Graph Theory and Optimization
Introduction on Linear Programming

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Thank you to F. Giroire for his slides

Outline

Motivation

Why linear programming is a very important tool?

- A lot of problems can be formulated as linear programmes, and
- There exist efficient methods to solve them
- or at least give good approximations.

- Solve difficult problems: e.g. original example given by Dantzig (1947). Best assignment of 70 people to 70 tasks.

$\rightarrow$ Magic algorithmic box.

Outline

What is a linear programme?

- Optimization problem consisting in
  - maximizing (or minimizing) a linear objective function
  - of *n* decision variables
  - subject to a set of constraints expressed by linear equations or inequalities.
- Originally, military context: "programme"="resource planning". Now "programme"="problem"
- Terminology due to George B. Dantzig, inventor of the Simplex Algorithm (1947)

Terminology

x_1, x_2 — Decision variables (generally: $\in \mathbb{R}$)

$$\max \quad 350x_1 + 300x_2$$

Objective function (linear!!)

subject to

$$\begin{aligned} x_1 + x_2 &\leq 200 \\ 9x_1 + 6x_2 &\leq 1566 \\ 12x_1 + 16x_2 &\leq 2880 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Constraints (linear!!)

Terminology

x_1, x_2 — Decision variables

$$\begin{array}{rcl} \max & 350x_1 + 300x_2 & \text{Objective function} \\ \text{subject to} & & \\ & x_1 + x_2 \leq 200 & \text{Constraints} \\ & 9x_1 + 6x_2 \leq 1566 & \\ & 12x_1 + 16x_2 \leq 2880 & \\ & x_1, x_2 \geq 0 & \end{array}$$

In linear programme: objective function + constraints are all linear

Typically (not always): variables are non-negative

If variables are integer: system called Integer Programme (IP)

Terminology

Linear programmes can be written under the standard form:

$$
\begin{array}{lrcll}
\text{Maximize} & \sum_{j=1}^{n} c_j x_j & & & \\
\text{Subject to:} & \sum_{j=1}^{n} a_{ij} x_j & \leq & b_i & \text{for all } 1 \leq i \leq m \\
 & x_j & \geq & 0 & \text{for all } 1 \leq j \leq n.
\end{array}
$$

- the problem is a maximization;
- all constraints are inequalities (and not equations);
- all variables are non-negative.

Outline

Example 1: a resource allocation problem

A company produces copper cable of 5 and 10 mm of diameter on a single production line with the following constraints:

- The available copper allows to produces 21 meters of cable of 5 mm diameter per week. Moreover, one meter of 10 mm diameter copper consumes 4 times more copper than a meter of 5 mm diameter copper.
- Due to demand, the weekly production of 5 mm cable is limited to 15 meters and the production of 10 mm cable should not exceed 40% of the total production.
- Cable are respectively sold 50 and 200 euros the meter.

What should the company produce in order to maximize its weekly revenue?

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Example 1: a resource allocation problem

Define two decision variables:

- x_1: the number of meters of 5 mm cables produced every week
- x_2: the number of meters of 10 mm cables produced every week

The revenue associated to a production (x_1, x_2) is

$$z = 50x_1 + 200x_2.$$

The capacity of production cannot be exceeded

$$x_1 + 4x_2 \leq 21.$$

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Example 1: a resource allocation problem

The demand constraints have to be satisfied

$$x_2 \leq \frac{4}{10}(x_1 + x_2)$$

$$x_1 \leq 15$$

Negative quantities cannot be produced

$$x1 \geq 0, x2 \geq 0.$$

Exercise: Write the above programme in standard form

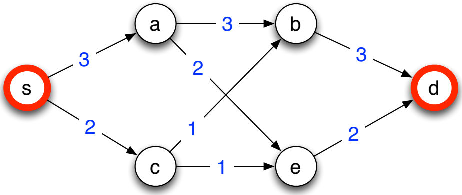
Example 1: a resource allocation problem

The model: To maximize the sell revenue, determine the solutions of the following linear programme x_1 and x_2:

$$
\begin{array}{rc}
\max & z = 50x_1 + 200x_2 \\
\text{subject to} & \\
 & x_1 + 4x_2 \leq 21 \\
 & -4x_1 + 6x_2 \leq 0 \\
 & x_1 \leq 15 \\
 & x_1, x_2 \geq 0
\end{array}
$$

Example 2: Maximum flow (Reminder on the Problem)

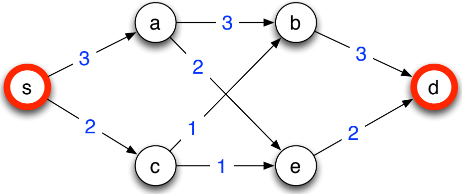
Directed graph: $D = (V, A)$, **source** $s \in V$, **destination** $d \in V$, capacity $c : A \to \mathbb{R}^+$.
$N^-(s) = \emptyset$ and $N^+(d) = \emptyset$



flow $f : A \to \mathbb{R}^+$ such that :

- capacity constraint: $\forall a \in A$, $f(a) \leq c(a)$
- conservation constraint: $\forall v \in V \setminus \{s, d\}$, $\sum_{w \in N^-(v)} f(wv) = \sum_{w \in N^+(v)} f(vw)$
- value of flow: $v(f) = \sum_{w \in N^+(s)} f(sw)$.

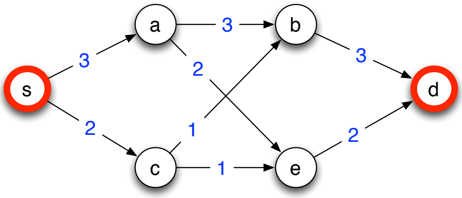
Example 2: Maximum flow (on an example)



Exercise: Give a LP computing a maximum flow in the above graph

hint: variables correspond to the expected solution

Example 2: Maximum flow (on an example)



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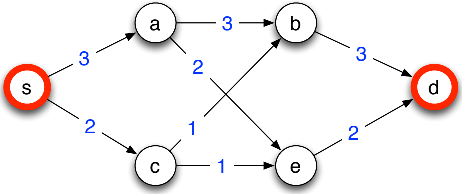
Solution: flow $f : A \rightarrow \mathbb{R}^+$

Variables: $f_x \in \mathbb{R}^+$ for each $x \in A$

Example 2: Maximum flow (on an example)



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Solution: flow $f : A \rightarrow \mathbb{R}^+$

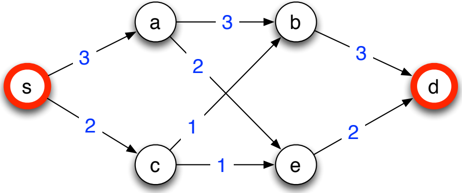
Objective: maximize the flow leaving s

subject to:

Variables: $f_x \in \mathbb{R}^+$ for each $x \in A$

Max. $f_{sa} + f_{sc}$

Example 2: Maximum flow (on an example)



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hint: variables correspond to the expected solution

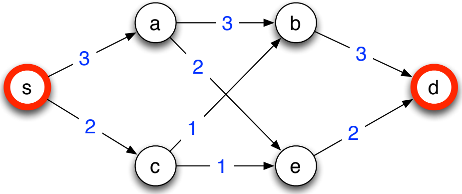
Solution: flow $f : A \to \mathbb{R}^+$ Variables: $f_x \in \mathbb{R}^+$ for each $x \in A$

Objective: maximize the flow leaving s Max. $f_{sa} + f_{sc}$

subject to:

Capacity constraints: $f_{sa} \leq 3$; $f_{sc} \leq 2$; $f_{ab} \leq 3$; $f_{ae} \leq 2$; $f_{cb} \leq 1$; $f_{ce} \leq 1$; $f_{bd} \leq 3$; $f_{ed} \leq 2$.

Example 2: Maximum flow (on an example)



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Conservation constraints: $f_{sa} = f_{ab} + f_{ae}$; $f_{sc} = f_{cb} + f_{ce}$; $f_{ae} + f_{ce} = f_{ed}$ and $f_{ab} + f_{cb} = f_{bd}$.

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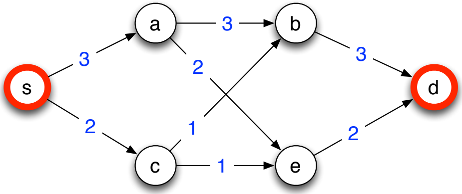

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Variables domain: $f_x \geq 0$ for any $x \in A$

Example 2: Maximum flow

$D = (V, A)$ be a graph with capacity $c : A \rightarrow \mathbb{R}^+$, and $s, t \in V$.
Problem: Compute a maximum flow from s to t.

Solution: $f : A \rightarrow \mathbb{R}^+$ $\Rightarrow$ variables f_a, for each $a \in A$

Objective function: maximize value of the flow $\sum_{u \in N^+(s)} f(su)$

Constraints:

- capacity constraints: $f(a) \leq c(a)$ for each $a \in A$
- flow conservation: $\sum_{u \in N^+(v)} f(vu) = \sum_{u \in N^-(v)} f(uv), \forall v \in V \setminus \{s, t\}$

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& \displaystyle\sum_{u \in N^+(v)} f(vu) & = & \displaystyle\sum_{u \in N^-(v)} f(uv) & \text{for all } v \in V \setminus \{s,t\} \\
& f(a) & \geq & 0 & \text{for all } a \in A
\end{array}
$$

Outline

Solving Difficult Problems

- Difficulty: Large number of solutions.
  - Choose the best solution among 2^n or $n!$ possibilities: all solutions cannot be enumerated.
  - Complexity of studied problems: often NP-complete.
    but Polynomial-time solvable when variables are real !!
- Solving methods:
  - Optimal solutions:
    - Graphical method (2 variables only).
    - Simplex method. exponential-time, work well in practice
    - interior point method polynomial-time
    - Ellipsoid polynomial-time
  - Approximations:
    - Theory of duality (assert the quality of a solution).
    - Approximation algorithms.

Graphical Method

- The constraints of a linear programme define a zone of solutions.
- The best point of the zone corresponds to the optimal solution.
- For problem with 2 variables, easy to draw the zone of solutions and to find the optimal solution graphically.

Graphical Method

Example:

$$
\begin{aligned}
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\text{subject to} \quad & \\
& x_1 + x_2 \leq 200 \\
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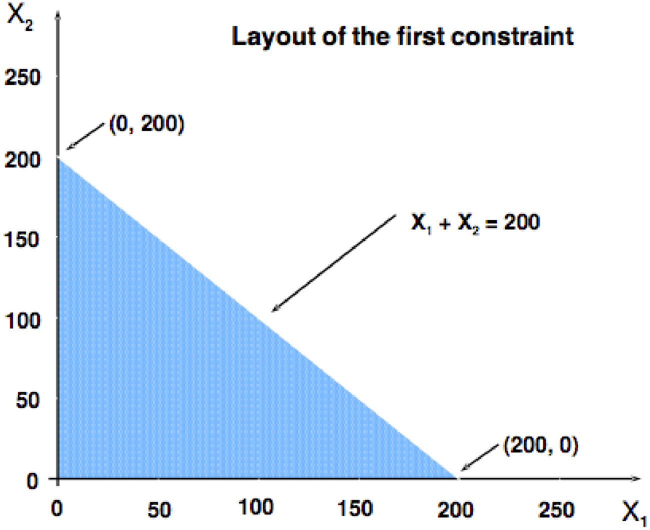
Graphical Method



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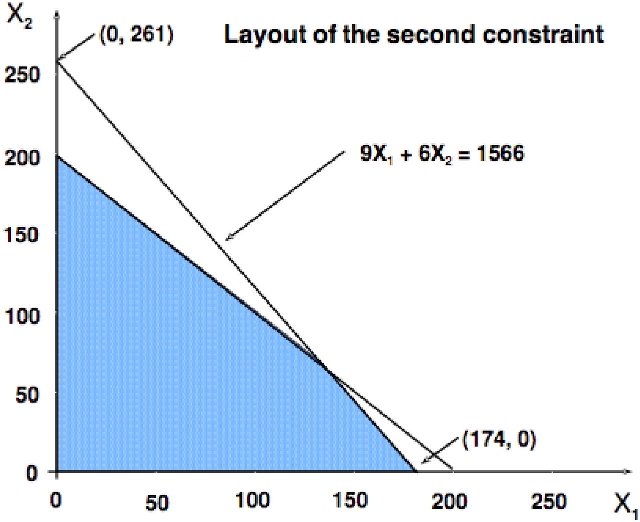

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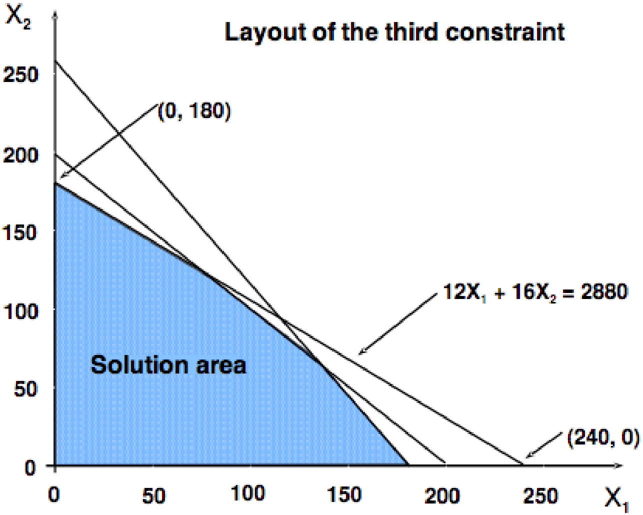

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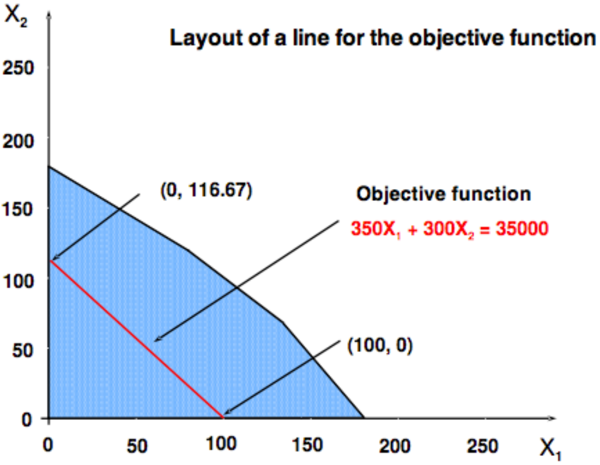

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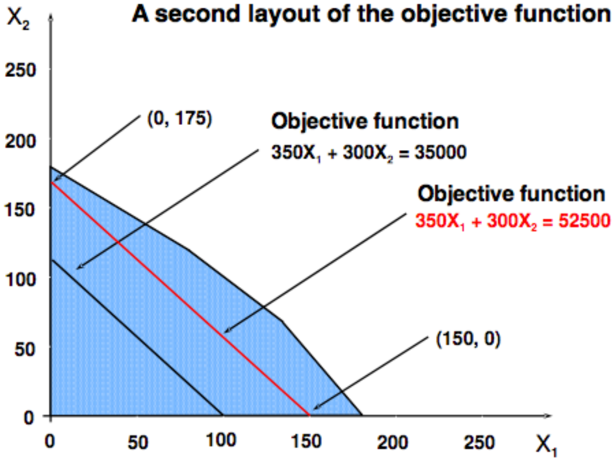

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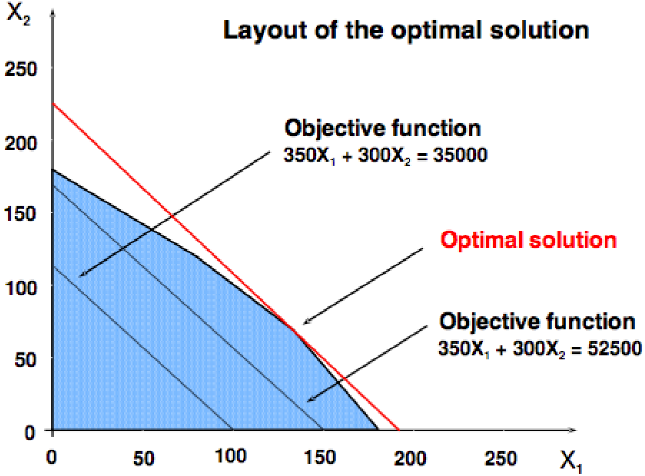


Computation of the optimal solution

The optimal solution is at the intersection of the constraints:

$$x_1 + x_2 = 200$$

$$9x_1 + 6x_2 = 1566$$

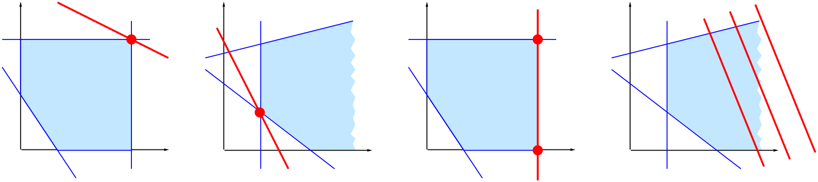
We get:

$$x_1 = 122$$

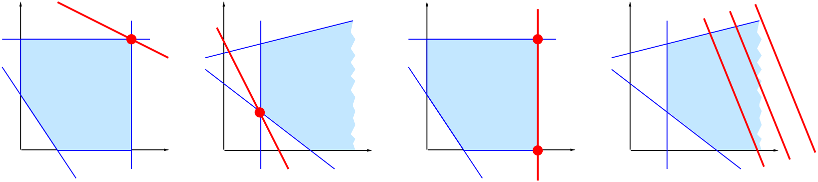
$$x_2 = 78$$

$$\text{Objective} = 66100.$$

Optimal Solutions: Different Cases



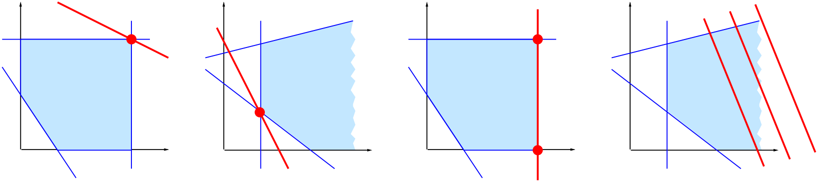
Optimal Solutions: Different Cases



Three different possible cases:

- a single optimal solution,
- an infinite number of optimal solutions, or
- no optimal solutions.

Optimal Solutions: Different Cases



Three different possible cases:

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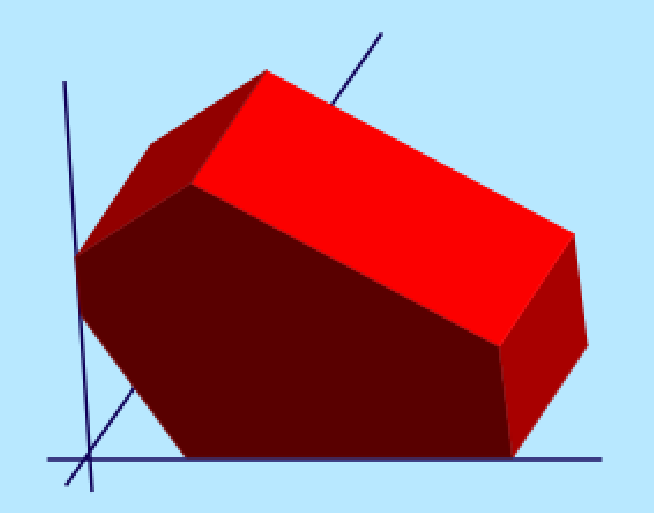
- no optimal solutions.

If an optimal solution exists, there is always a corner point optimal solution!

Solving Linear Programmes



Solving Linear Programmes

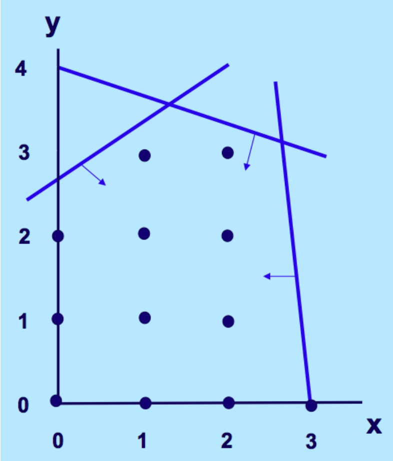
- The constraints of an LP give rise to a geometrical shape: a convex polyhedron.
- If we can determine all the corner points of the polyhedron, then we calculate the objective function at these points and take the best one as our optimal solution.
- The Simplex Method intelligently moves from corner to corner until it can prove that it has found the optimal solution.

Solving Linear Programmes

- Geometric method impossible in higher dimensions
- Algebraical methods:
  - Simplex method (George B. Dantzig 1949): skim through the feasible solution polytope.
    Similar to a "Gaussian elimination".
    Very good in practice, but can take an exponential time.
  - Polynomial methods exist:
    - Leonid Khachiyan 1979: ellipsoid method. But more theoretical than practical.
    - Narendra Karmarkar 1984: a new interior method. Can be used in practice.

But Integer Programming (IP) is different!

- Feasible region: a set of discrete points.
- Corner point solution not assured.
- No "efficient" way to solve an IP.
- Solving it as an LP provides a relaxation and a bound on the solution.



Summary: To be remembered

- What is a linear programme.
- The graphical method of resolution.
- Linear programs can be solved efficiently (polynomial).
- Integer programs are a lot harder (in general no known polynomial algorithms).
  In this case, we look for approximate solutions.