Graph Theory and Optimization
Examples of (Integer) Linear Programming

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Outline

Linear Programme (reminder)

Linear programmes can be written under the standard form:

$$\begin{array}{llll} \text{Maximize} & \sum_{j=1}^{n} c_j x_j & & \\ \text{Subject to:} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i & & \text{for all } 1 \leq i \leq m \\ & x_j \geq 0 & & \text{for all } 1 \leq j \leq n. \end{array}$$

- the problem is a maximization;
- all constraints are inequalities (and not equations);
- all variables $x_1, \cdots, x_n$ are non-negative.

Linear Programme (Real variables) can be solved in polynomial-time in the number of variables and constraints (e.g., ellipsoid method)

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LP Example: Maximum Flow

$D = (V, A)$ be a graph with capacity $c : A \to \mathbb{R}^+$, and $s, t \in V$.
Problem: Compute a maximum flow from s to t.

Solution: $f : A \to \mathbb{R}^+$ $\Rightarrow$ variables f_a, for each $a \in A$

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Objective function: maximize value of the flow $\sum_{u \in N^+(s)} f(su)$

Constraints:

- capacity constraints: $f(a) \leq c(a)$ for each $a \in A$
- flow conservation: $\sum_{u \in N^+(v)} f(vu) = \sum_{u \in N^-(v)} f(uv), \forall v \in V \setminus \{s,t\}$

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Integer Programme Example: Shortest path

$G = (V, E)$ be a graph with length $\ell : E \to \mathbb{R}^+$, and $s, t \in V$.
Problem: Compute a shortest path from s to t.

Solution: A path P from s to t $\Rightarrow$ variables x_e for each $e \in E$
$x_e = 1$ if $e \in E(P)$, $x_e = 0$ otherwise.

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$$
\begin{array}{llcll}
\text{Minimize} & \sum_{e \in E} \ell(e) x_e & & & \\
\text{Subject to:} & \sum_{u \in N(s)} x(su) & = & 1 & \\
 & \sum_{u \in N(t)} x(tu) & = & 1 & \\
 & \sum_{u \in N(v)} x(vu) & = & 2 & \text{for all } v \in V \setminus \{s,t\} \\
 & x(e) & \in & \{0,1\} & \text{for all } e \in E
\end{array}
$$

Integer Programme Example: Minimum Cut

$G = (V, E)$ be a graph with capacity $c : E \to \mathbb{R}^+$, and $s, t \in V$.
Problem: Compute a minimum s, t-cut

Solution: A partition (S, T) of V with $s \in S$ and $t \in T$

$\Rightarrow$ variables x_v for each $v \in V$

$x_v = 1$ if $v \in S$, $x_v = 0$ otherwise.

$$\text{Minimize} \quad \sum_{\{u,v\} \in E} c(\{u,v\})|x_u - x_v|$$

$$\begin{array}{lrcll} \text{Subject to:} & x_s & = & 1 & \\ & x_t & = & 0 & \\ & x_v & \in & \{0,1\} & \text{for all } v \in V \end{array}$$

Integer Programme Example: Minimum Spanning Tree

$G = (V, E)$ be a graph with weight $w : E \to \mathbb{R}^+$, and $s, t \in V$.
Problem: Compute a minimum spanning tree

Solution: A spanning tree T $\Rightarrow$ variables x_e for each $e \in E$
$x_E = 1$ if $e \in E(T)$, $x_e = 0$ otherwise.

$$\begin{array}{llcll}
\text{Minimize} & \sum_{e \in E} w(e) x_e & & & \\
\text{Subject to:} & \sum_{e=\{u,v\} \in E, u \in S, v \notin S} x_e & \geq & 1 & \text{for all } S \subseteq V \\
 & x_e & \in & \{0,1\} & \text{for all } e \in E
\end{array}$$

Remark: The number of constraints is exponential

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(Integer) Linear Programme Example: Exercises

$G = (V, E)$ be a graph with weight $w : E \rightarrow \mathbb{R}^+$, and $s, t \in V$.
What compute the following programmes?

$$\begin{array}{llcll} \text{Maximize} & \sum_{P \text{ path from s to t}} x_P & & & \\ \text{Subject to:} & \sum_{P, e \in E(P)} x_P & \leq & w(e) & \text{for all } e \in E \\ & x_P & \in & \{0,1\} & \text{for all paths } P \text{ from s to t} \end{array}$$

$$\begin{array}{llcll} \text{Maximize} & x_t & & & \\ \text{Subject to:} & x_s & = & 0 & \\ & x_v & \leq & x_u + w(\{u,v\}) & \text{for all } \{v,u\} \in E \\ & x_v & \geq & 0 & \text{for all } v \in V \end{array}$$

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Integer Programme Example: Maximum Matching

$G = (V, E)$ be a graph
Problem: Compute a maximum matching

Solution: a set $M \subseteq E$ of pairwise disjoint edges

$\Rightarrow$ variables x_e for each $e \in E$

$x_e = 1$ if $e \in M$, $x_e = 0$ otherwise.

$$
\begin{array}{llcll}
\text{Maximize} & \sum_{e \in E} x_e & & & \\
\text{Subject to:} & \sum_{e \in E, v \in e} x_e & \leq & 1 & \text{for all } v \in V \\
& x_e & \in & \{0,1\} & \text{for all } e \in E
\end{array}
$$