University of Oulu
Finland

School on Graph and Optimization
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Exercises : Weighted Graphs

Exercise 1 [Dijkstra]Apply the Dijkstra Algorithm to the graph depicted in Figure 1 to compute a shortest-path tree rooted in a and the distance between any vertex and vertex a. The first three steps of the algorithm must be detailed (**at most three or four lines per steps**). Moreover, indicate the order in which vertices are considered during the execution of the algorithm.

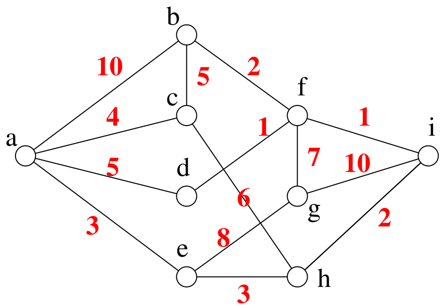

FIGURE 1 – A graph with 9 vertices. A number indicates the length of the arc it is close to.

Exercise 2 [Minimum spanning trees vs. shortest path trees]Let $d > 9$.

Let us consider the following graph $G = (V, E)$ with $d + 1$ nodes $V = \{v_0, v_1, \cdots, v_d\}$ such that, for any $1 \leq i \leq d$, there is an edge with weight/length d from v_0 to v_i, and, for any $1 \leq i < d$, there is an edge with weight/length i between v_i and v_{i+1}. Such a graph is depicted in Figure 2.

1. (2 points) Appy the Kruskal Algorithm on G. Explain how you proceed and what is the result that you obtain.
2. (1.5 points) Give d different minimum spanning trees of G.

   (*hint : consider the choices you had when applying the Kruskal Algorithm*)
3. (3 points) Let T be a spanning tree of G and let us assume that there are $1 \leq i < j \leq d$ such that $\{v_0, v_i\} \in E(T)$, $\{v_0, v_j\} \in E(T)$ and, for any $i < k < j$, $\{v_0, v_k\} \notin E(T)$. Show that T is not a minimum spanning tree of G.
4. (2 points) Prove that there are exactly d distinct minimum spanning trees in G.

   (*hint : use 2) and 3)*)
5. (*) (4 points) Show that no spanning tree of G is a shortest-path tree. That is, for any minimum spanning-tree T of G and for any $v \in V(G)$, T is not a shortest-path tree rooted in v.

   (*hint : the fact that $d > 9$ is important here*).

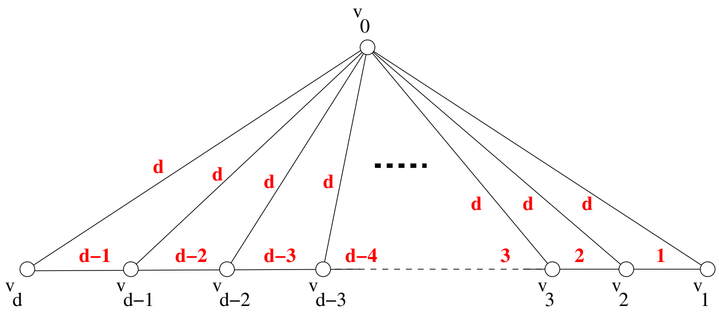

FIGURE 2 – A graph with $d+1$ vertices ($d > 9$). A number indicates the weight/length of the arc it is close to.

Exercise 3 [Modeling a problem as a shortest path problem in graphs] Four imprudent walkers are caught in the storm and nights. To reach the hut, they have to cross a canyon over a fragile rope bridge which can resist the weight of at most two persons. In addition, crossing the bridge requires to carry a torch to avoid to step into a hole. Unfortunately, the walkers have a unique torch and the canyon is too large to throw the torch across it. Due to dizziness and tiredness, the four walkers can cross the bridge in 1, 2, 5 and 10 minutes. When two walkers cross the bridge, they both need the torch and thus cross the bridge at the slowest of the two speeds.

With the help of a graph, find the minimum time for the walkers to cross the bridge.