Master 2 Informatique et Interactions Advanced graphs

First Exam, December 2021

2 hours.

No documents are allowed. No computers, cellphones.

Instruction and comments: the points awarded for your answer will be based on the correctness of your answer as well as the clarity of the main steps in your reasoning. All proposed solutions must be proved. **All the exercises are independent.** *The points are indicated so you may adapt your effort.*

Exercise 1 (Dijkstra. 3 points, 15 minutes) Consider the graph H depicted in Figure 1.

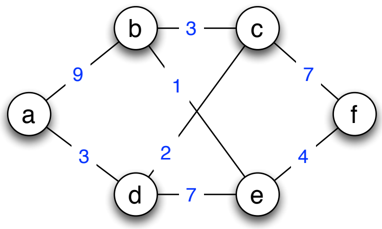

Figure 1: The weighted graph H. Integers on edges represent their length.

1. Give the definition of a shortest-path tree rooted in a.
2. Apply the Dijkstra algorithm on H to compute a shortest-path tree rooted in a and the distance between any vertex and the vertex a.

   Explain the execution of the algorithm (e.g., with a table as seen during the lecture). In particular, indicate the order in which vertices are considered during the execution of the algorithm.
3. Give the obtained shortest-path tree rooted in a.

Exercise 2 (Flow. 5 points, 20 minutes) Consider the elementary network flow N depicted in Figure 2 (left) and the initial flow f from s to t in Figure 2 (right).

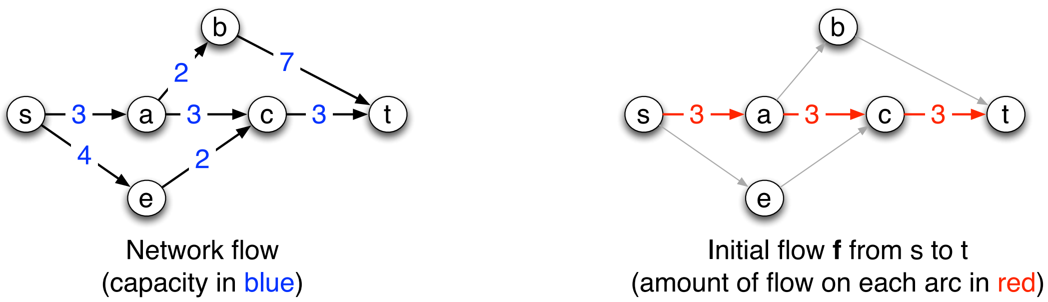

Figure 2: (left) Elementary network flow with arcs' capacity. (right) An initial flow f from s to t: a number close to an arc indicates the amount of flow along it.

1. What must be checked to show that f is a flow? What is the value of the initial flow f?
2. Apply the Ford-Fulkerson Algorithm to N starting from the flow f.

   For each iteration, draw the auxiliary graph, give the chosen path and the amount of flow that you will push, and draw the network with the new flow.
3. What is the final value of the flow? How to be sure that it is a maximum flow?

Exercise 3 (Modelization using Linear Programme. 4 points, 20 minutes)

1. Given a graph $G = (V, E)$ with a weight function $w : E \to \mathbb{R}^+$, a set $S \subseteq V$ of vertices is **stable** if no two vertices are adjacent (i.e., the vertices of S are pairwise non adjacent).

   Give an integer linear programme that computes a stable set with maximum weight. You must explain the meaning of the variables and of the constraints.

2. Let $G = (V, E)$ be an undirected graph, $w : E \to \mathbb{R}^+$, $x, y \in V$. What does the following programme compute? Explain the meaning of the variables and of the constraints.

$$\begin{array}{ll} \text{Minimize} & \sum_{\{i,j\}\in E} w(ij)e_{ij} \\ \text{Subject to:} & \\ & \begin{array}{rl} q_x + q_y = 1 & \\ q_i - q_j \leq e_{ij} & \forall \{i,j\} \in E \\ q_j - q_i \leq e_{ij} & \forall \{i,j\} \in E \\ q_v \in \{0,1\} & \forall v \in V \\ e_{ij} \geq 0 & \forall \{i,j\} \in E \end{array} \end{array}$$

Exercise 4 (Application of flows: Menger's theorem. 8 points, 60 minutes)
Let $D = (V, A)$ be a directed graph, $s, t \in V$ such that s is a source and t is a sink, and assume that every arc has capacity 1. Recall that two paths are arc-disjoint (resp., vertex-disjoint) if they do not share any arc (resp., vertex).

1. Explain briefly why there exists a maximum s-t-flow such that $f(a) \in \mathbb{N}$ for all $a \in A$.

2. On the digraph H of Figure 3 (left), draw an integral s-t-flow of value 3. Why is it maximum? Then give the "corresponding" three edge-disjoint directed paths from s to t in H.

3. Show that, there exists a s-t-flow of value at least 1, if and only if there exists a directed path from s to t in D.

4. Show that, there exists an integral s-t-flow of value $k \geq 1$, if and only if there exists k arc-disjoint directed paths from s to t in D. *Hint: induction on k*

5. Prove that the maximum number of arc-disjoint directed paths from s to t in D equals the minimum number of arcs that must be removed from D to disconnect s and t.

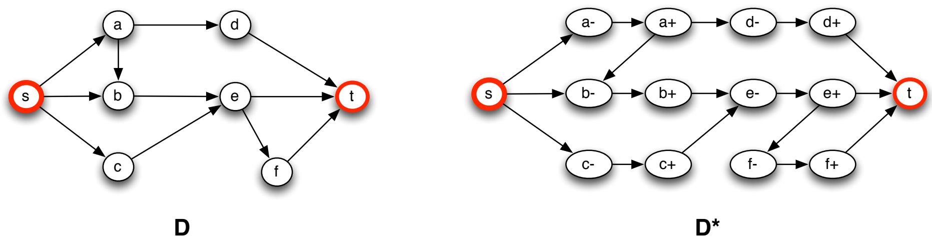

Figure 3: A digraph H (left) and the corresponding digraph H^* (right).

Let D^* be the graph obtained from D as follows. For each vertex $v \in V \setminus \{s, t\}$, replace the vertex v by two vertices v^- and v^+ such that there is an arc (v^-, v^+) and, for every arc $(a, b) \in A$ of D, add in D^* an arc (a^+, b^-). (We note $s = s^+$ and $t = t^-$). All arcs have capacity one. See an example in Fig. 3.

6. On the digraph H^* of Figure 3 (right), draw an integral s-t-flow of value 2. Why is it maximum? Then give the "corresponding" two vertex-disjoint directed paths from s to t in H (not in H^* !!).

7. Show that, there exists an integral s-t-flow of value $k \geq 1$ in D^*, if and only if there exists k vertex-disjoint directed paths from s to t in D.

8. **Menger's theorem:** Prove that the maximum number of vertex-disjoint directed paths from s to t in D equals the minimum number of vertices that must be removed from D to disconnect s and t.