Eternal Domination in Grids

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Abstract

In the eternal domination game played on graphs, an attacker attacks a vertex at each turn and a team of guards must move a guard to the attacked vertex to defend it. The guards may only move to adjacent vertices on their turn. The goal is to determine the eternal domination number of a graph which is the minimum number of guards required to defend against an infinite sequence of attacks.

This paper continues the study of the eternal domination game on strong grids $P_n \boxtimes P_m$. Cartesian grids $P_n \square P_m$ have been vastly studied with tight bounds existing for small grids such as $k \times n$ grids for $k \in \{2,3,4,5\}$. It was recently proven that the eternal domination number of these grids, for any n and m, is within an additive factor of $O(n+m)$ of their domination number which lower bounds the eternal domination number. We prove that, for all $n,m \in \mathbb{N}^*$ such that $m \geq n$, $\lceil \frac{nm}{9} \rceil + \Omega(n+m) = \gamma_{all}^{\infty}(P_n \boxtimes P_m) = \lceil \frac{nm}{9} \rceil + O(m\sqrt{n})$ (note that $\lceil \frac{nm}{9} \rceil$ is the domination number of $P_n \boxtimes P_m$).

Keywords: Eternal Domination, Combinatorial Games, Graphs, Graph Protection

1 Introduction

1.1 Background

The eternal domination game and its variants are graph protection models. The origins of the game date back to the 1990's where the military strategy of Emperor Constantine for defending the Roman Empire was studied in a mathematical setting [1, 24, 22, 23]. Roughly, a limited number of armies must be placed in such a way that an army can always move to defend against an attack by invaders.

Precisely, eternal domination is a 2-player game on graphs introduced in [6] and defined as follows. Initially, k guards are placed on some vertices of a graph $G = (V, E)$. Turn-by-turn, an *attacker* first chooses a non-occupied vertex $v \in V$ to attack. Then, if no guard is occupying a vertex adjacent to v, then the attacker wins. Otherwise, one guard must move along an edge to occupy v, and the next turn starts. If the attacker never wins whatever be its sequence of attacks, then the guards win. The aim in eternal domination is to minimize the number of guards that must be used in order to win. Hence, let $\gamma^{\infty}(G)$ be the minimum integer k such that there exists a strategy allowing k guards to win, regardless of what the attacker does [6].

In this paper, we consider the "all guards move" variant of eternal domination, proposed in [14], where, at their turn, every guard may move to a neighbour of its position (still satisfying that, at the end of the turn, the attacked vertex is occupied by a guard). Let $\gamma_{all}^{\infty}(G)$ be the minimum number of guards for which a winning strategy exists in this setting. By definition,

$\gamma(G) \leq \gamma_{all}^{\infty}(G) \leq \gamma^{\infty}(G)$ for any graph G where $\gamma(G)$ denotes the minimum size of a dominating set in G[1].

Variants of the eternal domination game also differ in the fact that one or more guards may simultaneously occupy a same vertex. In the initial variant where a single guard is allowed to move at every step, this is not a strong constraint [6]. That is, imposing that a vertex cannot be occupied by more than one guard does not increase the number of guards required to win. In the case when multiple guards may move each turn, there are some graphs where this constraint increases the number of guards. Let $\gamma_{all}^{*\infty}(G)$ be the minimum number of guards to win in G, moving several guards per step, and in such a way that a vertex cannot be occupied by several guards. See [20], for the construction of a graph G with $\gamma_{all}^{\infty}(G) < \gamma_{all}^{*\infty}(G)$.

Previous works mainly studied lower and upper bounds on $\gamma^{\infty}(G)$ and $\gamma_{all}^{\infty}(G)$ in function of other parameters of G, such as its domination number $\gamma(G)$ [14], independence number $\alpha(G)$[2] [6, 14], and clique cover number $\theta(G)$[3] [6]. Notably, these results give the following inequalities $\gamma(G) \leq \gamma_{all}^{\infty}(G) \leq \alpha(G) \leq \gamma^{\infty}(G) \leq \theta(G)$ [6]. Particular graph classes have also been studied such as paths and cycles [14], trees [18], and proper interval graphs [5]. In particular, the class of grids and graph products has been widely studied [4, 11, 13, 15, 20, 21, 25].

In this paper, we focus on the class of *strong grids SG* and provide an almost tight asymptotical value for $\gamma_{all}^{\infty}(SG)$. Our result also holds for $\gamma_{all}^{*\infty}(SG)$. Our main result is a new technique (in this area) to prove upper bounds that we believe can be generalized to many other "grid-like" graphs.

1.2 Related Work

The "all guards move" variant of the eternal domination game was shown to be NP-complete in Hamiltonian split graphs in [3]. Note that it is not known whether the problem of deciding γ_{all}^{∞} is in NP in general graphs. Moreover, given a graph G and an integer k as inputs, the problem of deciding if $\gamma^{\infty}(G) \leq k$ is coNP-hard [2].

Several graph classes have been studied. For a path P_n on n vertices, $\gamma_{all}^{\infty}(P_n) = \lceil \frac{n}{2} \rceil$ and for a cycle C_n on n vertices, $\gamma_{all}^{\infty}(C_n) = \lceil \frac{n}{3} \rceil$ [14]. In [18], the authors present a linear-time algorithm to determine $\gamma_{all}^{\infty}(T)$ for all trees T. In [5], it was proven that if G is a proper interval graph, then $\gamma_{all}^{\infty}(G) = \alpha(G)$. In the past few years, a lot of effort was put in by several authors to determine the eternal domination number of cartesian grids, $\gamma_{all}^{\infty}(P_n \Box P_m)$. Exact values were determined for $2 \times n$ cartesian grids in [15] and $4 \times n$ cartesian grids in [4]. Asymptotical tight bounds for $3 \times n$ cartesian grids were obtained in [13] and improved in [11]. Finally, bounds for $5 \times n$ cartesian grids were given in [25]. The best known lower bound for $\gamma_{all}^{\infty}(P_n \Box P_m)$ for values of n and m large enough, is the domination number with the latter only being recently determined in [16]. The best known upper bound for $\gamma_{all}^{\infty}(P_n \Box P_m)$ was determined recently in [21], where it was shown that $\gamma_{all}^{\infty}(P_n \Box P_m) = \gamma(P_n \Box P_m) + O(n + m)$.Note that all the results discussed in this subsection also hold for $\gamma_{all}^{*\infty}$.

There are also many other variants of the game that exist and here we give a brief description and references for some of them. Recently, the eternal domination game and a variant have been studied in digraphs, including orientations of grids and toroidal strong grids [2]. Eternal total domination was studied in [19], where a total dominating set must be maintained by the guards each turn. The eviction model of eternal domination was studied in [17], where a vertex containing a guard is attacked each turn, which forces the guard to move to an adjacent empty vertex with the condition that the guards must maintain a dominating set each turn. The authors of the current paper studied a generalization of eternal domination in [8, 9, 10]. For

[1]A set $D \subseteq V$ is a dominating set of G if every vertex is in D or adjacent to a vertex in D

[2]$\alpha(G)$ is the maximum size of a set $S \subseteq V(G)$ of pairwise non-adjacent vertices in a graph G.

[3]$\theta(G)$ is the minimum number of complete subgraphs of G whose union covers $E(G)$.

more information and results on the original eternal domination game and its variants, see the survey [20].

1.3 Our results

The main result of this paper is that, for all $n, m \in \mathbb{N}^*$ such that $m \geq n$,

$$\lceil \frac{nm}{9} \rceil + \Omega(n+m) = \gamma^{\infty}_{all}(P_n \boxtimes P_m) = \lceil \frac{nm}{9} \rceil + O(m\sqrt{n}).$$

Finally, we prove that this result also holds in the case when at most one guard may occupy each vertex.

Note that, in toroidal strong grids $C_n \boxtimes C_m$, the problem becomes trivial and $\gamma^{\infty}_{all}(C_n \boxtimes C_m) = \lceil \frac{nm}{9} \rceil$ for any n and m. However, in the strong grids, border-effects make the problem much harder. The upper bound is proven by defining a set of configurations for the guards and a strategy for the guards that bases the movements of the guards on the current configuration and the attacked vertex. The defined set of configurations dominate the grid and are invariant to the movements required by the defined strategy to defend against attacks. The attacks are separated into three types of attacks: horizontal, vertical, and diagonal. The strategy defined gives the movement of the guards based on the type of attack. Thus, it is shown that in each of the three cases of attacks, the guards are able to move from their current configuration to another configuration in the set of configurations that are invariant and hence, the guards can defend against an infinite sequence of attacks.

The lower bound is proven by showing that, in any winning configuration in eternal domination, there are some vertices that are dominated by more than one guard, and/or some guards dominate at most 6 vertices. By double counting, this leads to the necessity of having $\Omega(n+m)$ extra guards compared to the classical domination.

2 Preliminaries

We use the classical terminology of graph-theory [12]. In particular, given a graph $G = (V, E)$ and $S \subseteq V$, let $N(S) = \{v \in V \setminus S \mid \exists w \in S, \{v, w\} \in E\}$ denote the set of neighbours (not in S) of the vertices in S. Moreover, let $N[S] = N(S) \cup S$ denote the *closed neighbourhood* of S. For $v \in V$, let $N(v) = N(\{v\})$ and $N[v] = N(v) \cup \{v\}$.

Let $n, m \in \mathbb{N}^*$ be such that $m \geq n$ and let the $n \times m$ strong grid, denoted by $SG_{n\times m}$, be the strong product $P_n \boxtimes P_m$ of an n-node path with an m-node path. Precisely, $SG_{n\times m}$ is the graph with set of vertices $\{(i, j) \mid 1 \leq i \leq n, 1 \leq j \leq m\}$, and two vertices (i_1, j_1) and (i_2, j_2) are adjacent if and only if $max\{|i_2 - i_1|, |j_2 - j_1|\} = 1$. That is, the vertices are identified by their Cartesian coordinates, *i.e.*, the vertex (i, j) is the vertex in *row* i and *column* j. The vertex $(1, 1)$ is in the *bottom-left corner* and the vertex (n, n) is in the *top-right corner*.

Definition 1. The set of *border* vertices of $SG_{n\times m}$ is the set

$$B = \bigcup_{1\leq i\leq n, 1\leq j\leq m} \{(1, j), (n, j), (i, 1), (i, m)\}.$$

Definition 2. The set of *pre-border* vertices of $SG_{n\times m}$ is the set

$$PB = N(B) = \bigcup_{2\leq i\leq n-1, 2\leq j\leq m-1} \{(2, j), (n-1, j), (i, 2), (i, m-1)\}.$$

Essentially, the pre-border vertices are all the non-border vertices that are adjacent to the border vertices. Equivalently, PB is the set of border vertices of the strong grid induced by $V(SG_{n\times m}) \setminus B$.

We consider the Eternal Domination game which is a turn-by-turn 2-Player game in graphs. On every *turn*, each vertex of a graph $G = (V, E)$ may be occupied by one or more guards. Let $k \in \mathbb{N}^*$ be the total number of guards. The positions of the guards are formally defined by a multi-set C of vertices, called a *configuration*, where the number of occurrences of a vertex $v \in C$ corresponds to the number of guards at $v \in V$ and $k = |C|$. On every turn, given a current configuration $C = \{v_i \mid 1 \leq i \leq k\}$ of k guards, Player 1, the *attacker*, chooses to *attack* a vertex $v \in V$. Then, Player 2 (the *defender*) may move each of its *guards* to a neighbour of their current position, therefore, achieving a new configuration $C' = \{w_i \mid 1 \leq i \leq k\}$ such that $w_i \in N[v_i]$ for every $1 \leq i \leq k$ (we then say that C' is *compatible* with C, which is clearly a symmetric relation). If $v \notin C'$, then the attacker *wins*, otherwise, the game goes on with a next turn (given the new configuration C').

A *strategy* for k guards is defined by an initial configuration of size k and by a function that, for every current configuration C and every attacked vertex $v \in V$, specifies a new configuration C' compatible with C. A strategy $\mathcal{S}$ for the guards is *winning* if, for every sequence of attacked vertices, the attacker never wins when the defender plays according to $\mathcal{S}$.

Our main contribution is the design of a winning strategy for $\gamma(SG_{n\times m}) + o(\gamma(SG_{n\times m}))$ guards in $SG_{n\times m}$, where $\gamma(SG_{n\times m}) = \lceil \frac{nm}{9} \rceil$ is the *domination number* of $SG_{n\times m}$ [].

The next lemma is the key of the winning strategy that we design for $\gamma(SG_{n\times m})+o(\gamma(SG_{n\times m}))$ guards in $SG_{n\times m}$.

In our strategy, it will often be useful to move a guard from a node $u \in PB$ of the pre-border to another node $v \in PB$ such that u and v are not necessarily adjacent. For this purpose, the idea is to place a sufficient number of guards on the vertices of the border such that a "flow" of the guards on the border vertices will simulate the move of the guard from u to v in one turn.

More precisely, given a configuration C and $u, v \in V(SG_{n\times m})$ with $u \in C$, a guard is said to *jump* from u to v if the configuration $(C \setminus \{u\}) \cup \{v\}$ is compatible with C, *i.e.*, there are some moves of the guards such that it is possible, in one turn, to achieve the same configuration as C except that there is one guard less on u and one guard more on v. More generally, given $U \subset C$ and $W \subset V(SG_{n\times m})$, a set of guards is said to *jump* from U to W if the configuration $(C \setminus U) \cup W$ is compatible with the configuration C.

Lemma 3. *Let $\alpha, \beta \in \mathbb{N}^*$ such that $\beta \leq \alpha$. Let $U, W \in PB$ be two subsets of pre-border vertices such that $|U| = |W| = \beta$. In any configuration C such that $U \subseteq C$ and C contains at least α occurrences of each vertex in B (i.e., each border vertex is occupied by at least α guards), then β guards may "jump" from U to W in one turn.*

Proof. The proof is by induction on β. The inductive hypothesis is that if all the vertices in B contain α guards each, then $\beta \leq \alpha$ guards may "jump" from U to W in one turn such that at most β guards move off of each vertex $w \in B$ in this turn. For the base case, let us assume that $U = \{u\}$ and $W = \{w\}$. Let us show how 1 guard can "jump" from u to w in one turn. If $u = w$, the result holds trivially, so let $u \neq w$. Let $u' \in B$ (resp., w') be a neighbour of u (of w) that shares one coordinate with u (with w). Let $Q = (u' = v_0, v_1, \cdots, v_\ell = w')$ be a path from u' to w' induced by the border vertices. In one turn, one guard at u moves to u', for every $0 \leq i < \ell$, one guard at v_i moves to v_{i+1}, and a guard at v_ℓ moves to w.

Now, let us assume that the inductive hypothesis holds for $\beta \geq 1$. If $\beta = \alpha$, we are done. Then, let us assume that $\beta < \alpha$. Let $|U| = |W| = \beta + 1 \leq \alpha$ and let $u \in U$ and $w \in W$. By the inductive hypothesis, β guards may jump from $U \setminus \{u\}$ to $W \setminus \{w\}$ in one turn and in such a way that, for every vertex $b \in B$, at most β guards move off of b during this turn. Since, every vertex of B is occupied by $\alpha > \beta$ guards, at least one guard is unused on every vertex of B. During the same turn, it is then possible to use the same strategy as in the base case to make one guard jump from u to w. □

3 Upper bound strategy

This section is devoted to proving that for all $n, m \in \mathbb{N}^*$ such that $m \geq n$, $\gamma_{all}^{\infty}(SG_{n\times m}) = \lceil \frac{nm}{9} \rceil + O(m\sqrt{n})$.

Before considering the general case, let us first assume that $n-2 \mod 3 = 0$ and that there exists $k \in \mathbb{N}^*$ such that $k-2 \mod 3 = 0$, and $m \mod k = 0$. The $n \times m$ strong grid will be partitioned into *blocks* which are subgrids of size $n \times k$. More precisely, for all $1 \leq q \leq \frac{m}{k}$, the q^{th} block contains columns $(q-1)k+1$ through qk of $SG_{n\times m}$.

3.1 Horizontal attacks

In this section, we only consider one block of $SG_{n\times m}$. W.l.o.g., let us consider the block $SG_{n\times k}$ induced by $\{(i,j) \mid 1 \leq i \leq n, 1 \leq j \leq k\}$. Let us first define a family of parameterized configurations for this block.

Let $\mathcal{X} = \{(b, a_1, \dots, a_{\frac{n-2}{3}}) \mid b \in \{1,2,3\}, a_i \in \{1,2,3\}$ for $i = 1, \dots, \frac{n-2}{3}\}$.

Given $X = (b, a_1, \dots, a_{\frac{n-2}{3}}) \in \mathcal{X}$, let $x_i(X) = 3(i-1)+b+1$, and $y_{j,i}(X) = 3(j-1)+a_i+1$ for every $1 \leq i \leq \frac{n-2}{3}$ and $1 \leq j \leq \frac{k-2}{3}$. We set $x_i = x_i(X)$ and $y_{j,i} = y_{j,i}(X)$ when there is no ambiguity. Intuitively, b will represent the *vertical shift* of the positions of the guards in configuration X. Similarly, for every $1 \leq i \leq \frac{n-2}{3}$, a_i represents the *horizontal shift* of the positions of the guards in row $x_i(X)$ in configuration X (see Figure 1).

Horizontal Configurations. Let us define the set $\mathcal{C}_H$ of configurations as follows. For every $X \in \mathcal{X}$, let $C_H(X) = B \cup \{(x_i(X), y_{j,i}(X)) \mid 1 \leq i \leq \frac{n-2}{3}, 1 \leq j \leq \frac{k-2}{3}\}$ be the configuration where there is one guard at every vertex of B and one guard at each vertex $(x_i(X), y_{j,i}(X)) = (3(i-1)+b+1, 3(j-1)+a_i+1)$ for every $1 \leq i \leq \frac{n-2}{3}$ and $1 \leq j \leq \frac{k-2}{3}$. See an example in Figure 1. Then,

$$\mathcal{C}_H = \{C_H(X) \mid X \in \mathcal{X}\}.$$

Note that $|C_H(X)| = \frac{(n-2)(k-2)}{9} + 2(n+k) - 4 = \kappa_H$ for every $X \in \mathcal{X}$. That is, any horizontal configuration uses κ_H guards.

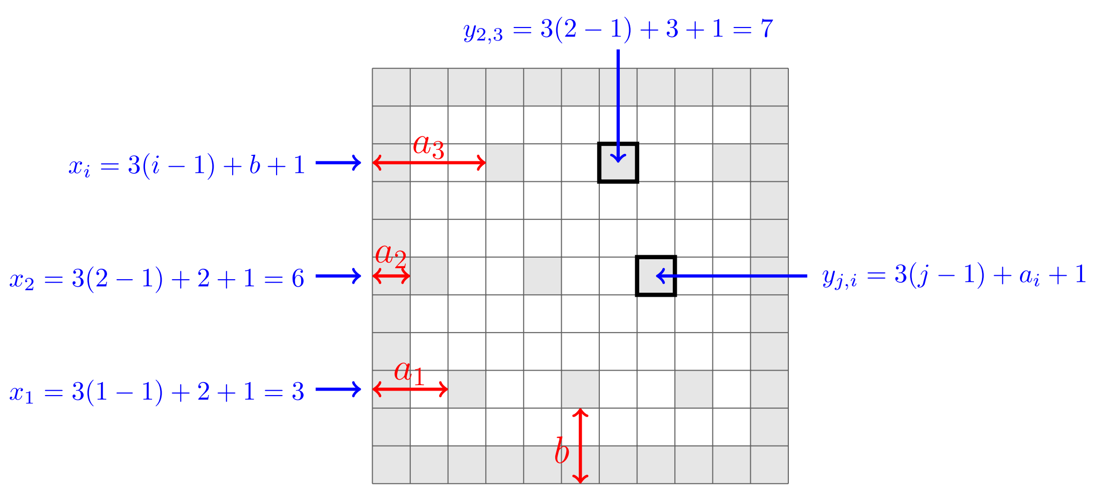

Figure 1: $P_{11} \boxtimes P_{11}$ where the squares are vertices. Example of a configuration $C_H(X)$ where $X = \{b=2, a_1=2, a_2=1, a_3=a_{\frac{n-2}{3}}=3\}$, there is one guard at each square in gray, and the white squares contain no guards.

Lemma 4. *Every configuration $C_H(X) \in \mathcal{C}_H$ is a dominating set of the block $SG_{n\times k}$.*

Proof. The pre-border and border vertices are dominated by the guards on the border vertices. For all $i, j \in \mathbb{N}^*$ such that $1 \le i \le \frac{n-2}{3}$ and $1 \le j \le \frac{k-2}{3}$, the guards on the vertices $(x_i, y_{j,i})$ dominate the vertices $\{(x_i+1, y_{j,i}), (x_i-1, y_{j,i}), (x_i, y_{j,i}-1), (x_i, y_{j,i}+1), (x_i+1, y_{j,i}+1), (x_i+1, y_{j,i}-1), (x_i-1, y_{j,i}-1), (x_i-1, y_{j,i}+1)\}$. □

In this subsection, we limit the power of the attacker by allowing it to attack only some predefined vertices (this kind of attack will be referred to as a *horizontal attack*). For every configuration $C_H(X) \in \mathcal{C}_H$ and for any such attack, we show that the guards may be moved (in one turn) in such a way to defend the attacked vertex and reach a new configuration in $\mathcal{C}_H$.

Horizontal Attacks. Let $X = (b, a_1, \dots, a_{\frac{n-2}{3}}) \in \mathcal{X}$ and $C_H(X) \in \mathcal{C}_H$. Let

$$A_H(X) = \{(x_i, y) \mid 1 \le i \le \frac{n-2}{3}, 1 \le y \le k\}.$$

A *horizontal attack with respect to* X is an attack at any vertex in $A_H(X)$, *i.e.*, an attack at any vertex of a row where some non-border vertex is occupied by a guard. Note that, for every vertex $v \in A_H(X)$, either v is occupied by a guard or there is a guard on the vertex to the left or to the right of v. In Figure 2, red squares represent the vertices of $A_H(X) \setminus C_H(X)$.

The next lemma proves that, from any horizontal configuration and against any horizontal attack (with respect to this current configuration), there is a possible strategy for the guards that defends against this attack and leads to a (new) horizontal configuration. Therefore, starting from any horizontal configuration, there is a strategy of the guards that wins against any sequence of horizontal attacks.

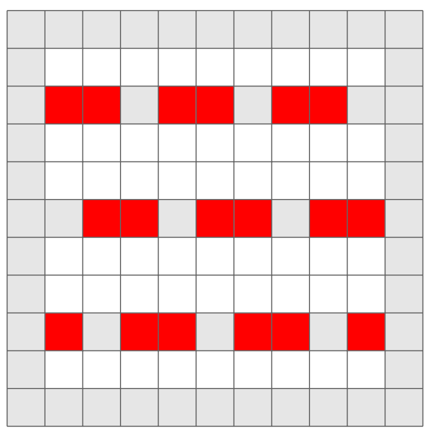

Figure 2: $P_{11} \boxtimes P_{11}$ where the squares are vertices. Example of the attackable vertices in red when only horizontal attacks are considered. The guards occupy a configuration $C_H(X)$ where $X = \{b = 2, a_1 = 2, a_2 = 1, a_3 = a_{\frac{n-2}{3}} = 3\}$, there is one guard at each square in gray, and the white squares contain no guards.

Lemma 5. *For any $X \in \mathcal{X}$ and any $v \in A_H(X)$, there exists $X' \in \mathcal{X}$ such that $v \in C_H(X')$ and configurations $C_H(X)$ and $C_H(X')$ are compatible. That is, in one turn, the guards may move from $C_H(X)$ to $C_H(X')$ and defend against an attack at v.*

Proof. Initially, κ_H guards are in a configuration $C_H(X)$ (see Figure 1). Consider an attack at some vertex $v \in A_H(X)$. If $v \in C_H(X)$, all guards may remain idle. Hence, let us assume that $v \in A_H(X) \setminus C_H(X)$.

Let us assume that $v = (x_\ell(X), y_{w,\ell}(X) - 1) = (x_\ell, y_{w,\ell} - 1)$ for some integers $1 \leq \ell \leq \frac{n-2}{3}$ and $1 \leq w \leq \frac{k-2}{3}$ (note that if $a_\ell = 1$ then $w > 1$ since v is not a border vertex), that is v is to the left of the vertex $(x_\ell, y_{w,\ell})$ that is occupied by a guard. The cases of attacks at $(x_\ell(X), y_{w,\ell}(X) + 1)$ (v is to the right of an occupied vertex) or $(x_\ell(X), 2)$, or $(x_\ell(X), k - 1)$ (the attacked vertex v is adjacent to a border vertex), are similar, by symmetry, to at least one of the two cases below.

The guards will move from the configuration $C_H(X)$ to a configuration $C_H(X')$ that defends against the attack at v, *i.e.*, $v \in C_H(X')$, where $X' = \{b', a'_1, \ldots, a'_{\frac{n-2}{3}}\}$ as defined below.

Intuitively, for the guards to move from the configuration $C_H(X)$ to a configuration $C_H(X')$ that defends against this attack at v, all the guards in row x_ℓ will shift left except for perhaps the guards on the border vertices (it depends on the value of a_ℓ). Hence, the only difference between X and X' will be the value of the horizontal shift related to row x_ℓ.

Precisely, by the definition of $C_H(X)$, there is a guard at $(x_\ell, y_{w,\ell})$. There are two cases of how the guards will move in response to the attack, depending on the three possible values of $a_\ell \in \{1, 2, 3\}$.

Case i) $a_\ell \in \{2, 3\}$**.** To defend against the attack, all the guards in row x_ℓ except those that occupy border vertices, shift one vertex to the left. That is, the guard at $(x_\ell, y_{j,\ell})$ moves to $(x_\ell, y_{j,\ell} - 1)$ for all $j \in \mathbb{N}^*$ such that $1 \leq j \leq \frac{k-2}{3}$. Since the positions of the other guards did not change, the guards occupy a configuration $C_H(X')$ where $b' = b$, $a'_i = a_i$ for all $1 \leq i \leq \frac{n-2}{3}$ such that $i \neq \ell$, but $a'_\ell = a_\ell - 1$.

Case ii) $a_\ell = 1$**.** To defend against the attack, all the guards in row x_ℓ except the one at $(x_\ell, 1)$, shift one vertex to the left. That is, the guard at $(x_\ell, y_{j,\ell})$ moves to $(x_\ell, y_{j,\ell} - 1)$ for all $j \in \mathbb{N}^*$ such that $1 < j \leq \frac{k-2}{3}$. Also, the guard at $(x_\ell, 2)$ jumps to $(x_\ell, k - 1)$ which is possible by Lemma 3 and since none of the border guards have to move for any other purpose. Since the positions of the other guards did not change, the guards occupy a configuration $C_H(X')$ where $b' = b$, $a'_i = a_i$ for all $1 \leq i \leq \frac{n-2}{3}$ such that $i \neq \ell$, but $a'_\ell = 3$. See Figure 3.

□

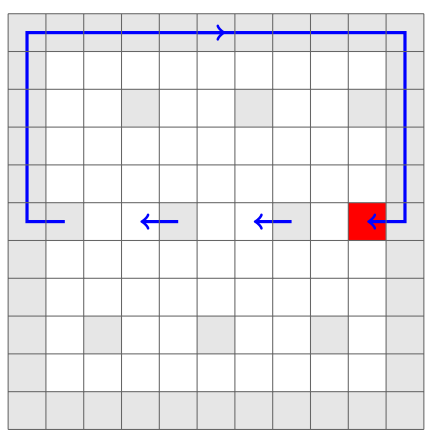

Figure 3: $P_{11} \boxtimes P_{11}$ where the squares are vertices. Example of an attack in Case ii) at the red square. The guards occupy a configuration $C_H(X)$ where $X = \{b = 2, a_1 = 2, a_2 = 1, a_3 = a_{\frac{n-2}{3}} = 3\}$, there is one guard at each square in gray, and the white squares contain no guards. The arrows (in blue) show the movements of the guards in response to the attack.

3.2 Vertical attacks

In this section, we consider the entire strong grid $SG_{n\times m}$ partitioned into $\frac{m}{k}$ blocks $SG_{n\times k}$ with block q, for $1 \le q \le \frac{m}{k}$, being induced by $\{(i, j + (q-1)k) \mid 1 \le i \le n, 1 \le j \le k\}$. Let us first define a family of parameterized configurations for this graph.

A configuration for the whole grid will be defined as the union of some configurations for each of the q blocks. Formally, for every $1 \le q \le \frac{m}{k}$, let us first define:

$$\mathcal{X}^q = \{(b^q, a_1^q, \dots, a_{\frac{n-2}{3}}^q) \mid b^q \in \{1,2,3\}, a_i^q \in \{1,2,3\} \text{ for } i = 1, \dots, \frac{n-2}{3} \text{ and } q = 1, \dots, \frac{m}{k}\}.$$

Given $X^q = (b^q, a_1^q, \dots, a_{\frac{n-2}{3}}^q) \in \mathcal{X}^q$, let $x_i^q(X^q) = 3(i-1) + b^q + 1$, and $y_{j,i}^q(X^q) = (q-1)k + 3(j-1) + a_i^q + 1$ for every $1 \le i \le \frac{n-2}{3}$, $1 \le j \le \frac{k-2}{3}$, and $1 \le q \le \frac{m}{k}$. We set $x_i^q = x_i^q(X^q)$ and $y_{j,i}^q = y_{j,i}^q(X^q)$ when there is no ambiguity.

That is, intuitively, b^q will represent the *vertical shift* of the positions of the guards in configuration X^q in the q^{th} block. Similarly, for every $1 \le i \le \frac{n-2}{3}$, a_i^q represents the *horizontal shift* of the positions of the guards in row $x_i(X)$ in configuration X^q in the q^{th} block.

Finally, let $\mathcal{Y} = \{(X^1, \dots, X^{\frac{m}{k}}) \mid X^q \in \mathcal{X}^q \text{ for } q = 1, \dots, \frac{m}{k}\}$.

Vertical Configurations. In order to properly define the following set of configurations, we require the following notation. For a set S of vertices in a configuration $\mathcal{C}$ and an integer $x > 0$, let $S^{[x]}$ be the multi-set of vertices that consists of x copies of each vertex in S. Intuitively, $S^{[x]}$ will be used to define a configuration where x guards occupy each vertex of S. Let us now define the set $\mathcal{C}_V$ of configurations as follows.

For every $Y = (X^1, \dots, X^{\frac{m}{k}}) \in \mathcal{Y}$, let $C_V(Y) = B^{[\frac{k-2}{3}]} \cup \bigcup_{q=1}^{\frac{m}{k}} C_H(X^q)$ be the configuration obtained as follows. First, for any $1 \le q \le \frac{m}{k}$, guards are placed in configuration $C_H(X^q)$ in the q^{th} block. Then, $\frac{k-2}{3}$ guards are added to every border vertex. Note that overall, there are $\frac{k-2}{3} + 1$ guards at each vertex of B. See an example in Figure 4. Then,

$$\mathcal{C}_V = \{C_V(Y) \mid Y \in \mathcal{Y}\}.$$

Note that $|C_V(Y)| = \frac{m}{k}\kappa_H + 2(\frac{k-2}{3})(n+m-2) = \kappa_V$ for every $Y \in \mathcal{Y}$. That is, any vertical configuration uses κ_V guards.

Lemma 6. *Every configuration $C_V(Y) \in \mathcal{C}_V$ is a dominating set of $SG_{n\times m}$.*

Proof. Since $C_V(Y) \in \mathcal{C}_V$, by definition, for all $1 \le q \le \frac{m}{k}$, there exists $X^q \in \mathcal{X}^q$ such that the vertices of $C_H(X^q)$ are occupied by guards. Therefore, each of the $\frac{m}{k}$ blocks $SG_{n\times k}$ is dominated by the guards within it by Lemma 4. □

In this subsection, we limit the power of the attacker by allowing it to attack only some predefined vertices (this kind of attack will be referred to as a *vertical* attack). For every configuration $C_V(X) \in \mathcal{C}_V$ and for any such attack, we show that the guards may be moved (in one turn) in such a way to defend the attacked vertex and reach a new configuration in $\mathcal{C}_V$.

Vertical Attacks. Let $Y = (X^1, \dots, X^{\frac{m}{k}}) \in \mathcal{Y}$ and $C_V(Y) \in \mathcal{C}_V$. Let

$$\begin{aligned} A_V(Y) = &\{(x_i^q - 1, y_{j,i}^q), (x_i^q + 1, y_{j,i}^q) \mid 1 \le i \le \frac{n-2}{3}, 1 \le j \le \frac{k-2}{3}, 1 \le q \le \frac{m}{k}\} \\ &\cup \{(2, y_{j,n-1}^q) \mid 1 \le j \le \frac{k-2}{3}, 1 \le q \le \frac{m}{k} \textit{ and } b^q = 3\} \\ &\cup \{(n-1, y_{j,2}^q) \mid 1 \le j \le \frac{k-2}{3}, 1 \le q \le \frac{m}{k} \textit{ and } b^q = 1\} \end{aligned}$$

A *vertical attack with respect to* Y is an attack at any vertex in $A_V(Y)$, *i.e.*, an attack at any non-border vertex above or below a guard not on a border vertex. Moreover, if the vertical shift b^q of the q^{th} block equals 3, then some vertices of the second row of the q^{th} block may also be attacked (depending on the horizontal shift a^q_{n-1}). Finally, if the vertical shift b^q of the q^{th} block equals 1, then some vertices of the $(n-1)^{th}$ row of the q^{th} block may also be attacked (depending on the horizontal shift a^q_2).

Note that $A_V(Y) \cap C_V(Y) = \emptyset$, and $A_V(Y) \cap A_H(X^q) = \emptyset$ for any $X^q \in Y$, *i.e.*, any vertical attack with respect to Y is not a horizontal attack with respect to $X^q \in Y$ and vice versa. In Figure 5, red squares represent the vertices of $A_V(Y)$.

The next lemma proves that, from any vertical configuration and against any vertical attack (with respect to this current configuration), there is a possible strategy for the guards that defends against this attack and leads to a (new) vertical configuration. Therefore, starting from any vertical configuration, there is a strategy of the guards that wins against any sequence of vertical attacks.

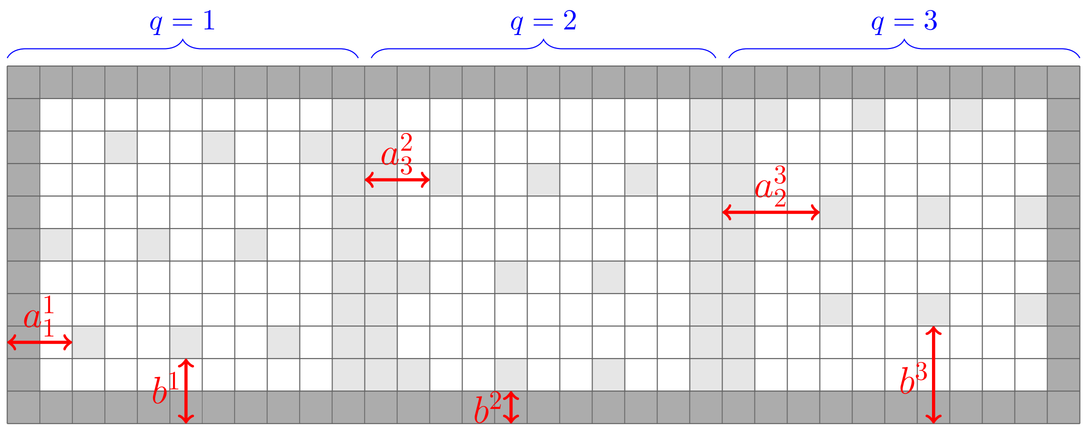

Figure 4: $P_{11} \boxtimes P_{33}$ where the squares are vertices. Example of a configuration $C_V(Y)$ where $k = 11$, $Y = \{X^1, X^2, X^3\}$, $X^1 = \{2, 2, 1, 3\}$, $X^2 = \{1, 1, 1, 2\}$, $X^3 = \{3, 3, 3, 1\}$, there are $\frac{k-2}{3} + 1 = 4$ guards at each square in dark gray, 1 guard at each square in light gray, and the white squares contain no guards.

Lemma 7. *For any* $Y \in \mathcal{Y}$ *and any* $v \in A_V(Y)$, *there exists* $Y' \in \mathcal{Y}$ *such that* $v \in C_V(Y')$ *and configurations* $C_V(Y)$ *and* $C_V(Y')$ *are compatible. That is, in one turn, the guards may move from* $C_V(Y)$ *to* $C_V(Y')$ *and defend against an attack at* v.

Proof. Let $Y = (X^1, \cdots, X^{\frac{m}{k}})$. Initially, κ_V guards are in a configuration $C_V(Y)$ (see Figure 4).

Consider an attack at some vertex $v \in A_V(Y)$. Let us assume that $v = (x^z_\ell(X^z)-1, y^z_{w,\ell}(X^z))$ for some $1 \leq z \leq \frac{m}{k}$, $1 \leq \ell \leq \frac{n-2}{3}$, and $1 \leq w \leq \frac{k-2}{3}$ (note that if $b^z = 1$, then $\ell > 1$ since v is not a border vertex). That is, v is a vertex of the z^{th} block is below the vertex $(x^z_\ell(X^z), y^z_{w,\ell}(X^z))$ that is occupied by a guard.

The cases of attacks at $(x^z_\ell(X^z)+1, y^z_{w,\ell}(X^z))$ (v is above an occupied vertex), $(2, y^z_{w,n-1}(X^z))$ (v is above a border vertex), and $(n-1, y^z_{w,2}(X^z))$ (v is below a border vertex), are similar, by symmetry, to at least one of the two cases below.

The guards will move from the configuration $C_V(Y)$ to a configuration $C_V(Y')$ that defends against the attack at v, *i.e.*, $v \in C_V(Y')$, where $Y' = \{X'^1, \ldots, X'^{\frac{m}{k}}\}$ as defined below.

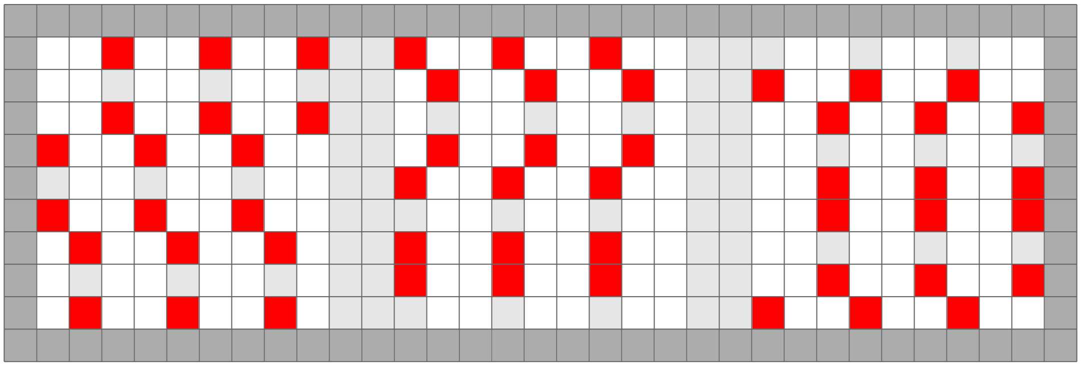

Figure 5: $P_{11} \boxtimes P_{33}$ where the squares are vertices. Example of the attackable vertices in red when only vertical attacks are considered. The guards occupy a configuration $C_V(Y)$ where $k = 11$, $Y = \{X^1, X^2, X^3\}$, $X^1 = \{2, 2, 1, 3\}$, $X^2 = \{1, 1, 1, 2\}$, $X^3 = \{3, 3, 3, 1\}$, there are $\frac{k-2}{3} + 1 = 4$ guards at each square in dark gray, 1 guard at each square in light gray, and the white squares contain no guards.

Intuitively, for the guards to move from the configuration $C_V(Y)$ to a configuration $C_V(Y')$ that defends against this attack at v, all the guards in the block z will shift down except for perhaps the guards on the border vertices (it depends on the value of b^z).

Precisely, by the definition of $C_V(Y)$, there is a guard at $(x^z_\ell, y^z_{w,\ell})$. There are two cases of how the guards will move in response to the attack, depending on the three possible values of $b^z \in \{1, 2, 3\}$.

Case i) $b^z \in \{2, 3\}$**.** To defend against the attack, all the guards in the block z that contains the attacked vertex except those that occupy border vertices of the block z, shift one vertex downwards. That is, for all $i, j \in \mathbb{N}$ such that $1 \leq i \leq \frac{n-2}{3}$ and $1 \leq j \leq \frac{k-2}{3}$, the guard at $(x^z_i, y^z_{j,i})$ moves to $(x^z_i - 1, y^z_{j,i})$.
Since the positions of the other guards did not change, the guards occupy a configuration $C_V(Y')$ where $X^p = X'^p$ for all $1 \leq p \leq \frac{m}{k}$ such that $p \neq z$, and $X'^z = (b'^z, a'^z_1, \cdots, a'^z_{\frac{n-2}{3}})$ with $a'^z_i = a^z_i$ for all $1 \leq i \leq \frac{n-2}{3}$, but $b'^z = b^z - 1$.

Case ii) $b^z = 1$**.** To defend against the attack, all the guards in the block z shift one vertex downwards, except those that occupy the vertices of the border of the block z and the guards just above the bottom border of the block. Using the guards on the border of the (whole) grid, the guards just above the bottom border of the block jump to the row just below the top border of the block z.
That is, for all $i, j \in \mathbb{N}$ such that $1 < i \leq \frac{n-2}{3}$ and $1 \leq j \leq \frac{k-2}{3}$, the guard at $(x^z_i, y^z_{j,i})$ moves to $(x^z_i - 1, y^z_{j,i})$. Also, the guard at $(2, y^z_{j,i})$ jumps to $(n-1, y^z_{j,i})$ which is possible by Lemma 3 since a total of $\frac{k-2}{3}$ guards jump, $\frac{k-2}{3} + 1$ guards occupy each vertex of the border of the grid, and since none of the border guards have to move for any other purpose.
Since the positions of the other guards did not change, the guards occupy a configuration $C_V(Y')$ where $X^p = X'^p$ for all $1 \leq p \leq \frac{m}{k}$ such that $p \neq z$, and $X'^z = (b'^z, a'^z_1, \cdots, a'^z_{\frac{n-2}{3}})$ with $a'^z_i = a^z_i$ for all $1 \leq i \leq \frac{n-2}{3}$, but $b'^z = 3$. See Figure 6.

□

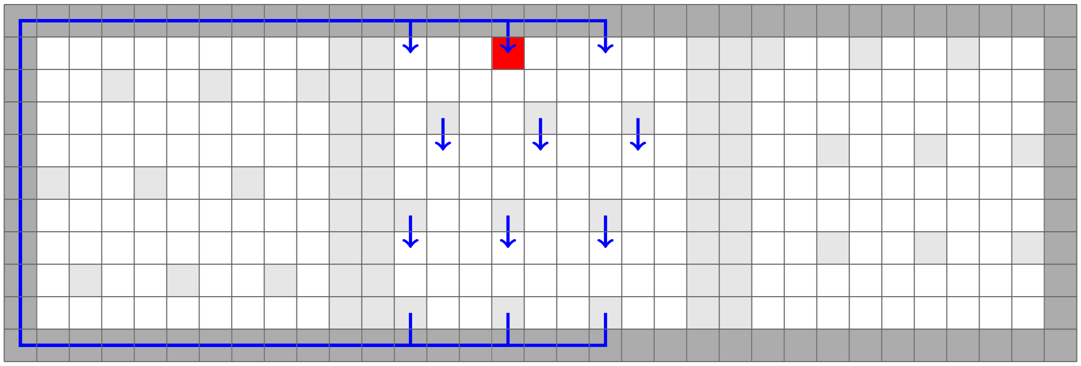

Figure 6: $P_{11} \boxtimes P_{33}$ where the squares are vertices. Example of an attack in Case iii) at the red square. The guards occupy a configuration $C_V(Y)$ where $k = 11$, $Y = \{X^1, X^2, X^3\}$, $X^1 = \{2, 2, 1, 3\}$, $X^2 = \{1, 1, 1, 2\}$, $X^3 = \{3, 3, 3, 1\}$, there are $\frac{k-2}{3} + 1 = 4$ guards at each square in dark gray, 1 guard at each square in light gray, and the white squares contain no guards. The arrows (in blue) show the movements of the guards in response to the attack.

3.3 Diagonal attacks

The same $n \times m$ strong grid $SG_{n\times m}$, notations, and configurations for the guards used in subsection 3.2 will be used here.

In this subsection, we limit the power of the attacker by allowing it to attack only some *diagonal* vertices. For every configuration $C_V(X) \in \mathcal{C}_V$ and for any such attack, we show that the guards may be moved (in one turn) in such a way to defend the attacked vertex and reach a new configuration in $\mathcal{C}_V$.

Diagonal Attacks. Let $Y = (X^1, \dots, X^{\frac{m}{k}}) \in \mathcal{Y}$ and $C_V(Y) \in \mathcal{C}_V$. Let $A_D(Y) = V(SG_{n\times m}) \setminus (B \cup A_H(Y) \cup A_V(Y))$. That is, $A_D(Y)$ covers all possible attacks that are neither horizontal nor vertical.

A *diagonal attack with respect to* Y is an attack at any vertex in $A_D(Y)$. Note that, for every vertex $v \in A_D(Y)$, there is a guard on a vertex adjacent to v and neither in the same column nor in the same row as v. In Figure 7, red squares represent the vertices of $A_D(Y)$.

The next lemma proves that, from any vertical configuration and against any diagonal attack (with respect to this current configuration), there is a possible strategy for the guards that defends against this attack and leads to a (new) vertical configuration. Therefore, starting from any vertical configuration, there is a strategy of the guards that wins against any sequence of diagonal attacks.

Lemma 8. *For any $Y \in \mathcal{Y}$ and any $v \in A_D(Y)$, there exists $Y' \in \mathcal{Y}$ such that $v \in C_V(Y')$ and configurations $C_V(Y)$ and $C_V(Y')$ are compatible. That is, in one turn, the guards may move from $C_V(Y)$ to $C_V(Y')$ and defend against an attack at v.*

Proof. Let $Y = (X^1, \cdots, X^{\frac{m}{k}})$. Initially, κ_V guards are in a configuration $C_V(Y)$ (see Figure 4).

Consider an attack at some vertex $v \in A_D(Y)$. Let us assume that $v = (x^z_\ell(X^z) - 1, y^z_{w,\ell}(X^z) + 1)$ for some $1 \leq \ell \leq \frac{n-2}{3}$, $1 \leq w \leq \frac{k-2}{3}$, and $1 \leq z \leq \frac{m}{k}$ (Note that, if $b^z = 1$, then $\ell > 1$ and if $a^z_\ell = 3$, then $w < \frac{k-2}{3}$ since v is not a border vertex). All other cases are similar by symmetry (see Figures 9 and 10).

The guards will move from a configuration $C_V(Y)$ to a configuration $C_V(Y')$ that defends against the attack at v, *i.e.*, $v \in C_V(Y')$, where $Y' = \{X'^1, \dots, X'^{\frac{m}{k}}\}$ as defined below.

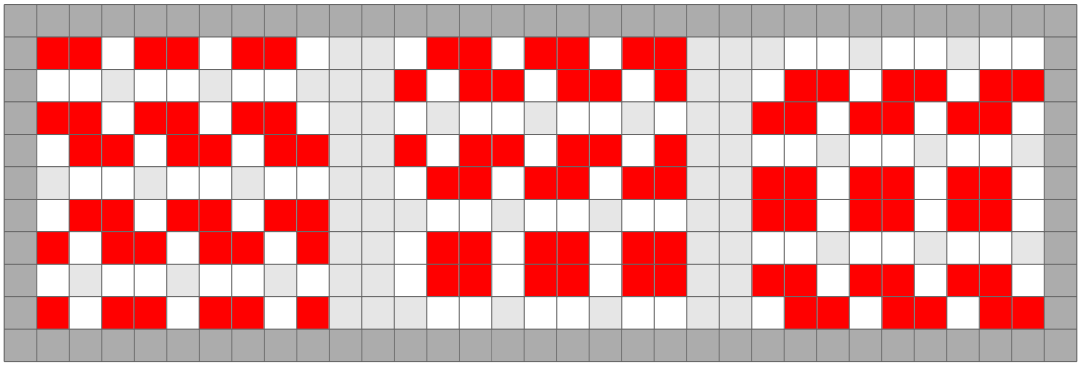

Figure 7: $P_{11} \boxtimes P_{33}$ where the squares are vertices. Example of the attackable vertices in red when only diagonal attacks are considered. The guards occupy a configuration $C_V(Y)$ where $k = 11$, $Y = \{X^1, X^2, X^3\}$, $X^1 = \{2, 2, 1, 3\}$, $X^2 = \{1, 1, 1, 2\}$, $X^3 = \{3, 3, 3, 1\}$, there are $\frac{k-2}{3} + 1 = 4$ guards at each square in dark gray, 1 guard at each square in light gray, and the white squares contain no guards.

Intuitively, for the guards to move from a configuration $C_V(Y)$ to a configuration $C_V(Y')$ that defends against this attack at v, in the block z that contains the attacked vertex, the guards in row x^z_ℓ will move as they would in response to a horizontal attack and a vertical attack but simultaneously, so moving diagonally down and to the right, and the remainder of the guards in the block z will move as they would in response to a vertical attack, so moving down.

In particular, if $b^z = 1$ (there are guards on the row above the bottom border of the block q), the guards on row 2 in the block z will jump to the row below the top border of block z using the border of the grid (as specified in Lemma 7). Moreover, if $a^z_\ell = 3$, the guard on vertex $(x^z_\ell, zq - 1)$ jumps to vertex $(x^z_\ell - 1, z(q - 1) + 2)$ using the border of the block z. So, a total of at most $\frac{k-2}{3} + 1$ guards jump which is possible (by Lemma 3) since enough guards are present on each vertex of the border of the grid.

Precisely, after their moves, the guards occupy a configuration $C_V(Y')$ where $X^p = X'^p$ for all $1 \leq p \leq \frac{m}{k}$ such that $p \neq z$, and $X'^z = (b'^z, a'^z_1, \cdots, a'^z_{\frac{n-2}{3}})$ with $a'^z_i = a^z_i$ for all $1 \leq i \leq \frac{n-2}{3}$ such that $i \neq \ell$, but

Case $b^z \in \{2, 3\}$ and $a^z_\ell \in \{1, 2\}$. $a'^z_\ell = a^z_\ell + 1$ and $b'^z = b^z - 1$.

Case $b^z \in \{2, 3\}$ and $a^z_\ell = 3$. $a'^z_\ell = 1$ and $b'^z = b^z - 1$.

Case $b^z = 1$ and $a^z_\ell \in \{1, 2\}$. $a'^z_\ell = a^z_\ell + 1$ and $b'^z = 3$. See Figure 8.

Case $b^z = 1$ and $a^z_\ell = 3$. $a'^z_\ell = 1$ and $b'^z = 3$.

□

3.4 Upper Bound in Strong Grids

Note that, for any $Y = (X^1, \cdots, X^{\frac{m}{k}}) \in \mathcal{Y}$, then $A_D(Y) \cup A_V(Y) \cup \bigcup_{q=1}^{\frac{m}{k}} A_H(X^q) \cup B = V(SG_{n\times m})$. That is, any attack by the attacker in $SG_{n\times m}$ is either an attack at an occupied vertex or a horizontal, vertical or diagonal attack.

Hence, the previous lemmas hold for any possible attack and we are ready to prove our main theorem.

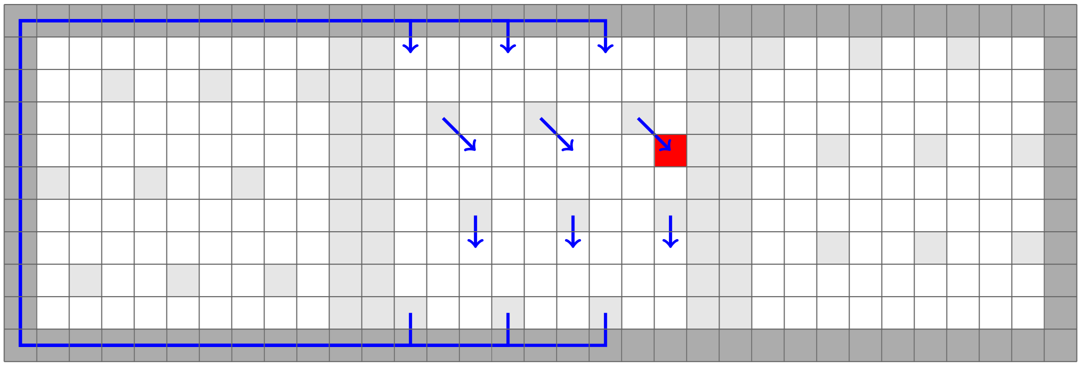

Figure 8: $P_{11} \boxtimes P_{33}$ where the squares are vertices. Example of a diagonal attack at the red square. The guards occupy a configuration $C_V(Y)$ where $k = 11$, $Y = \{X^1, X^2, X^3\}$, $X^1 = \{2, 2, 1, 3\}$, $X^2 = \{1, 1, 3, 2\}$, $X^3 = \{3, 3, 3, 1\}$, there are $\frac{k-2}{3} + 1 = 4$ guards at each square in dark gray, 1 guard at each square in light gray, and the white squares contain no guards. The arrows (in blue) show the movements of the guards in response to the attack.

Theorem 9. *For all $n, m \in \mathbb{N}^*$ such that $m \geq n$,*

$$\gamma^{\infty}_{all}(SG_{n\times m}) = \lceil \frac{nm}{9} \rceil + O(m\sqrt{n}) = (1 + o(1))\gamma(SG_{n\times m}).$$

Proof. Let k be the integer closest to $\sqrt{n}$ such that $k - 2 \mod 3 = 0$.

First, we prove that we can restrict our study to the case when n, m, and k satisfy the hypothesis of the previous lemmas, *i.e.*, $n - 2 \mod 3 = 0$ and $m \mod k = 0$. For this purpose, some guards are placed at each of the vertices of a few columns and rows (and these guards will never move) such that what remains to be protected is an $a \times b$ subgrid H such that $a - 2 \mod 3 = 0$ and $b \mod k = 0$.

If $n - 2 \mod 3 = 0$, then $a = n$. Otherwise, if $n - 2 \mod 3 = 1$ (resp., 2) then, place one guard at every vertex of the first (resp., the first two) row(s) of $SG_{n\times m}$ and $a = n - 1$ (resp., $a = n - 2$). Then, place one guard at every vertex of the $x < k$ first columns of $SG_{n\times m}$, such that $b = m - x$ and $b \mod k = 0$. Overall, $O(m + kn) = O(m + n\sqrt{n})$ guards have been placed, so proving that $\gamma^{\infty}_{all}(H) = \lceil \frac{ab}{9} \rceil + O(b\sqrt{a})$ will be sufficient to prove the theorem.

Hence, from now on, let us assume that n and m satisfy $n-2 \mod 3 = 0$ and $m \mod k = 0$.

Let $Y \in \mathcal{Y}$ be any configuration. The guards initially occupy the configuration $C_V(Y)$. By Lemma 6, the guards occupy a dominating set. We show that, whatever be the attack at any vertex v, there is $Y' \in \mathcal{Y}$ such that $v \in C_V(Y')$ and $C_V(Y')$ is compatible with $C_V(Y)$.

Let the attacker attack some unoccupied vertex $v \in V(SG_{n\times m})$. As mentioned in subsection 3.3, the vertex v is in $A_H(Y)$ or $A_V(Y)$ or $A_D(Y)$ (or already contains a guard since every border vertex contains at least one guard). If $v \in C_V(Y)$, all guards remain idle. Hence, let us assume that $v \notin C_V(Y)$. If $v \in A_H(X^q)$ for some $X^q \in Y$, then the guards in the block q that contains v will respond as in Lemma 5 (only the guards in the same block and in the same row as v will move, plus some guards on the border of this block if some jump is needed). If $v \in A_V(Y)$, then the guards in the block q that contains v will respond as in Lemma 7. If $v \in A_D(Y)$, then the guards in the block q that contains v will respond as in Lemma 8. By Lemma 5, Lemma 7, and Lemma 8, after the attack, the guards occupy a configuration $C_V(Y')$ for some $Y' \in \mathcal{Y}$ and thus, can defend against an infinite sequence of attacks.

The above strategy uses $\kappa_V = \lceil \frac{m}{k} \rceil \kappa_H + 2(\frac{k-2}{3})(m + n - 2)$ guards (see Subsection 3.2). Since $\kappa_H = \lceil \frac{(n-2)(k-2)}{9} \rceil + 2(n + k) - 4$ (see Subsection 3.1) and $k = \Theta(\sqrt{n})$, the strategy uses

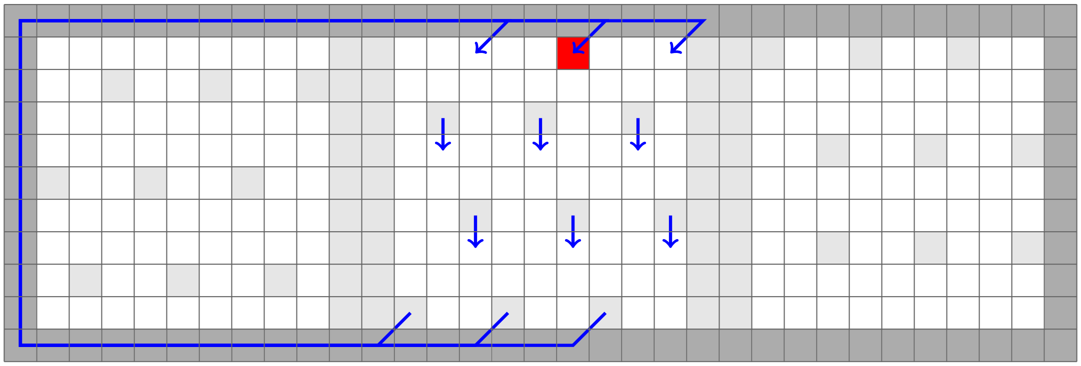

Figure 9: $P_{11} \boxtimes P_{33}$ where the squares are vertices. Example of a diagonal attack at the red square. The guards occupy a configuration $C_V(Y)$ where $k = 11$, $Y = \{X^1, X^2, X^3\}$, $X^1 = \{2, 2, 1, 3\}$, $X^2 = \{1, 1, 3, 2\}$, $X^3 = \{3, 3, 3, 1\}$, there are $\frac{k-2}{3} + 1 = 4$ guards at each square in dark gray, 1 guard at each square in light gray, and the white squares contain no guards. The arrows (in blue) show the movements of the guards in response to the attack. To be consistent with the strategy described, the guards in the second row of the middle block move diagonally down and to the left when entering and leaving the border but they may clearly just move vertically down in both instances if they move far enough along the border.

$\kappa_V = \lceil \frac{nm}{9} \rceil + O(m\sqrt{n})$ guards, which concludes the proof of the theorem. $\square$

4 Lower Bound in Strong Grids

So far, the best lower bound for $\gamma_{all}^{\infty}(SG_{n\times m})$ was the trivial lower bound $\gamma(SG_{n\times m})$. In this section, we slightly increase this lower bound, reducing the gap with the new upper bound of the previous section.

Theorem 10. *For all $n, m \in \mathbb{N}^*$, $\gamma_{all}^{\infty}(SG_{n\times m}) = \gamma(SG_{n\times m}) + \Omega(n + m)$.*

Proof. $\gamma_{all}^{\infty}(SG_{n\times m})$ is clearly increasing with n and m, thus, it is sufficient to prove the theorem for $n \mod 3 = 0$ and $m \mod 3 = 0$. Hence, let us assume that $n \mod 3 = 0$ and $m \mod 3 = 0$.

Note that, if n and m are divisible by 3, there is a unique minimum dominating set of $SG_{n\times m}$ and, in this dominating set, each vertex is dominated by exactly one guard. The idea of the proof is that, in any winning configuration in eternal domination, there are some vertices that are dominated by more than one guard, and/or some guards dominate at most 6 vertices. By double counting, this leads to the necessity of having $\Omega(n + m)$ extra guards compared to the classical domination.

The following claim shows that, whatever be the guards' strategy, at every step, every 4×5 subgrid that includes 5 border vertices must have at least two guards in it or else the attacker wins.

Claim 11. *Consider any configuration of the guards in $SG_{n\times m}$. If there is a 4×5 subgrid that includes 5 border vertices with only one guard in it, the attacker can win in at most two turns.*

Proof of the claim. W.l.o.g. let the 4×5 subgrid include border vertices from the left column of $SG_{n\times m}$. Also, for some integer $1 \leq x \leq m - 4$, let $\{(x, 1), \ldots, (x + 4, 1)\}$ be the 5 border vertices. If there is only one guard in this subgrid, then the guard must be at $(x + 2, 2)$ in order

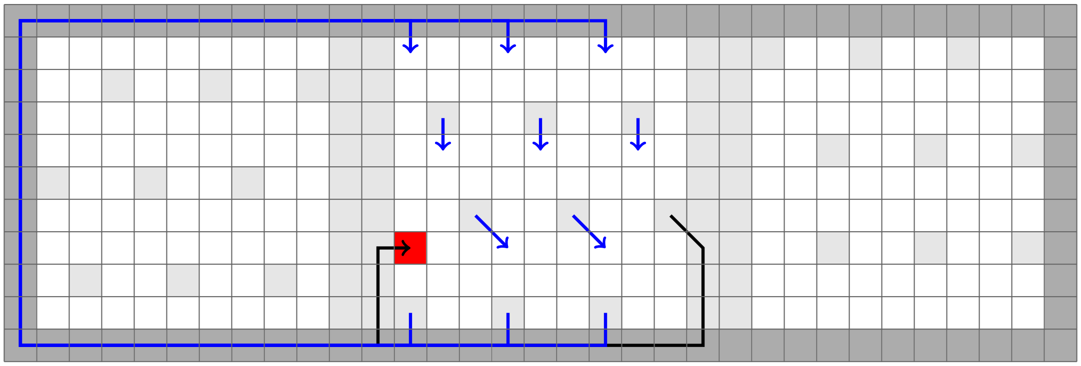

Figure 10: $P_{11} \boxtimes P_{33}$ where the squares are vertices. Example of a diagonal attack at the red square. The guards occupy a configuration $C_V(Y)$ where $k = 11$, $Y = \{X^1, X^2, X^3\}$, $X^1 = \{2, 2, 1, 3\}$, $X^2 = \{1, 1, 3, 2\}$, $X^3 = \{3, 3, 3, 1\}$, there are $\frac{k-2}{3} + 1 = 4$ guards at each square in dark gray, 1 guard at each square in light gray, and the white squares contain no guards. The arrows (in blue) show the movements of the guards in response to the attack. The arrow in black is to differentiate between the different guards jumping.

to prevent the attacker from winning in one turn as otherwise, it is not possible to dominate all the vertices of the subgrid. Then, the attacker attacks $(x + 2, 3)$ which forces the guard at $(x + 2, 2)$ to move to $(x + 2, 3)$ as he is the only guard adjacent to that vertex since, initially, there was only one guard in the 4×5 subgrid. Now the attacker attacks $(x+2, 1)$ and wins since every guard is at distance at least 2 from this vertex after the previous moves of the guards since, initially, there was only one guard in the 4×5 subgrid. ⋄

In any configuration C, let $x = x(C)$ be the number of 4×5 subgrids with at least one vertex dominated by two guards and $y = y(C)$ be the number of 4×5 subgrids where one guard dominates exactly 6 vertices.

Using the previous claim, it can be proved that:

Claim 12. *There is $\delta > 0$ such that, for any configuration C of the guards in $SG_{n \times m}$ in any winning strategy for the guards, $x + y = \delta(n + m)$ where $x = x(C)$ and $y = y(C)$ are defined as above.*

Proof of the claim. Consider the subgraph induced by rows 1 through 4 and columns 6 through $m - 5$ of $SG_{n \times m}$.

Considering columns 6 through 13, there must exist a 4×5 subgrid that includes 5 border vertices and has a guard in its center column as otherwise, there are no guards in the four center columns of the 8 considered (columns 9 through 12 in this case) which means that $SG_{n \times m}$ is not dominated and hence, this configuration is not part of any winning strategy for the guards. Therefore, by considering the columns eight by eight from the first to the last column in rows 1 through 4 of the subgraph described above, there are at least $\lfloor \frac{m-10}{8} \rfloor$ 4×5 subgrids that fit the profile of the subgrid in Claim 11. Hence, there are at least two guards in each of these subgrids as otherwise, the attacker wins by Claim 11. Moreover, since there is a guard in the center column of each of these subgrids, there is at least one vertex in each of these subgrids that is dominated by two guards, unless there is a guard on the border in the center column and the other guard(s) are in row 4. However, in the latter case, the guard on the border in the center column only dominates 6 vertices. By symmetry, this is true for the first and last 4 columns and the topmost 4 rows as well. Therefore, there are at least $\lfloor \frac{m-10}{4} \rfloor + \lfloor \frac{n-10}{4} \rfloor$ subgrids in $SG_{n\times m}$ that fit the profile of the subgrid in Claim 11. Then, $\lfloor \frac{m-10}{4} \rfloor + \lfloor \frac{n-10}{4} \rfloor \leq x + y$. $\diamond$

Let us consider any winning strategy using k guards. Let x and y be the same as in Claim 12. At every step, these k guards dominate at most $9k - 3y$ vertices (with multiplicity, *i.e.*, a vertex is counted once for each guard that dominates it). By the definition of x, at least $nm+x$ vertices (with multiplicity) must be dominated. Hence, $9k-3y \geq nm+x$. It follows that $k \geq \frac{nm}{9} + \frac{x}{9} + \frac{y}{3}$. By Claim 12, $\frac{x}{9} + \frac{y}{3} = \delta'(n+m)$ for some $\delta' > 0$ and so $k = \frac{nm}{9} + \Omega(n+m)$. $\square$

5 At Most One Guard at each Vertex

This section is devoted to proving that the two main results presented thus far are also true for the variant of the eternal domination game where at most one guard may occupy a vertex. The corresponding eternal domination number for this variant will be denoted by $\gamma^{*\infty}_{all}$. This variant is also considered in, e.g.,[21, 18].

A generalization of Lemma 3 will be the key to generalizing Theorem 9 to this variant of the game. The following definitions are required to properly state this new lemma.

For $t \in \mathbb{N}^*$, the set of vertices of the *t-thick border* of $SG_{n\times m}$ is the set

$$TB_t = \bigcup_{1\leq i\leq n, 1\leq j\leq m, 1\leq \ell\leq t} \{(\ell, j), (n+1-\ell, j), (i, \ell), (i, m+1-\ell)\}$$

and $TB_0 = \emptyset$. In other words, $TB_1 = B(SG_{n\times m})$ is the border of $SG_{n\times m}$, and $TB_t = TB_{t-1} \cup B(SG_{n\times m} \setminus TB_{t-1})$ for any $t \geq 1$. Essentially, the t-thick border vertices are the vertices of the t leftmost and rightmost columns and the t top and bottom rows of $SG_{n\times m}$.

Recall that $PB = TB_2 \setminus TB_1$ is the set of pre-border vertices. Two vertex-disjoint sets $U, W \subseteq PB$ are said to be *non-overlapping*, if there is a path Q induced only by vertices of PB such that $U \subseteq V(Q)$ and $V(Q) \cap W = \emptyset$.

Let $PB_\alpha = TB_{\alpha+1} \setminus TB_\alpha$ be the pre-border vertices of $SG_{n\times m} \setminus (TB_{\alpha-1})$.

Lemma 13. *Let $\alpha, \beta \in \mathbb{N}^*$ such that $\beta \leq \alpha$. Let $U, W \subseteq PB_\alpha$ be two non-overlapping subsets of pre-border vertices of $SG_{n\times m} \setminus (TB_{\alpha-1})$ such that $|U| = |W| = \beta$. In any configuration C such that $U \subseteq C$, if the α-thick border of $SG_{n\times m}$ contains one guard at each of its vertices, then β guards may "jump" from U to W in one turn.*

Proof. Let $U = \{u_1, \cdots, u_\beta\}$ and $W = \{w_1, \cdots, w_\beta\}$ where the vertices of U and W are ordered according to the order in which they appear when going clockwise along the cycle induced by PB_α. Because $\alpha \geq \beta$, and U and W are non-overlapping, there exist vertex-disjoint paths $P_1, \cdots, P_\beta$ such that, for any $1 \leq i \leq \beta$, P_i is a path from u_i to $w_{\beta-i+1}$ whose internal vertices are in TB_α (see Figure 11 for an example with $\alpha = \beta = 4$). Since each vertex in TB_α contains one guard, there is a guard at each vertex of the paths $P_1, \cdots, P_\beta$ except for at the end vertices $w_1, \cdots, w_\beta$. For the guards to jump from U to W, in one turn, for all $1 \leq i \leq \beta$, each guard on each of the vertices of the path P_i moves to its neighbour in the direction of $w_{\beta-i+1}$. $\square$

Theorem 14. *For all $n, m \in \mathbb{N}^*$ such that $m \geq n$,*

$$\gamma(SG_{n\times m}) + \Omega(n+m) \leq \gamma^{*\infty}_{all}(SG_{n\times m}) \leq \gamma(SG_{n\times m}) + O(m\sqrt{n}).$$

Proof. The lower bound simply follows from Theorem 10 and the fact that $\gamma^{*\infty}_{all}(G) \geq \gamma^{\infty}_{all}(G)$ for any graph G.

Let us prove the upper bound. The strategy that we propose follows the same principles as the one of Theorem 9 but the border vertices occupied by several guards are replaced by several layers of vertices, each one occupied by a single guard.

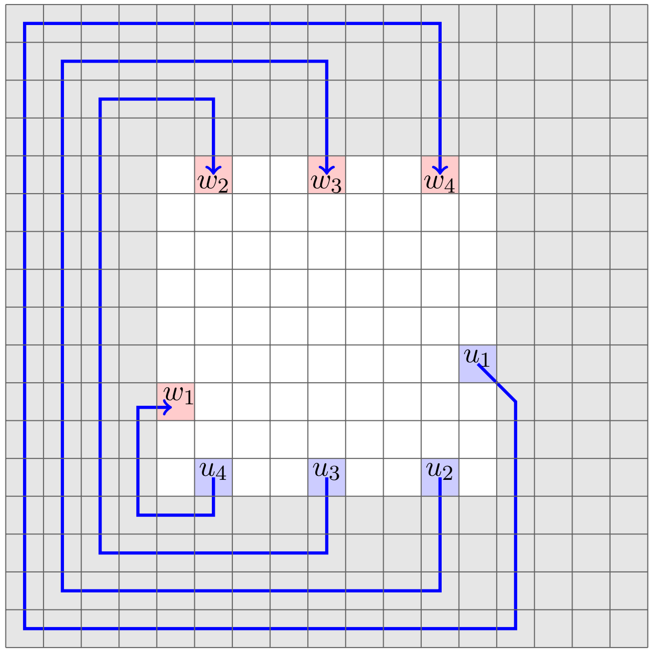

Figure 11: $P_{17} \boxtimes P_{17}$ where the squares are vertices. Example of how the guards jump in Lemma 13. The vertices of $U = \{u_1, u_2, u_3, u_4\}$ are in blue and the vertices of $W = \{w_1, w_2, w_3, w_4\}$ are in red. Note that when $|U| = |W| = \beta = \alpha$. The arrows (in blue) show the vertex-disjoint paths $P_1, \ldots, P_\beta$ that allow the guards to jump from U to W. There is 1 guard at each square in light gray and each vertex of U (in blue), and the white squares contain no guards.

Let k be the integer closest to $\sqrt{n}$ such that $k-2 \mod 3 = 0$. Let $SG_{n^* \times m^*}$ be the remaining subgrid that excludes the first and last (topmost and bottommost resp.) $\frac{k-2}{3}$ columns (rows resp.). As in Theorem 9, we may assume that $n = n^* + 2(\frac{k-2}{3})$ and $m = m^* + 2(\frac{k-2}{3})$ are such that $n^* - 2 \mod 3 = 0$ and $m^* \mod k = 0$. Indeed, otherwise, it is sufficient to "fill" (place one guard at every vertex) at most two rows and at most $k = O(\sqrt{n})$ columns with one guard per vertex (see proof of Theorem 9).

Hence, from now on, let us assume that n and m satisfy $n - 2(\frac{k-2}{3}) - 2 \mod 3 = 0$ and $m - 2(\frac{k-2}{3}) \mod k = 0$.

Instead of there being $\frac{k-2}{3} + 1$ guards occupying each of the border vertices of the grid like in Theorem 9, there is one guard at each vertex of the first $\frac{k-2}{3} + 1$ and last $\frac{k-2}{3} + 1$ columns and rows.

The strategy for the guards remains the same as the strategy used in Theorem 9 except for in the case when a guard or guards have to jump from one vertex to another in which case they move as in Lemma 13 with a small exception. The exception is that one of the paths between a vertex being jumped from and a vertex being jumped to in a block z, may consist of vertices in one of the columns that forms a border of block z. Figure **??** shows an example of a response to a diagonal attack that forces guards to jump and shows that this exception is trivial to deal with. $\square$

6 Further Work

Our results in the strong grid leave the open problem of tightening the bounds. Also, for which other grid graphs can our techniques used in obtaining the upper bound be applied? The technique of considering subgrids where only certain attacks are permitted and packing the borders of these subgrids as well as the entire grid with guards should allow to prove that $\gamma_{all}^{\infty}(G) = \gamma(G) + o(nm)$ for any type of $n \times m$ grid. This should be true since, for all Cayley graphs H obtainable from abelian groups, $\gamma_{all}^{\infty}(H) = \gamma(H)$ [14]. Thus, many grid graphs can be represented as Cayley graphs obtained from abelian groups which are truncated. The technique described above should permit the additional $o(nm)$ guards to suffice due to this truncation. Also, as mentioned in the introduction, it is known that given a graph G and an integer k as inputs and asking whether $\gamma_{all}^{\infty}(G) \leq k$ is NP-hard in general [3] but the exact complexity of the decision problem is still open.

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