomin,Giroire,Mazauric,Jean-Marie,Nisse 12]



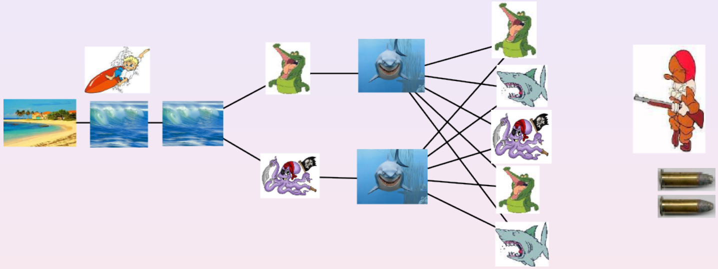
An Observer must ensure that a Surfer never reaches a dangerous node

# Surveillance game [Fomin,Giroire,Mazauric,Jean-Marie,Nisse 12]



Turn by turn: Observer marks $k = 2$ nodes

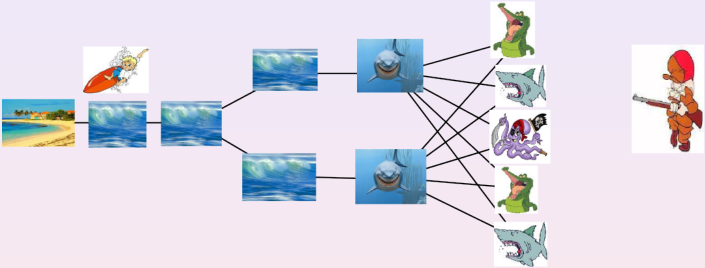
# Surveillance game [Fomin,Giroire,Mazauric,Jean-Marie,Nisse 12]



Turn by turn: Observer marks $k = 2$ nodes

**then** Surfer may move on a adjacent node

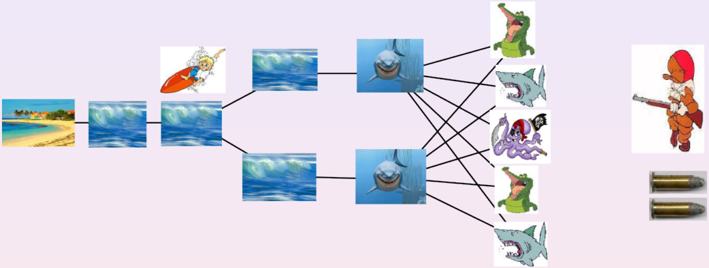
# Surveillance game [Fomin,Giroire,Mazauric,Jean-Marie,Nisse 12]



Turn by turn: Observer marks $k = 2$ nodes

**then** Surfer may move on a adjacent node

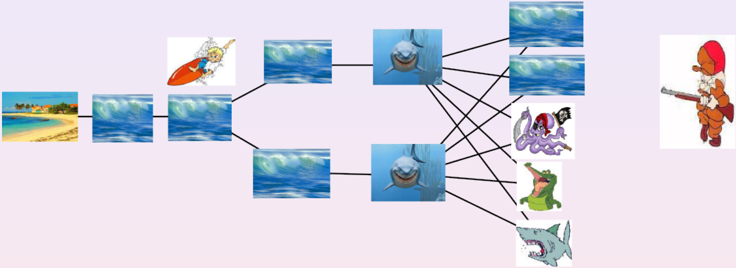
# Surveillance game [Fomin,Giroire,Mazauric,Jean-Marie,Nisse 12]



Turn by turn: Observer marks $k = 2$ nodes

**then** Surfer may move on a adjacent node

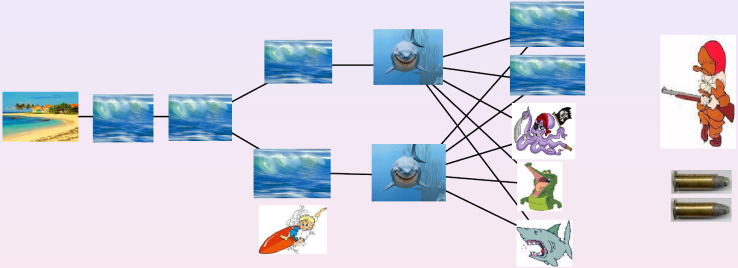
# Surveillance game [Fomin,Giroire,Mazauric,Jean-Marie,Nisse 12]



Turn by turn: Observer marks $k = 2$ nodes

**then** Surfer may move on a adjacent node

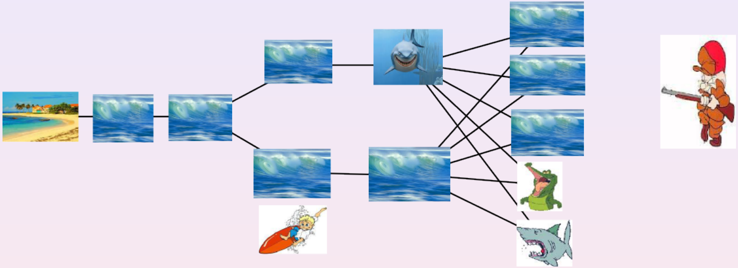
# Surveillance game [Fomin,Giroire,Mazauric,Jean-Marie,Nisse 12]



Turn by turn: Observer marks $k = 2$ nodes

**then** Surfer may move on a adjacent node

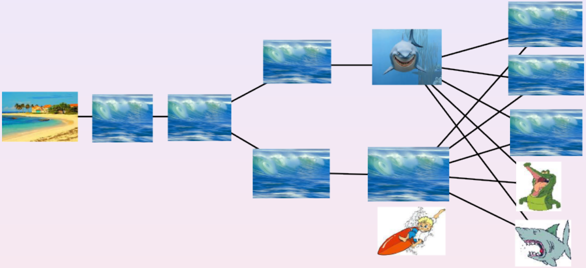
# Surveillance game [Fomin,Giroire,Mazauric,Jean-Marie,Nisse 12]



Turn by turn: Observer marks $k = 2$ nodes

**then** Surfer may move on a adjacent node

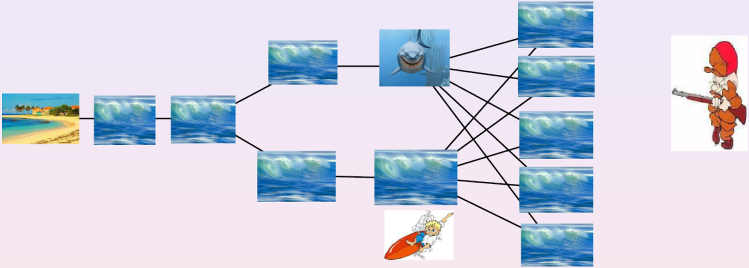
# Surveillance game [Fomin,Giroire,Mazauric,Jean-Marie,Nisse 12]



Turn by turn: Observer marks $k = 2$ nodes

**then** Surfer may move on a adjacent node

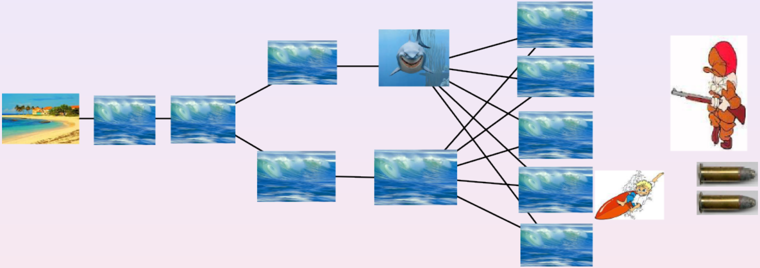
# Surveillance game [Fomin,Giroire,Mazauric,Jean-Marie,Nisse 12]



Turn by turn: Observer marks $k = 2$ nodes

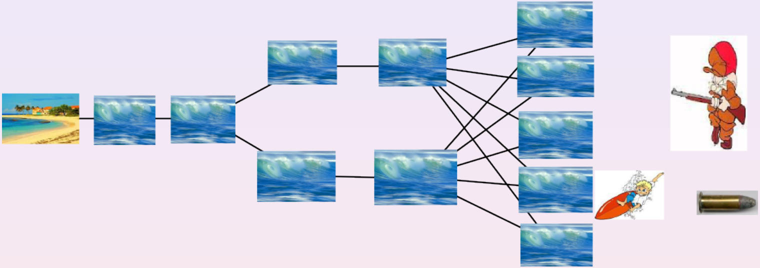
**then** Surfer may move on a adjacent node

# Surveillance game [Fomin,Giroire,Mazauric,Jean-Marie,Nisse 12]



Turn by turn: Observer marks $k = 2$ nodes

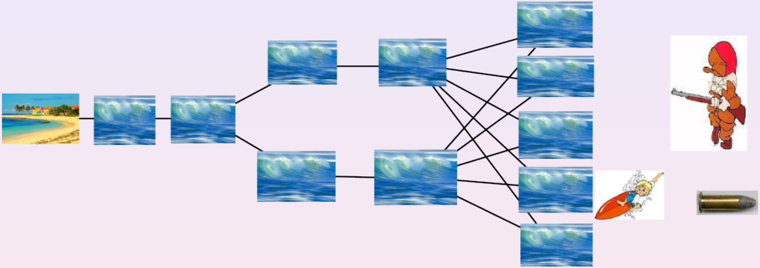
**then** Surfer may move on a adjacent node



Turn by turn: Observer marks $k = 2$ nodes

**then** Surfer may move on a adjacent node

# Surveillance game [Fomin,Giroire,Mazauric,Jean-Marie,Nisse 12]

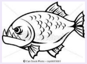


In this example, all nodes are marked

Victory of the Observer using 2 marks per turn

# Model: another Two players game

- a *Surfer*  starts from safe homebase $v_0$ 

  in $G$, a dangerous graph     

- a *Guard*  with some amount $k$ of bullets 

**Turn by turn:**

1. the guard secures $\leq k$ nodes;
2. then, the Surfer may move to an adjacent node.

**Defeat:** Surfer in unsafe node  **Victory:** $G$ safe 

Minimize amount of bullets to win for any Surfer's trajectory

Surveillance number of $G$ (**connected**) from $v_0$: $sn(G, v_0)$

# Two Players Combinatorial Games

- Two players play a game on a graph.
- Game is played turn-by-turn.
- Players play by moving and/or adding tokens on vertices of the graph.
- Optimization problem:

  minimizing number of tokens to achieve some goal

# All these games are hard

- Cops and Robber: $k$ cops are enough?
  - PSPACE-complete in general graphs [Mamino, 2012].
- Surveillance Game: $k$ marks per turn are enough?
  - $k = 2$ NP-complete for Chordal/Bipartite Graphs [Fomin et al, 2012].
  - $k = 4$ PSPACE-complete for DAGS [Fomin et al, 2012].
- Angels and Devils: Does an Angel of power $k$ wins?
  - 1-Angel loses in (infinite) grids [Conway, 1982].
  - 2-Angel wins in (infinite) grids [Máthé, 2007].
- Eternal Dominating set.
- Eternal Vertex Cover.
  - NP-hard [Fomin et al, 2010].

# New tools/approaches are required

**Several questions remain open**

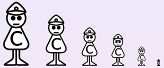
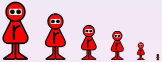
- Meyniel conjecture: $cn(G) = O(\sqrt{n})$ in any $n$-node graphs?
- Polynomial-time Approximation algorithms?

less difficult but still intriguing

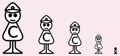
- number of cops to capture fast robber in grids?
- cost of connectivity in surveillance game?
- etc.

Here, we present preliminary results of our new approach

- Fractional games:

  both players can use “fractions” of tokens. 
  

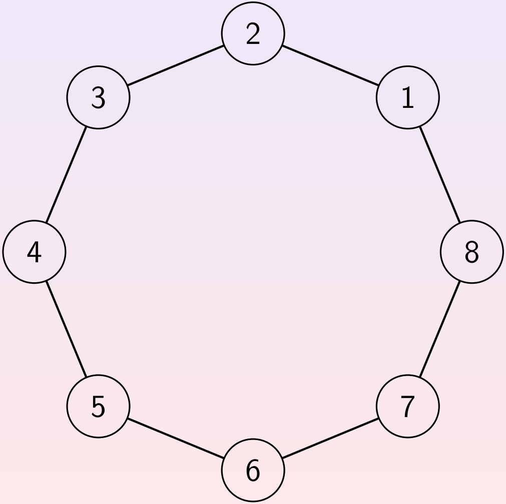
- Semi-Fractional games:

  only one player (Player C) can use fractions of tokens. 
  

- Integral games: classical games, token are unsplittable

# Example: Fractional Cops and Robber

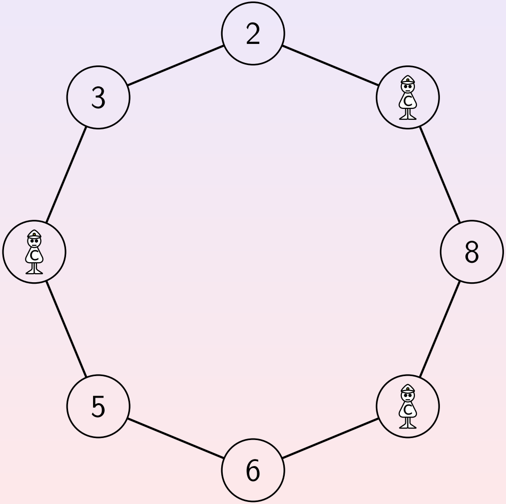
integral game:
cop-number $= 2$



# Example: Fractional Cops and Robber

integral game:
cop-number $= 2$

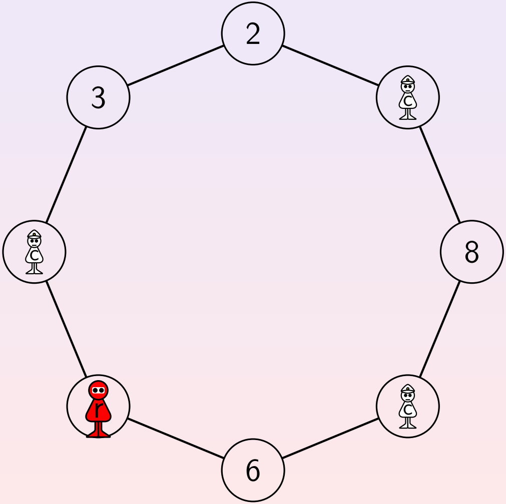
semi fractional:
cop-number $\leq 3/2$



# Example: Fractional Cops and Robber

integral game:
cop-number $= 2$

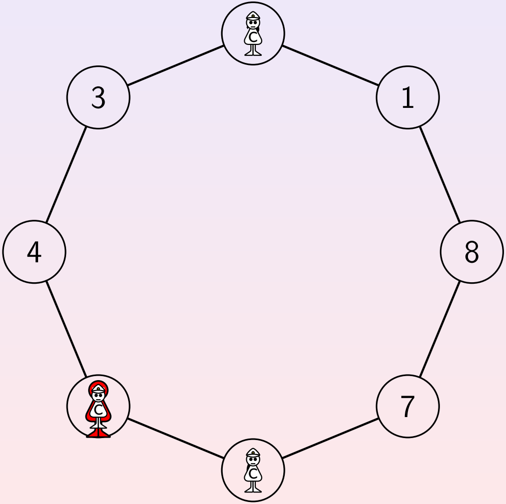
semi fractional:
cop-number $\leq 3/2$



# Example: Fractional Cops and Robber

integral game:
cop-number $= 2$

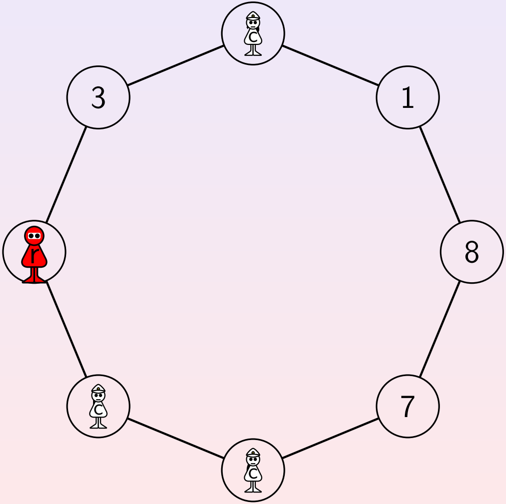
semi fractional:
cop-number $\leq 3/2$



# Example: Fractional Cops and Robber

integral game:
cop-number $= 2$

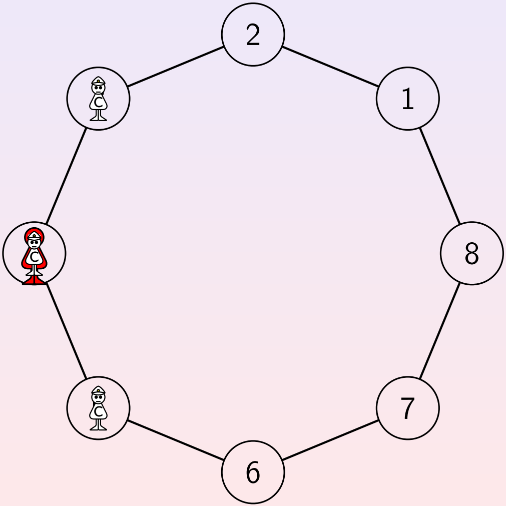
semi fractional:
cop-number $\leq 3/2$



# Example: Fractional Cops and Robber

integral game:
cop-number $= 2$

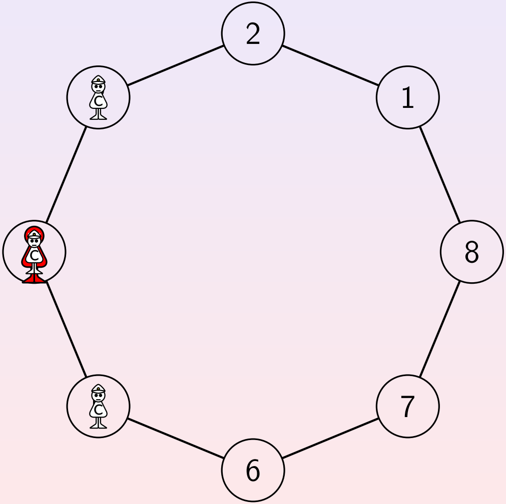
semi fractional:
cop-number $\leq 3/2$



# Example: Fractional Cops and Robber

integral game:
cop-number $= 2$

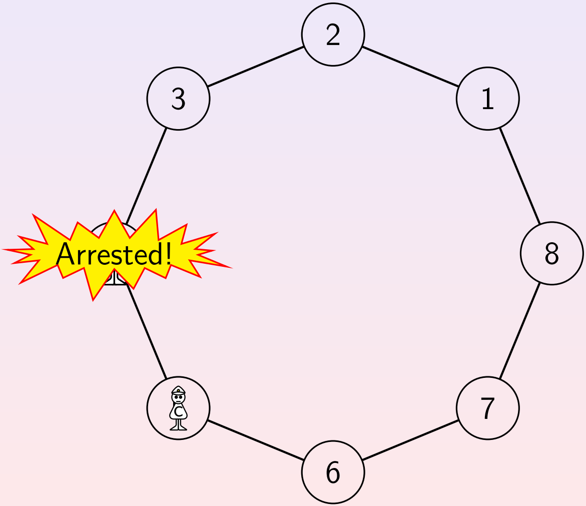
semi fractional:
cop-number $\leq 3/2$



# Example: Fractional Cops and Robber

integral game:
cop-number $= 2$

semi fractional:
cop-number $\leq 3/2$



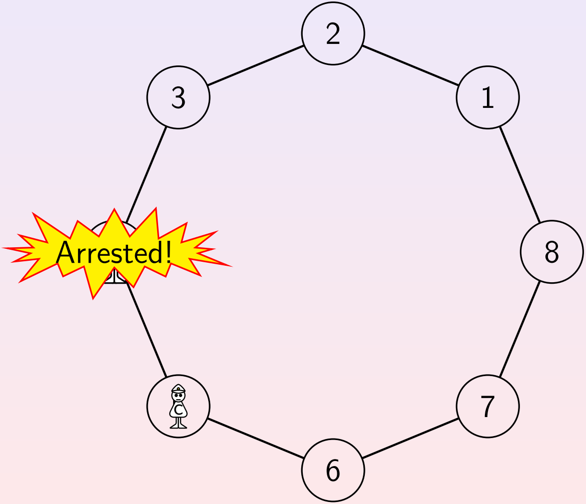
# Example: Fractional Cops and Robber

integral game:
cop-number $= 2$

semi fractional:
cop-number $\leq 3/2$

**Remark:**
by definition:
semi-fractional $\leq$ integral

gap?
relationship with fractional?



# Preliminary results

**Fractional games**

*general framework: fractional relaxation of turn-by-turn games*
**important property:** *"convexity" of winning states*

**Semi-fractional = fractional** (properties of robber's moves)

*solutions of fractional games provides lower bounds for integral games*

**Algorithm $\mathcal{A}$ to decide which player wins**

*tools: linear programming techniques.*

**Bad news:** *one step of $\mathcal{A}$ is exponential (exponent: length of the game)*
**Hope:** *use specifities of games to reduce time-complexity*

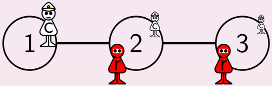
**Integrality gap**

**Bad news:** *fractional cop-number $\leq 1+\epsilon$ for any graph and any $\epsilon > 0$*
**Hope:** *surveillance game: fractional game gives a probabilistic* $\log n$*-approximation*

# States of the Game

In $n$-node graph

- $c \in \mathbb{R}_+^n$ represents the tokens of Player C.
- $r \in \mathbb{R}_+^n$ represents the tokens of Player R.
- $(c, r) \in \mathbb{R}_+^{2n}$ represents the state of the game.




$c = (0.7, 0.2, 0.1)$ and $r = (0, 0.5, 0.5)$;

**set of states = polytope** Examples:

cops and robber: $\sum_{i \leq n} r_i = 1$ and $\sum_{i \leq n} c_i = k$ (# of cops)
surveillance game: $\sum_{i \leq n} r_i = 1$

# Winning states and moves

**winning states = convex subset of states**

cops and robber: $\{(c, r) \mid c_i \geq r_i, i = 1 \cdots, n\}$
surveillance game: $\{(c, r) \mid c_i \geq 1, i = 1 \cdots, n\}$

**moves**

**slide tokens** along edges = multiplication by stochastic matrix in

$$\left\{ [\alpha_{i,j}]_{1 \leq i,j \leq n} \;\middle|\; \begin{array}{r} \forall 1 \leq i, j \leq n, \alpha_{i,j} \geq 0, \text{ and} \\ \forall j \leq n, \sum_{1 \leq i \leq n} \alpha_{i,j} = 1, \text{ and} \\ \text{if } \{i, j\} \notin E(G) \text{ then } \alpha_{i,j} = 0 \end{array} \right\}$$

if $ij \in E$, an amount $\alpha_{i,j}$ of the token in $v_j$ goes to $v_i$

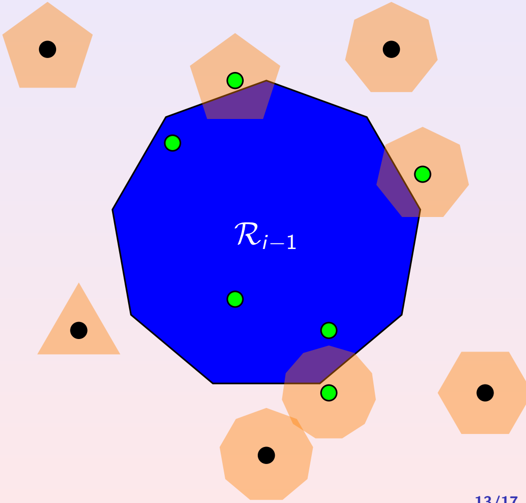
**mark nodes** = add to $c$ a vector in

$$\{(m_1, \cdots, m_n) \mid \sum_{i \leq n} m_i \leq k\}$$

# Main Idea of Algorithm 1/2

$\mathcal{R}_{i-1}$: states from which Player C always wins in at most $i-1$ rounds, when Player R is the first to play.

$\mathcal{C}_i = \{(c, r) \mid \exists move, (move(c), r) \in \mathcal{R}_{i-1}\}$



# Main Idea of Algorithm 1/2

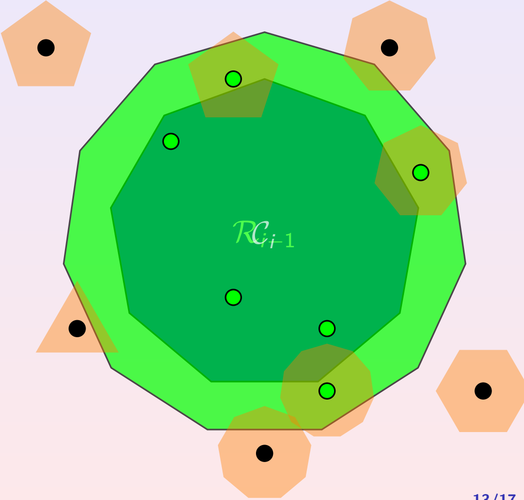
$\mathcal{R}_{i-1}$: states from which Player C always wins in at most $i-1$ rounds, when Player R is the first to play.

$\mathcal{C}_i = \{(c, r) \mid \exists move, (move(c), r) \in \mathcal{R}_{i-1}\}$

$\mathcal{C}_i$: states from which Player C always wins in at most $i$ rounds when playing first

$\mathcal{C}_i$ = polytope, computable from $\mathcal{R}_{i-1}$, polynomial-size but in higher dimension

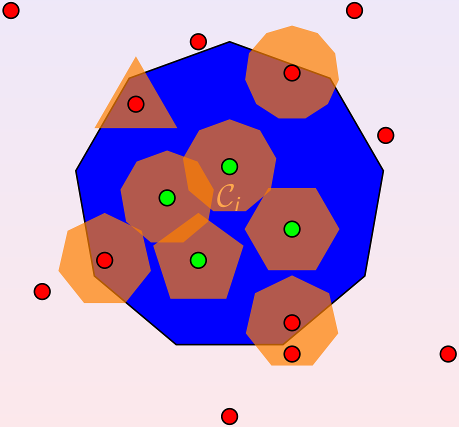
**Problem:** projection



# Main Idea of Algorithm 2/2

$\mathcal{C}_i$: states from which Player C always wins in at most $i$ rounds when playing first

$\mathcal{R}_i = \{(c, r) \mid \forall move, (c, move(r)) \in \mathcal{C}_i\}$



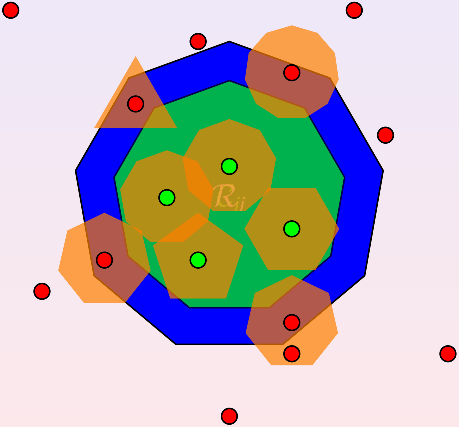
# Main Idea of Algorithm 2/2

$\mathcal{C}_i$: states from which Player C always wins in at most $i$ rounds when playing first

$\mathcal{R}_i = \{(c, r) \mid \forall move, (c, move(r)) \in \mathcal{C}_i\}$

$\mathcal{R}_i$: states from which Player C always wins in at most $i$ rounds, when Player R is the first to play.

$\mathcal{R}_i$ = polytope, computable from $\mathcal{C}_i$, polynomial-size.
**Trick:** “just" have to reenforce each constraint



# Bad news and Good news

**Fractional cop-number is one** :(

**Strategy:** place $f_1 = 1/n$ cop per node
At step $i$,

1. $h_i = \sum_{j \leq i} f_j$ cop "follows" the robber.
   $1 - h_i$ cop remains
2. place $f_{i+1} = \frac{1-h_i}{n}$ cop per node. $h_i \to_{i \to \infty} 1$

Meyniel conjecture seems safe...

**approximation for surveillance number?** :)

Surveillance game: inequality defining the polytopes are similar to set cover
Proof based on approximation of set cover
for the moment: only probabilistic strategy

# Conclusion and Future Work

Promising framework (we hope)

Lot of work remains:

- polynomial algorithm? (it is if length of game is bounded)
- approximation?
- fast robber?
- careful analysis of polytopes in each game
- etc.

Thank you