Master 2 Advanced Graphs

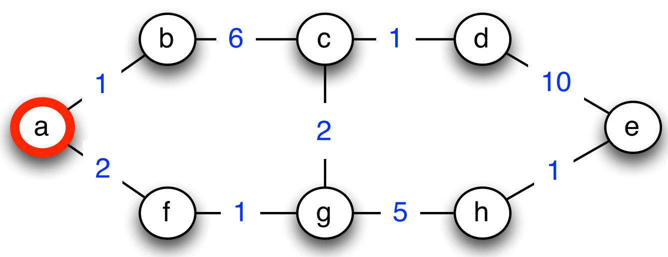
Mid term Exam, November 2020

1 hour 30

Online exam.

Instruction and comments: the points awarded for your answer will be based on the correctness of your answer as well as the clarity of the main steps in your reasoning. All proposed solutions must be proved. **All the exercises are independent.** *The points and durations are indicated so you may adapt your effort.*

Exercise 1 (Dijkstra. (5 points, 15 minutes)) Consider the following graph H.



Give the definition of a shortest path tree rooted in a in the graph H.

Applying the Dijkstra's algorithm on H, compute a shortest path tree rooted in a. Use the representation with a table (as seen during the lecture) to describe the execution of the algorithm.

In particular, indicate the order in which vertices are considered during the execution of the algorithm, and give the final solution.

Exercise 2 (Flow. (5 points, 30 minutes)) Let $\mathcal{N}$ be the network flow defined by a directed graph $D = (V, A)$, with a source $s \in V$ and a target $t \in V$, and an **integral** capacity function $c : A \to \mathbb{N}$ over the arcs.

1. Give the definition of a flow function from s to t in $\mathcal{N}$.
2. Let $f : A \to \mathbb{R}^+$ be a flow, give the definition of the value $v(f)$ of f.
3. Prove that, because the capacities are integers, the maximum value of a flow between s and t is an integer, and there exists a flow $f : A \to \mathbb{N}$ (i.e., $f(a)$ is an integer for all $a \in A$) with maximum value.

As a concrete example, consider the network flow (with source s and target t) and the initial flow f_0 described in Figure 1.

4. Prove that the initial flow f_0 (in red) is actually a flow and give its value.
5. Applying the Ford-Fulkerson's algorithm, **starting from the initial given flow** f_0, compute a maximum flow from s to t. For each iteration of the algorithm, you must give the **auxiliary digraph, the path on which the flow will be increased and the resulting flow after the iteration.**
6. Give a minimum s-t cut and explain why this provides a certificate proving that the flow that you have computed is maximum.

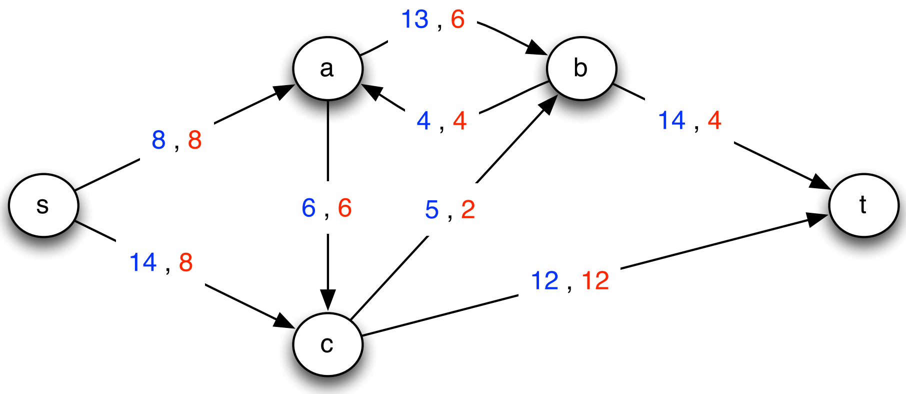


Figure 1: A network flow $\mathcal{N}$ with an initial flow f_0 (in red). The integers in blue denote the arc-capacities. The integers in red denote the value (on each arc) of the initial flow f_0.

Exercise 3 (Modeling as a linear program (5 points, 25 minutes)) *The production manager of a chemical plant is attempting to devise a shift pattern for his workforce. Each day of every working week is divided into three eight-hour shift periods (00:01-08:00, 08:01-16:00, 16:01-24:00) denoted by night, day and late respectively. The plant must be manned at all times and the minimum number of workers required for each of these shifts over any working week is as below:*

	Mon	*Tues*	*Wed*	*Thur*	*Fri*	*Sat*	*Sun*
Night	*5*	*3*	*2*	*4*	*3*	*2*	*2*
Day	*7*	*8*	*9*	*5*	*7*	*2*	*5*
Late	*9*	*10*	*10*	*7*	*11*	*2*	*2*

The union agreement governing acceptable shifts for workers is as follows:

- *Each worker is assigned to work either a night shift or a day shift or a late shift and once a worker has been assigned to a shift they must remain on the same shift every day that they work.*
- *Each worker works four consecutive days during any seven day period.*

In total there are currently 60 workers.

Formulate the production manager's problem as a linear program. In particular, give the meaning of all variables and constraints you will use.

Exercise 4 (Traveling Salesman (5 points, 20 minutes)) A travelling salesman must visit n different locations. For every $1 \leq i, j \leq n$, let $d_{i,j} \geq 0$ denote the distance between locations i and j.

Formulate a linear program to determine a visit sequence starting and ending at location 1, which minimizes the travelled distance. In particular, give the meaning of all variables and constraints you will use.

Hint: A solution corresponds to a cycle visiting exactly once each vertex of the graph whose vertices are the locations and every pair of locations is linked by an edge with length given by d.