Graphs :

2 hours

Handwritten documents are allowed, no more. You may answer in french if you prefer. Each of your answers must be explained/detailled.

1 Notations and definitions

Let $G = (V, E)$ be a graph, where V is the set of vertices and E is the set of edges. For any $v \in V$, let $N(v) = \{w \in V \mid \{v, w\} \in E\}$ be the set of neighbours of v.

A **matching** in G is a set $M \subseteq E$ of pairwise disjoint edges, i.e., for every $e, f \in M$, $e \cap f = \emptyset$. Note that $\emptyset$ is a matching in G. Let $\mu(G)$ be the maximum size of a matching in G.

A **vertex cover** in G is a set $K \subseteq V$ of vertices that "touch" all edges, i.e., for every $e \in E$, there exists $v \in K$ such that $v \in e$. Note that V is a vertex cover in G. Let $vc(G)$ be the minimum size of a vertex cover in G.

An **independent set** in G is a set $I \subseteq V$ of vertices that are pairwise non-adjacent, i.e., for every $u, v \in I$, $\{u, v\} \notin E$. Note that $\emptyset$ is an independent set in G. Let $\alpha(G)$ be the maximum size of an independent set in G.

A graph $G = (V, E)$ is **bipartite** if its vertex-set can be partionned into two independent sets, i.e., $V = A \cup B$, $A \cap B = \emptyset$ such that A is an independent set and B is an independent set.

Let $n, m \in \mathbb{N}^*$ be two non-zero integers. The $n \times m$ **grid** $G_{n \times m}$ is the graph defined as follows. Its vertex set is $V(G_{n\times m}) = \{(i, j) \mid 1 \leq i \leq n, 1 \leq j \leq m\}$ and two vertices (i, j) and (i', j') are adjacent if and only if $|i - i'| + |j - j'| = 1$. See Figure 1 for various examples.

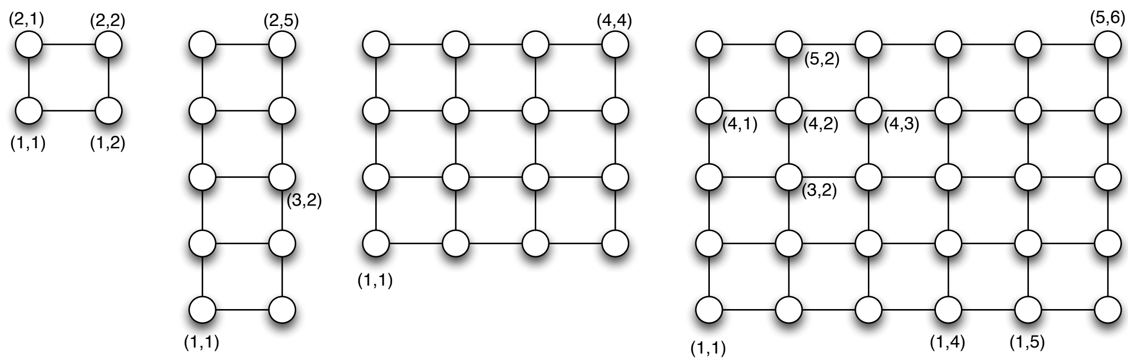

FIGURE 1 – From left to right, the grids $G_{2\times 2}$, $G_{5\times 2}$, $G_{4\times 4}$ and $G_{5\times 6}$

Question 1 *Prove that the 4 grids in Figure 1 are bipartite. To do so, describe a possible partition (A, B) of the vertex-set into two independent sets by coloring the vertices.*

Question 2 *Prove that the grid $G_{n\times m}$ is bipartite for any $n, m \in \mathbb{N}^*$ (Describe a possible partition of the vertices into two independent sets).*

The goal of the following is to revise the basics on graphs and algorithms and, more precisely, to compute $\alpha(G_{n\times m})$ for every $n, m \in \mathbb{N}^*$.

2 Maximum matching in bipartite graphs

Let $G = (V, E)$ be a graph and $M \subseteq E$ be a matching. A vertex $v \in V$ is **covered** by M if it is "touched" by some edge of M, i.e., if there exists $e \in M$ such that $v \in e$. Otherwise, v is said **uncovered by** M. An M**-augmenting path** is a path $(v_1, \cdots, v_p)$ $(p \geq 2)$ such that v_1 and v_p are uncovered by M and, for every $1 \leq i \leq \lfloor p/2 \rfloor$, $\{v_{2i}, v_{2i+1}\} \in M$.

Question 3 *Prove that if there exists an M-augmenting path, then the matching M is not maximum, i.e., $|M| < \mu(G)$.*

Algorithm 1

Require: A bipartite graph $G = (A \cup B, E)$, i.e., A and B are independent sets, a matching $M \subseteq E$ of G and $a \in A$ not covered by M.

$X = \{a\}$.
$Z = \emptyset$.
$Continue = true$.
while $Continue$ **do**
    $Continue = false$ and $X' = \emptyset$.
    for every $v \in X$ **do**
        for every $w \in N(v) \setminus Z$ **do**
            if w not covered by M **then**
                return an M-augmenting path from a to w
            else
                $Continue = true$ and let $u \in A$ such that $\{u, w\} \in M$.
                Add w to Z and u to X'.
    $X \leftarrow X'$
return $False$.

Question 4 *Apply Algorithm 1 on the instance depicted in Figure 2(left). Describe precisely the evolution of the variables X, Z and $Continue$ after each iteration of the while-loop.*

Question 5 *Apply Algorithm 1 on the instance depicted in Figure 2(right). Describe precisely the evolution of the variables X, Z and $Continue$ after each iteration of the while-loop.*

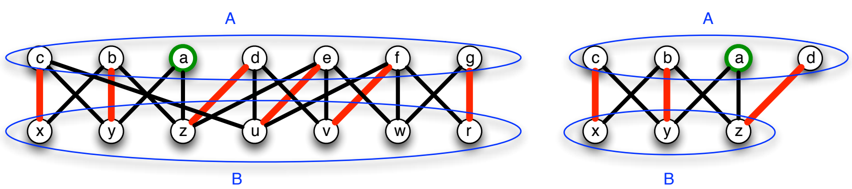

FIGURE 2 – Two examples of bipartite graphs with a given matching M (in bold red).

Question 6 *(*) What is the goal of Algorithm 1. Argue your answer.*

The next two questions are independent from above questions.

Question 7 *Give (draw) a maximum matching in each of the grids depicted in Figure 3. Give arguments that the matchings you propose are maximum.*

Question 8 *Express $\mu(G_{n \times m})$ for every $n, m \in \mathbb{N}^*$ (and prove it by exhibiting a matching of this size and showing that no larger matching exists).*

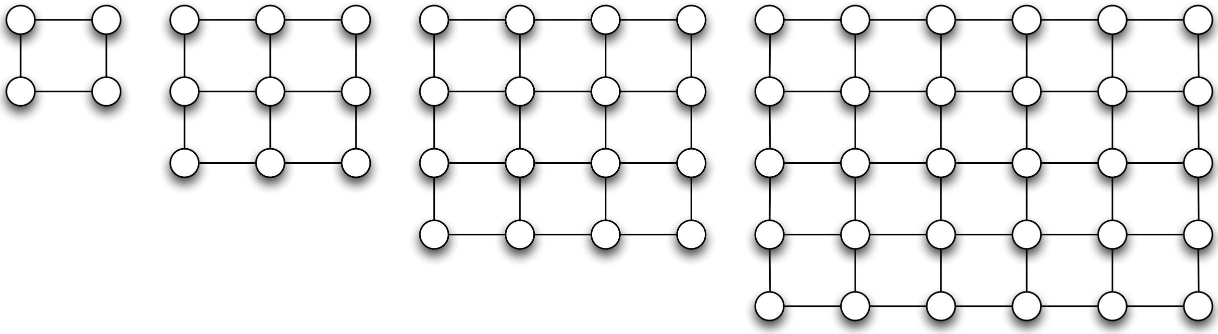

FIGURE 3 – Examples of grids

3 Minimum vertex cover in bipartite graphs

Question 9 *Prove that, in any graph G, $vc(G) \geq \mu(G)$.*

Question 10 *Let $G = (V = (A \cup B), E)$ be a bipartite graph with $A \cap B = \emptyset$ and A and B being independent sets. Let $M \subseteq E$ be a maximum matching of G, i.e. $|M| = \mu(G)$. Show that, if all vertices of A are covered by M, then A is a vertex cover of G and then, $vc(G) = |A| = \mu(G)$.*

Given a matching M, an **M-alternating path** is a path $(v_1, \cdots, v_p)$ $(p \geq 1)$ such that, for every $1 \leq i < p - 1$, exactly one edge among $\{v_i, v_{i+1}\}$ and $\{v_{i+1}, v_{i+2}\}$ belongs to M.

Question 11 *Let $G = (V = (A \cup B), E)$ be a bipartite graph with $A \cap B = \emptyset$ and A and B being independent sets. Let $M \subseteq E$ be a maximum matching of G, i.e. $|M| = \mu(G)$. Assume that M does not cover all vertices in A and let $X \subseteq A$ be the vertices in A not covered by M.*

Let U be the set of vertices linked to some vertex in X by an M-alternating path. Let $U_A = U \cap A$ and $U_B = U \cap B$. Note that $X \subseteq U_A$.

1. *Prove that $|M| = |A \setminus X|$.*
2. *Prove that all vertices in U_B are covered by M.*
3. *Let $N(U_A) = \{u \in B \mid \exists v \in U_A, \{u, v\} \in E\}$. Prove that $U_B = N(U_A)$.*
4. *(*) Prove that $U_B \cup (A \setminus U_A)$ is a vertex cover of size $|M| = \mu(G)$.*

Question 12 *Deduce that, in any bipartite graph G, $vc(G) = \mu(G)$.*

4 Maximum independent sets in grids

Question 13 *Prove that, in any graph $G = (V, E)$, if $K \subseteq V$ is a vertex cover in G, then $V \setminus K$ is an independent set in G.*

Question 14 *Prove that, in any graph G with n vertices, $\alpha(G) = n - vc(G)$.*

Question 15 *For every $n, m \in \mathbb{N}^*$, express $\alpha(G_{n\times m})$ (and prove your statement).*