Graph Theory and Optimization
Introduction on Linear Programming

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October 2018

Thank you to F. Giroire for his slides
Outline

1. Motivations
2. Linear Programmes
3. First examples
4. Solving Methods: Graphical method, simplex...
Motivation

Why linear programming is a very important tool?

- A lot of problems can be formulated as linear programmes, and
- There exist efficient methods to solve them
- or at least give good approximations.

- Solve difficult problems: e.g. original example given by Dantzig (1947). Best assignment of 70 people to 70 tasks.

→ Magic algorithmic box.
What is a linear programme?

- **Optimization problem** consisting in
  - maximizing (or minimizing) a linear objective function
  - of \( n \) decision variables
  - subject to a set of constraints expressed by linear equations or inequalities.

- Originally, military context: "programme"="resource planning". Now "programme"="problem"

- Terminology due to George B. Dantzig, inventor of the Simplex Algorithm (1947)
Motivations
Linear Programmes
First examples
Solving Methods: Graphical method, simplex...

Terminology

\( x_1, x_2 \)

Decision variables (generally: \( \in \mathbb{R} \))

\[
\begin{align*}
\text{max} & \quad 350x_1 + 300x_2 \\
\text{subject to} & \quad x_1 + x_2 \leq 200 \\
& \quad 9x_1 + 6x_2 \leq 1566 \\
& \quad 12x_1 + 16x_2 \leq 2880 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

Objective function (linear!!)

Constraints (linear!!)
### Terminology

Decision variables:

\[ x_1, x_2 \]

Objective function:

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In linear programme: **objective function + constraints are all linear**

Typically (not always): **variables are non-negative**

If variables are integer: system called **Integer Programme (IP)**
Terminology

Linear programmes can be written under the standard form:

Maximize \[ \sum_{j=1}^{n} c_j x_j \]

Subject to: \[ \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \text{for all } 1 \leq i \leq m \]

\[ x_j \geq 0 \quad \text{for all } 1 \leq j \leq n. \]

- the problem is a maximization;
- all constraints are inequalities (and not equations);
- all variables are non-negative.
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Example 1: a resource allocation problem

A company produces copper cable of 5 and 10 mm of diameter on a single production line with the following constraints:

- The available copper allows to produces 21 meters of cable of 5 mm diameter per week. Moreover, one meter of 10 mm diameter copper consumes 4 times more copper than a meter of 5 mm diameter copper.
- Due to demand, the weekly production of 5 mm cable is limited to 15 meters and the production of 10 mm cable should not exceed 40% of the total production.
- Cable are respectively sold 50 and 200 euros the meter.

What should the company produce in order to maximize its weekly revenue?
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Example 1: a resource allocation problem

Define two decision variables:

- $x_1$: the number of meters of 5 mm cables produced every week
- $x_2$: the number of meters of 10 mm cables produced every week

The revenue associated to a production $(x_1, x_2)$ is

$$z = 50x_1 + 200x_2.$$ 

The capacity of production cannot be exceeded

$$x_1 + 4x_2 \leq 21.$$
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Example 1: a resource allocation problem

The demand constraints have to be satisfied

\[ x_2 \leq \frac{4}{10}(x_1 + x_2) \]

\[ x_1 \leq 15 \]

Negative quantities cannot be produced

\[ x_1 \geq 0, x_2 \geq 0. \]

**Exercise:** Write the above programme in standard form
Example 1: a resource allocation problem

The model: To maximize the sell revenue, determine the solutions of the following linear programme $x_1$ and $x_2$:

\[
\begin{align*}
\text{max} \quad z &= 50x_1 + 200x_2 \\
\text{subject to} \quad x_1 + 4x_2 &\leq 21 \\
-4x_1 + 6x_2 &\leq 0 \\
\quad x_1 &\leq 15 \\
\quad x_1, x_2 &\geq 0
\end{align*}
\]
Example 2: Maximum flow  (Reminder on the Problem)

Directed graph: \( D = (V, A) \), \textbf{source} \( s \in V \), \textbf{destination} \( d \in V \), \textbf{capacity} \( c : A \rightarrow \mathbb{R}^+ \).

\( N^{-}(s) = \emptyset \) and \( N^{+}(d) = \emptyset \)

![Graph diagram](image)

flow \( f : A \rightarrow \mathbb{R}^+ \) such that:

- capacity constraint: \( \forall a \in A, f(a) \leq c(a) \)
- conservation constraint: \( \forall v \in V \setminus \{s, d\}, \sum_{w \in N^{-}(v)} f(wv) = \sum_{w \in N^{+}(v)} f(vw) \)
- value of flow: \( v(f) = \sum_{w \in N^{+}(s)} f(sw) \).
Example 2: Maximum flow (on an example)

Exercise: Give a LP computing a maximum flow in the above graph

*hint: variables correspond to the expected solution*
Example 2: Maximum flow  
(on an example)

Exercise: Give a LP computing a maximum flow in the above graph  
\textit{hint: variables correspond to the expected solution}

Solution: flow $f : A \to \mathbb{R}^+$  
Variables: $f_x \in \mathbb{R}^+$ for each $x \in A$
Example 2: Maximum flow (on an example)

Exercise: Give a LP computing a maximum flow in the above graph

hint: variables correspond to the expected solution

Solution: flow $f : A \rightarrow \mathbb{R}^+$
Objective: maximize the flow leaving $s$

subject to:

Variables: $f_x \in \mathbb{R}^+$ for each $x \in A$
Max. $f_{sa} + f_{sc}$
Example 2: Maximum flow
(on an example)

Exercise: Give a LP computing a maximum flow in the above graph

hint: variables correspond to the expected solution

Solution: flow $f : A \rightarrow \mathbb{R}^+$

Objective: maximize the flow leaving $s$

subject to:

Capacity constraints: $f_{sa} \leq 3; f_{sc} \leq 2; f_{ab} \leq 3; f_{ae} \leq 2; f_{cb} \leq 1; f_{ce} \leq 1; f_{bd} \leq 3; f_{ed} \leq 2$. 

Variables: $f_x \in \mathbb{R}^+$ for each $x \in A$

Max. $f_{sa} + f_{sc}$
Example 2: Maximum flow (on an example)

Exercise: Give a LP computing a maximum flow in the above graph

**hint:** variables correspond to the expected solution

Solution: flow \( f : A \rightarrow \mathbb{R}^+ \)

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subject to:

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**Conservation constraints:** \( f_{sa} = f_{ab} + f_{ae}; f_{sc} = f_{cb} + f_{ce}; f_{ae} + f_{ce} = f_{ed} \) and \( f_{ab} + f_{cb} = f_{bd}. \)
Example 2: Maximum flow (on an example)

Exercise: Give a LP computing a maximum flow in the above graph

hint: variables correspond to the expected solution

Solution: flow $f : A \rightarrow \mathbb{R}^+$

Variables: $f_x \in \mathbb{R}^+$ for each $x \in A$

Max. $f_{sa} + f_{sc}$

Objective: maximize the flow leaving $s$

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Conservation constraints: $f_{sa} = f_{ab} + f_{ae}; f_{sc} = f_{cb} + f_{ce}; f_{ae} + f_{ce} = f_{ed}$ and $f_{ab} + f_{cb} = f_{bd}$.

Variables domain:

$f_x \geq 0$ for any $x \in A$
Example 2: Maximum flow

\( D = (V, A) \) be a graph with capacity \( c : A \rightarrow \mathbb{R}^+ \), and \( s, t \in V \).

**Problem:** Compute a maximum flow from \( s \) to \( t \).

**Solution:** \( f : A \rightarrow \mathbb{R}^+ \)

**Objective function:** maximize value of the flow

**Constraints:**

- capacity constraints:
  \[ f(a) \leq c(a) \text{ for each } a \in A \]

- flow conservation:
  \[ \sum_{u \in N^+(v)} f(vu) = \sum_{u \in N^-(v)} f(uv), \forall v \in V \setminus \{s, t\} \]
Example 2: Maximum flow

\( D = (V, A) \) be a graph with capacity \( c : A \rightarrow \mathbb{R}^+ \), and \( s, t \in V \).

**Problem:** Compute a maximum flow from \( s \) to \( t \).

<table>
<thead>
<tr>
<th>Maximize</th>
<th>[ \sum_{u \in N^+(s)} f(su) ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subject to:</td>
<td>[ f(a) \leq c(a) \quad \text{for all } a \in A ]</td>
</tr>
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<td></td>
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Solving Difficult Problems

• **Difficulty:** Large number of solutions.
  - Choose the best solution among $2^n$ or $n!$ possibilities: all solutions cannot be enumerated.
  - Complexity of studied problems: often NP-complete.
    - but Polynomial-time solvable when variables are real !!

• **Solving methods:**
  - Optimal solutions:
    - Graphical method (2 variables only).
    - Simplex method. exponential-time, work well in practice
    - interior point method polynomial-time
    - Ellipsoid polynomial-time
  - Approximations:
    - Theory of duality (assert the quality of a solution).
    - Approximation algorithms.
The constraints of a linear programme define a zone of solutions.

The best point of the zone corresponds to the optimal solution.

For problem with 2 variables, easy to draw the zone of solutions and to find the optimal solution graphically.
Example:

$$\begin{align*}
\text{max} & \quad 350x_1 + 300x_2 \\
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& \quad 9x_1 + 6x_2 \leq 1566 \\
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\end{align*}$$
Graphical Method

Layout of the first constraint

\[ X_1 + X_2 = 200 \]

Graphical Method

Motivations

Linear Programmes

First examples

Solving Methods: Graphical method, simplex...
Graphical Method

Layout of the second constraint

\[ 9X_1 + 6X_2 = 1566 \]

Points:
- \((0, 261)\)
- \((174, 0)\)
Graphical Method
Graphical Method

Layout of a line for the objective function

Objective function

\[ 350X_1 + 300X_2 = 35000 \]

Points:

- \((0, 116.67)\)
- \((100, 0)\)
Graphical Method

A second layout of the objective function

Objective function

$350X_1 + 300X_2 = 35000$

$(0, 175)$

Objective function

$350X_1 + 300X_2 = 52500$

$(150, 0)$
Graphical Method

Objective function
350X₁ + 300X₂ = 35000

Optimal solution

Objective function
350X₁ + 300X₂ = 52500
Computation of the optimal solution

The optimal solution is at the intersection of the constraints:

\[ x_1 + x_2 = 200 \]

\[ 9x_1 + 6x_2 = 1566 \]

We get:

\[ x_1 = 122 \]

\[ x_2 = 78 \]

Objective \( = 66100 \).
Optimal Solutions: Different Cases
Optimal Solutions: Different Cases

Three different possible cases:

- a single optimal solution,
- an infinite number of optimal solutions, or
- no optimal solutions.
Optimal Solutions: Different Cases

Three different possible cases:

- a single optimal solution,
- an infinite number of optimal solutions, or
- no optimal solutions.

If an optimal solution exists, there is always a corner point optimal solution!
Solving Linear Programmes
Solving Linear Programmes

• The constraints of an LP give rise to a geometrical shape: a convex polyhedron.

• If we can determine all the corner points of the polyhedron, then we calculate the objective function at these points and take the best one as our optimal solution.

• The Simplex Method intelligently moves from corner to corner until it can prove that it has found the optimal solution.
Solving Linear Programmes

- Geometric method impossible in higher dimensions
- Algebraical methods:
  - **Simplex method** (George B. Dantzig 1949): skim through the feasible solution polytope.
    Similar to a "Gaussian elimination".
    Very good in practice, but can take an exponential time.
  - **Polynomial methods** exist:
But Integer Programming (IP) is different!

- Feasible region: a set of discrete points.
- Corner point solution not assured.
- No "efficient" way to solve an IP.
- Solving it as an LP provides a relaxation and a bound on the solution.
Summary: To be remembered

- What is a linear programme.
- The graphical method of resolution.
- Linear programs can be solved efficiently (polynomial).
- Integer programs are a lot harder (in general no known polynomial algorithms).
  In this case, we look for approximate solutions.