Short exam : 1 hour

No electronic/manuscript document is allowed. You can answer in french or english. Most of the questions may be answered in at most one sentence.

1 Graph colouring

Question 1 *Give the definition of the degree of a vertex $v \in V$ in a graph $G = (V, E)$.*

A graph is **cubic** if all its vertices have degree exactly 3.

Question 2 *Let $k \in \mathbb{N}$. Give the definition of a proper k-colouring of a graph $G = (V, E)$.*

The **chromatic number** $\chi(G)$ of a graph G is the smallest integer k such that G admits a proper k-colouring.

Question 3
- *Draw a cubic graph G with $\chi(G) = 4$.*
- *Draw a cubic graph G different from K_4 (the complete graph with 4 vertices).*
- *Prove that, for every cubic graph G, $\chi(G) \leq 4$.*

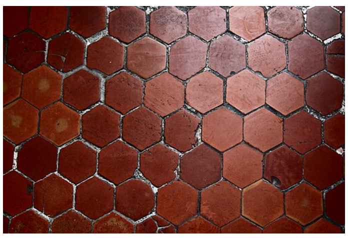
2 Planar graphs

Question 4
- *Give the definition of a connected graph.*
- *Give the definition of an outerplanar graph.*
- *Draw $K_{3,3}$ on the torus (i.e., on a donuts) without crossing edges.*

Let $G = (V, E)$ be a connected planar graph with $|F|$ faces.

Question 5 *Give (without proof) the Euler's Formula linking $|F|, |V|$ and $|E|$ in connected planar graphs.*

The Euler's Formula is actually also true for graphs embedded (drawn) without crossing on a sphere (not only on a plane).



(a) Tiling of the plane with hexagones.

(b) Soccer ball.

FIGURE 1 – Examples.

The goal of the following questions is to explain how a soccer ball is built. Did you notice that a soccer ball consists of hexagons (generally in white) and of pentagons (generally in black)? Why not only with hexagons? How many pentagons?

Question 6 *For purpose of contradiction, let us assume that it is possible to tile the sphere only with hexagons (as in Figure 1a, but on a sphere). Let $x = |F|$ be the number of faces (of hexagons) of such a tiling.*

- *Noticing that each face has 6 vertices and that each vertex appears in 3 faces, what is the number of vertices $|V|$ in function of x ?*
- *Noticing that each face has 6 edges and that each edge appears in 2 faces, what is the number of edges $|E|$ in function of x ?*
- *Apply the Euler's Formula. Conclusion ?*

Question 7 *Let us consider a tiling of the sphere only with x hexagons and y pentagons (and each vertex appears in exactly 3 faces as in Figure 1b).*

- *Give the number of faces in function of x and y.*
- *What is the number of vertices $|V|$ in function of x and y ?*
- *What is the number of edges $|E|$ in function of x and y ?*
- *Apply the Euler's Formula. Conclusion ? (what must be the number of pentagons ?)*

3 Vertex cover in trees and dynamic programming

Question 8 — *Give the definition of a path between two vertices a and b in a graph.*

- *Give the definition of a tree.*

In what follows, we consider **rooted trees** (T, r). That is, given a tree $T = (V, E)$, we fix an (arbitrary) vertex $r \in V$ that is called the **root** of T. The root r being fixed, the **descendents** of a vertex $v \in V$ are all vertices $w \in V$ (including v) such that the (unique) path from w to r contains v. The **children** of v are its neighbours that also are its descendents. If $v \neq r$, the **parent** of v is its (unique) neighbour that is not its descendent. Finally, let T_v denote the subtree of T (rooted in v) induced by the descendents of v (so $v \in V(T_v)$). In particular, $T_r = T$. A **leaf** of (T, r) is a vertex with no children. See Figure 2.

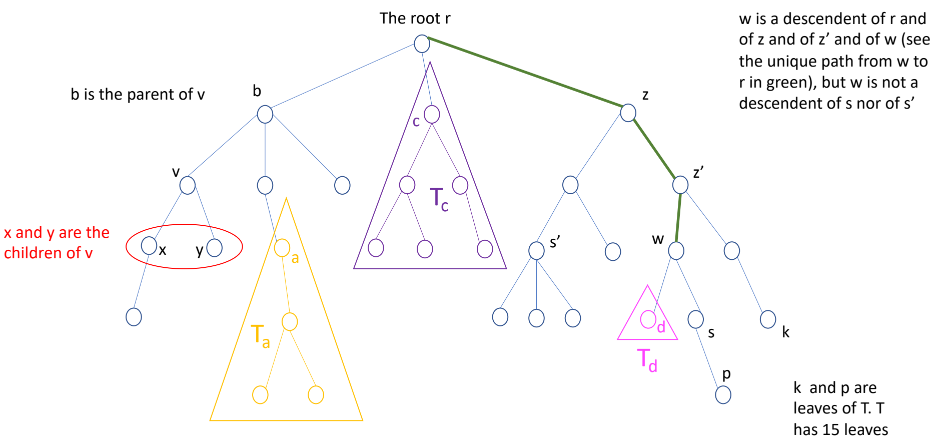


FIGURE 2 – Illustrations for the definition and notations related to rooted trees.

Recall that a **vertex cover** of a graph $G = (V, E)$ is a set $Q \subseteq V$ of vertices that "touches" all edges. Formally, $Q \subseteq V$ is a vertex cover of G if, for every edge $e \in E$ (seen as a set of 2 vertices), $e \cap Q \neq \emptyset$.

In what follows, we consider a weight function $w : V \to \mathbb{R}^+$ over the vertices of G. The weight of a subset $Q \subseteq V$ of vertices is $w(Q) = \sum_{v \in Q} w(v)$, i.e., the sum of the weights of the vertices of Q.

The goal of what follows is to design an algorithm that computes a vertex cover of minimum weight in a weighted tree. From now on, let $T = (V, E)$ be a tree rooted in a vertex $r \in V$ and with a weight function $w : V \to \mathbb{R}^+$.

Question 9 *Let $v \in V$ with children $v_1, \cdots, v_d$ and let Q be a vertex cover of T that does not contain v (i.e., $v \notin Q$). Explain (in one sentence) why $\{v_1, \cdots, v_d\} \subseteq Q$.*

Given a vertex v of the rooted tree (T, r) with weight w, let $vc(T_v)$ denote the smallest weight of a vertex cover of T_v. Let $vc^+(T_v)$ denote the smallest weight of a vertex cover of T_v that contains v (i.e., consider all vertex covers of T_v that contains v and consider one such vertex cover with smallest weight), and let $vc^-(T_v)$ denote the smallest weight of a vertex cover of T_v that does not contain v (i.e., consider all vertex covers of T_v that do not contain v and consider one such vertex cover with smallest weight).

Question 10 *For every vertex $v \in V$, explain (in one sentence) why $vc(T_v) = \min\{vc^+(T_v), vc^-(T_v)\}$.*

Question 11 *Let v be a leaf of T (so, T_v is the tree that contains only the vertex v). Give the values of $vc(T_v)$, of $vc^+(T_v)$ and of $vc^-(T_v)$ as a function of $w(v)$.*

Question 12 *Let $v \in V$ with children $v_1, \cdots, v_d$. Using question 9, explain (in one sentence) why $vc^-(T_v) = \sum_{1 \leq i \leq d} vc^+(T_{v_i})$.*

Question 13 *Let $v \in V$ with children $v_1, \cdots, v_d$. Explain (in one/two sentences) why $vc^+(T_v) = w(v) + \sum_{1 \leq i \leq d} vc(T_{v_i})$.*

Question 14 *From previous questions, design an algorithm (write the pseudo code) that takes a weighted tree as input and computes the minimum weight of a vertex cover of T.*

Question 15 *What is the time-complexity of your algorithm (as a function of the number n of vertices of the input tree) ?*