Allowing each node to communicate only once in a distributed system

F. Becker$^1$  A. Kosowski$^2$  M. Matamala$^3$  N. Nisse$^4$
I. Rapaport$^3$  K. Suchan$^5$  I. Todonca$^1$

$^1$ LIFO, Univ. Orléans, France  
$^2$ Inria, LIAFA, Paris, France  
$^3$ DIM, Universidad de Chile, Santiago, Chile  
$^4$ COATI, Inria, I3S, CNRS, UNS, Sophia Antipolis, France  
$^5$ Universidad Adolfo Ibáñez, Santiago, Chile

ANR DISPLEXITY
Bordeaux, Sept. 3rd 2015

Adding a Referee to an Interconnection Network: What Can(not) Be Computed in One Round. IPDPS 2011
Allowing each node to communicate only once in a distributed system: shared whiteboard models SPAA 2012
Models of distributed computing

**Goal:** Monitoring properties in large-scale distributed networks

### Property testing

A central entity:

1. queries some nodes (typically $o(n)$)
2. extracts some local information from each query (typically $O(\log n)$ bits)
   
   ex: “What is your $i^{th}$ neighbor?”
3. decides some (global) property about the graph or about node labeling
   
   (Is it planar? connected? what is its diameter? Is the coloring proper?...)

   this talk is not about Property testing...

### Distributed decision

Each node

1. gathers some (local) information by exchanging messages with its neighborhood
2. decides some (global) property about the graph or about node labeling

   Node’s Output $\in \{0, 1\}$

   If YES instance: all nodes must answer 1
   
   otherwise, at least one node must answer 0

   this talk is not about Distributed decision either... but ...

Becker et al.  What Can(not) Be Computed in One Round?
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*Becker et al.*

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Becker *et al.* What Can(not) Be Computed in One Round?
Let $G = (V, E)$ be an $n$-node graph, with a labeling on nodes. This defines a message-passing system where:

- Nodes have distinct IDs.
- There are $t$ rounds (possibly a function of $n$).
- The system is synchronous.
- There are no faults.
- At each round, nodes exchange messages with their neighbors.
- Messages have arbitrary size.
- Nodes have arbitrary computational power.

If $t \geq \text{diameter}(G)$, any (decidable) property can be decided.
$G = (V, E)$ be a $n$-node graph, with a labeling on nodes
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LOCAL model in $O(1)$ rounds

Some trivial examples in 1 round

Deciding if a coloring is proper, Deciding a MIS,...

Proper coloring: $c : V \rightarrow \mathbb{N}$ s.t. any two adjacent vertices have $\neq$ colors

Maximal Independent Set (MIS): $S \subseteq V$ s.t. no adjacent vertices in $S$.

Seminal(?) work Cole,Vishkin 86 + Linial 92

Computing a 3-coloring in rings requires $\Theta(\log^* n)$ rounds

What cannot be computed Kuhn,Moscibroda,Wattenhoffer 04

Minimum Vertex Cover (or Max Matching)
in $k$ rounds: $\Omega(n^\varepsilon/k^2/k)$-approx.
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Minimum Dominating Set (or Max Indep. Set)
in $k$ rounds: $\Omega(\Delta^{1/k}/k)$-approx.
$O(1)$-approx requires $\Omega(\log \Delta / \log \log \Delta)$ rounds
## LOCAL model in $O(1)$ rounds

### Some trivial examples in 1 round

- Deciding if a coloring is proper, Deciding a MIS, ...

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**Minimum Vertex Cover (or Max Matching)**

- in $k$ rounds: $\Omega(n^{c/k^2/k})$-approx.
  
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**Minimum Dominating Set (or Max Indep. Set)**

- in $k$ rounds: $\Omega(\Delta^{1/k} / k)$-approx.
  
  - $O(1)$-approx requires $\Omega(\log \Delta / \log \log \Delta)$ rounds
LOCAL model in $O(1)$ rounds

Locally Checkable Labeling (LCL)  
Naor, Stockmeyer 95

- in graph with bounded degree
- non-decidable in general
- randomization does not help
- does not depend on ID, but on local ordering

Decision Problem  
Fraigniaud, Korman, Peleg 2013

- randomization helps
- non-determinism helps
- randomization + non-determinism = All
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CONGEST model

Let $G = (V, E)$ be a $n$-node graph, with a labeling on nodes. Consider a message-passing system where:

- nodes have distinct IDs
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- no faults
- at each round, nodes exchange messages with their neighbors
- messages have size $O(\log n)$
- nodes have arbitrary computational power

If $t \geq \text{diameter}(G)$, any (decidable) property can be decided.
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CONGEST model: What cannot be computed...

... in diameter number of rounds

$B$ upper bound on the size of messages
D: diameter of graph

Frischknecht, Holzer, Wattenhofer, 2012

Any distributed randomized $\epsilon$-error algorithm requires:

- **Diameter**:
  - $\Omega(n/B)$ rounds to decide if $D \leq 4$ (even for $D \leq 5$)
  - $\Omega(\sqrt{n}/B + D)$ rounds to $c$-approx of diameter, $c < 3/2$

- **Girth**: $\Omega(\sqrt{n}/B + D)$ rounds to $c$-approx of girth, $c < 2$

proofs rely on Two-Party Communication Complexity:

Alice: $a \in \{0, 1\}^k$ and Bob: $b \in \{0, 1\}^k$
need to compute $f: \{0, 1\}^k \times \{0, 1\}^k \to \{0, 1\}$
what number of bits (as a function of $k$) do they need to exchange?

Disjointness problem: any algorithm using public randomness requires $\Omega(k)$

$f(a, b) = 0$ iff $\exists i \leq k$ such that $a[i] = b[i] = 1$

idea: $\forall(v(G) = A \cup B$: A represents a, B represents b, Required info \leq$ #Rounds
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Distributed testing

So far

- **property testing**: central entity gather structured local info from some nodes

- **distributed decision**: all nodes gather information and answer 0 or 1
  - LOCAL: (unbounded size) message passing
  - CONGEST: $O(\log n)$ bits message passing
  - # of rounds: $O(1)$ or function of $n$
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In distributed testing setting, LOCAL model and $O(1)$ rounds:

testing Cycle-Freeness requires messages of size $\Omega(\log d)$ per node with degree $d$

[Arfaoui,Fraigniaud,Ilcinkas,Mathieu, 2014]
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- **our model**: all nodes send some "message" to a central entity
Graph with $n$ nodes.

A node knows: its ID and the IDs of its neighbors.

Nodes have distinct identifiers in $\{1, \cdots, n\}$. 

Our model
Our model

A node has arbitrary computation power.

Goal: encode its local knowledge in a small message (typically $O(\log n)$ bits)
Each node sends its (unique) message to a central entity.

**Remark:** If $|message| = n$ bits, then node gives its whole neighborhood.
The referee has arbitrary computation power and uses the $n$ messages to...
Our model

... answer a question about the graph (typically: "does $G$ has some property?")
Our model

... answer a question about the graph (typically: "does \( G \) has some property?")

Related with *Number-in-hand model* with simultaneous messages

\[ f(x_1, \cdots, x_k) \]

[Babai, Kimmel, Lokam 2004]
To sum up: Model of distributed computing

**Principle**

Does $G$ belong to $\mathcal{P}$?

- each node encodes its local knowledge
  
  \[
  \text{message} : \text{ID of } v \times \text{IDs of } N(v) \to \text{message}(v), \text{ and} \\
  |\text{message}(v)| = O(\log n) \text{ bits}
  \]

- the referee decodes the $n$ messages to answer
  
  \[
  \text{answer} : (\text{message}(v_i))_{i \leq n} \to \{0, 1\}^*
  \]

**Hypothesis**

- arbitrary computational power: \text{message} and \text{answer} are arbitrary functions
- IDs are distinct in \{1, \cdots, n\}

**Remark:** if bounded maximum degree: each node may send its full adjacency list.

**Problem**

in total: $O(n \log n)$ bits of local information

What kind of question can be answered?

Becker et al. What Can(not) Be Computed in One Round?
Example: Does $G$ is a tree? If yes, compute its adjacency matrix.
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\[ \sum_{v} \text{degree}(v) = 2(n - 1)? \]

Problem if not connected
No adjacency matrix

Becker et al.
What Can(not) Be Computed in One Round?
Example: Does $G$ is a tree? If yes, compute its adjacency matrix.

Each node sends its ID, its degree and the sum of the IDs of its neighbors.
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The referee iteratively “prunes” the one-degree nodes in the whiteboard.
In parallel, he re-builds the tree.
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The referee iteratively “prunes” the one-degree nodes in the whiteboard. In parallel, he re-builds the tree.
$G$ degeneracy $k$: $\exists v \in V(G)$ with degree $\leq k$ and $G \setminus v$ degeneracy $\leq k$.

Possible: BUILD graphs with degeneracy $\leq k$, using messages of size $O(k \cdot \log n)$

Decide if a graph has bounded degeneracy (include planar graphs, bounded genus graphs, bounded treewidth graphs...). If yes, build their adjacency matrix.

**proof:** generalization of the "pruning process" of trees.

Each node $v$ sends

- its ID
- its degree
- $\sum_{w \in N(v)} ID(w)$
- $\sum_{w \in N(v)} ID^2(w)$
- ...
- $\sum_{w \in N(v)} ID^k(w)$

The referee can compute neighborhood of nodes with degree $\leq k$

( unique integral solution )
What Can(not) Be Computed in One Round

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| Decide if the graph contains a triangle, a (induced or not) square.  
Decide if the graph has diameter at most 3 | |

**Recently:** characterization of induced subgraphs whose containment can be decided  
[Kari, Matamala, Rapaport, Salo 2015]

**proof:** Kind of reduction.  
If possible $\Rightarrow$ Possible to build adjacency matrix of bipartite graphs.  
$2^{\Omega(n^2)}$ such graphs $\Rightarrow$ impossible to distinguish all of them with $O(n \log n)$ bits.  
$\Rightarrow$ contradiction
Not possible to decide if $G$ contains a triangle using messages of size $O(\log n)$.

Assume $\mathcal{A}$ solves TRIANGLE: use $\mathcal{A}$ to BUILD bipartite graphs.
Assume \( A \) solves TRIANGLE: 

Assume \( A \) solves TRIANGLE: use \( A \) to BUILD bipartite graphs

\[ \{i, j\} \in E(G) \iff G(i, j) \text{ contains a triangle} \]
Not possible to decide if $G$ contains a triangle using messages of size $O(\log n)$.

Each node $j$ sends 2 messages:

- $A_1(j)$ encodes $N_G(j)$ (same as $N_{G(i,k)}(v)$ for any $i, k \in V \setminus \{j\}$)
- $A_2(j)$ encodes $N_{G(j,i)}(j) = N_G(j) \cup \{R\}$ (the same for any $w \in V \setminus \{j\}$)
Not possible to decide if $G$ contains a triangle using messages of size $O(\log n)$.

Combining the $2n$ messages, the referee simulates $A$ in each $G(i,j)$, $i,j \in V$.

$\Rightarrow$ able to decide if $\{i,j\} \in E(G)$ for any $i,j \in V$. 

Becker et al.

What Can(not) Be Computed in One Round?
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Randomized version: private/public coins

Not possible  [Becker, Montealegre, Rapaport, Todinca, 2014]

For any $\epsilon < 1/2$, any public coins randomized protocol for TRIANGLE (resp., Diameter $\leq 3$) with $\epsilon$ two-sided errors requires messages of size $\Omega(n)$ bits.

Proof: Reduction to INDEX function:
Alice: $m$-bits vector $x$, Bob: integer $q \leq m$
INDEX($x$, $q$) = $x_q$, the $q^{th}$ bit of $x$.
Requires $\Omega(m)$ bits in randomized protocol
[Kermer, Nisan, Ron, 1999]

Encoding of $x$ of size $m = n^2$ as a bipartite graph with $n$ vertices.
protocol for TRIANGLE with $g(n)$ bits/message
\[ \Rightarrow \] protocol for INDEX using $O(g(n) \cdot n)$ bits.

Hierarchy of randomized protocol  [Becker, Montealegre, Rapaport, Todinca, 2014]
Public coins $\preceq$ Private coins $\preceq$ Deterministic
Randomized version: private/public coins

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Hierarchy of randomized protocol [Becker, Montealegre, Rapaport, Todinca, 2014]
Public coins $\prec$ Private coins $\prec$ Deterministic
Open Problem: Connectivity

### Possible

CONNECTIVITY can be decided by a randomized protocol using public coins and messages of size $O(\log^2 n)$ bits

[Ahm, Guha, McGregor, 2012]

### Randomized, using small messages

What about a deterministic protocol using messages of $o(n)$ bits??

Even in simpler case:

- $G$, $n - 1$-regular with $2n$ nodes
- $G$ connected or 2 disjoint cliques.
Generalization

Until now:

All nodes write **simultaneously** on the whiteboard
Don't take advantage of what is written by other nodes.

Now:

Nodes can also read the whiteboard.
Can use previous messages to compute their own message

![Diagram showing nodes and whiteboard interactions](Image)
4 Models

**SimAsync**

All nodes write *simultaneously* on the whiteboard

**SimSync**

Nodes write *sequentially*.

*Worst ordering:* order chosen by an adversary

**Async**

Nodes *rise hand to speak*.

If several nodes rise hand, all write *simultaneously*.

**Sync**

Nodes *rise hand to speak*.

If several nodes rise hand, they write *sequentially* in worst ordering.
**Hierarchy of models**

\[ \text{SimAsync}(\log n) \prec \text{SimSync}(\log n) \]

Maximal Independent Set (MIS) containing \( v_1 \).

- \( MIS \notin \text{SimAsync}(\log n) \) (Simultaneous)
- \( MIS \in \text{SimSync}(\log n) \) (Sequential, adversarial ordering)

trivial algo.
Hierarchy of models

**SimSync**(log $n$) $\preceq$ **Async**(log $n$)

BFS-tree in Bipartite graphs

$\notin$ **SimSync**(log $n$)

$G = (V, E)$ Bipartite graph $V = \{v_2, \cdots, v_n\}$ (blue and green nodes)

Matching between $V$ and $W = \{v_{n+1}, \cdots, v_{2n-2}\}$ (red nodes)

$G_i$: Blue edge from 1 to $n + i \in W$ (ex: $n = 6$ and $i = 5$: $G_5$)
Hierarchy of models

\[ \text{SimSync}(\log n) \prec \text{Async}(\log n) \]

BFS-tree in Bipartite graphs \( \not\in \text{SimSync}(\log n) \)

- \( N_G(v_i) = \) "level 3" of BFS-tree in \( G_i \)
- SimSync: Adversarial ordering: first nodes of \( V \)
- Solving BFS \( \Rightarrow \) BUILD Bipartite, impossible if \( O(\log n) \) bits/message
Hierarchy of models

**SimSync**($\log n$) $\prec$ **Async**($\log n$)

BFS-tree in Bipartite graphs $\in$ **Async**($\log n$)

Initially, only the root (v1) speaks

Central entity

Degree (root)
Hierarchy of models

$\textbf{SimSync}(\log n) \prec \textbf{Async}(\log n)$

BFS-tree in Bipartite graphs $\in \textbf{Async}(\log n)$

Key point: bipartiteness allows that nodes know when all nodes of previous level have sent their message.
Hierarchy of models

**SimSync**(log n) ⊊ **Async**(log n)

BFS-tree in Bipartite graphs ∈ **Async**(log n)

Then, all nodes down to **Level 1** speak

- 5: up=1, down=2
- 8: up=1, down=1
- 3: up=1, down=1
- 4: up=1, down=1
- 7: up=1, down=2

**Level 2**
- 9: up=4, down=1
- 8: up=3, down=1

Key point: bipartiteness allows that nodes know when all nodes of previous level have sent their message.
Hierarchy of models

\[ \text{SimSync} (\log n) \prec \text{Async} (\log n) \]

BFS-tree in Bipartite graphs \( \in \text{Async} (\log n) \)

Key point: bipartiteness allows that nodes know when all nodes of previous level have sent their message.

- **Level 1**
  - 5: up=1, down=2
  - 8: up=1, down=1
  - 3: up=1, down=1
  - 4: up=1, down=1
  - 7: up=1, down=2

- **Level 2**
  - 9: up=4, down=1
  - 8: up=3, down=1

- **Level 3**
  - 6: up=2, down=0

For any node \( v \) in the graph:

- \( \text{up}(v) = \text{degree} \) "toward" the root
- \( \text{down}(v) = \text{degree}(v) - \text{up}(v) \)
Hierarchy of models

$\text{SimSync}(\log n) \prec \text{Async}(\log n)$

BFS-tree in Bipartite graphs

Then, all nodes down to Level 2 speak

Level 1
- 5: up=1, down=2
- 8: up=1, down=1
- 3: up=1, down=1
- 4: up=1, down=1
- 7: up=1, down=2

Level 2
- 9: up=4, down=1
- 8: up=3, down=1

Level 3
- 6: up=2, down=0

up(v) = degree "toward" the root

down(v) = degree(v) - up(v)

Similar protocol works in model Sync in general graphs.
SimAsync($\log n$) $\prec$ SimSync($\log n$) $\prec$ Async($\log n$) $\preceq$ Sync($\log n$)

<table>
<thead>
<tr>
<th>message: $O(\log n)$ bits</th>
<th>SimAsync</th>
<th>SimSync</th>
<th>Async</th>
<th>Sync</th>
</tr>
</thead>
<tbody>
<tr>
<td>BUILD k-degenerate</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>rooted MIS</td>
<td>no</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>SQUARE</td>
<td>no</td>
<td>no</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>Bipartite-BFS</td>
<td>no</td>
<td>no</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>BFS</td>
<td>?</td>
<td>?</td>
<td>?</td>
<td>yes</td>
</tr>
<tr>
<td>Connectivity</td>
<td>?</td>
<td>?</td>
<td>?</td>
<td>yes</td>
</tr>
</tbody>
</table>

Orthogonal criteria

Let $f(n) = o(n)$ and $g(n) = o(f(n))$.

There exist problems solvable in SimAsync($f(n)$) and not in Sync($g(n)$).
Further works

Probabilistic algorithms?

What if graph partially known?  (Distributed testing)

Connectivity?

What is a realistic (useful) model?

...