Master 2, Informatique et Interactions

Final Exam Advanced Graphs, February 2022

3 hours

Documents are allowed, but, NO computers, NO cellphones!!

Instruction and comments: the points awarded for your answer will be based on the correctness of your answer as well as the clarity of the main steps in your reasoning. All proposed solutions must be proved. **All the 3 sections are independent.**

1 Diameter of trees. (5.5 points, 40 minutes)

Reminder. A *tree* is a connected graph without cycle. A *leaf* in a tree is any vertex with degree one (with a single neighbour). A *path* $(v_1, \cdots, v_p)$ between $v_1 \in V$ and $v_p \in V$ in a graph $G =: (V, E)$ is a sequence of distinct vertices ($v_i \in V$ for every $1 \leq i \leq p$ and $v_i \neq v_j$, for $1 \leq i < j \leq p$) such that $\{v_i, v_{i+1}\} \in E$ for every $1 \leq i < p$. The *length* of a path $(v_1, \cdots, v_p)$ is its number of edges $p - 1$. The *distance* $dist_G(u, v)$ between two vertices $u, v \in V$ is the minimum length of a path between u and v in G. The *diameter* $diam(G)$ of a graph G is the maximum distance between two vertices of G, i.e., $diam(G) = \max_{u,v \in V} dist_G(u, v)$.

Given a graph $G = (V, E)$ and $r \in V$, a BFS (Breadth First Search) rooted in r is an algorithm that computes, in time $O(|E|)$, the distance between r and every vertex in G, i.e., that computes $dist_G(u, r)$ for every $u \in V$.

Question 1 (0.25 point) *What is the diameter of a tree with one vertex? with two vertices?*

Question 2 (1 points) *Let $T = (V, E)$ be a tree with $|V| \geq 3$. Show that there exists $v \in V$ that is not a leaf.* *hint: by contradiction*

The goal of this problem is to understand the goal and the time-complexity of the following algorithm.

Algorithm 1

Require: A tree $T = (V, E)$ with $|V| \geq 3$

1: Let $r \in V$ which is not a leaf (r is then called the *root*) and do a BFS of G rooted in r.
2: Let $u \in V$ be any vertex that maximizes $dist_T(u, r)$.
3: Do a BFS of G rooted in u.
4: Let $w \in V$ be any vertex that maximizes $dist_T(w, u)$.
5: **return** $dist_T(w, u)$.

Let T_0 be the tree with 9 vertices defined as follows. $V(T_0) = \{a, b, c, d, e, f, g, h, i\}$ and $E(T_0) = \{ab, bc, cd, de, ef, cg, gh, hi\}$.

Question 3 (1 point) *Draw T_0. Apply Algorithm 1 on the tree T_0 with root $r = c$. Give the results of each line of the algorithm (in particular, what is u? what is w? what is the result of the algorithm?).*

Question 4 (1.5 points) *What is the time-complexity of Algorithm 1 in function of $|V|$?*

Question 5 (1.5 points) *Show that, during the execution of Alg. 1, the vertex u is a leaf of T and the vertex w is a leaf of T.*

Question 6 (0.25 point) *What does compute Algorithm 1? (no proof is required)*

2 Coloring in graphs on surfaces. (6.5 points, 40 minutes)

Reminder. A *planar* graph is any graph that can be drawn in the plane without crossing edges. Given $k \in \mathbb{N}^*$, a *k-colouring* of a graph $G = (V, E)$ is a function $c : V \to \{1, \cdots, k\}$ such that $c(u) \neq c(v)$ for all $\{u, v\} \in E$. The *chromatic number* $\chi(G)$ of a graph G is the smallest k such that G admits a k-colouring.

Question 7 (0.5 point) *Draw a graph G such that $\chi(G) > 4$ (explain).*

Question 8 (1 point) (planar Euler's formula) *Given a connected planar graph $G = (V, E)$ with planar embedding (drawing) with f faces, prove that $|V| - |E| + f = 2$.*

Question 9 (1 point) *Prove that K_5 (the complete graph with 5 vertices and all possible edges) is not planar.*

Question 10 (0.5 point) *What is the "four colours theorem"?*

Now, we want to go beyond planar graphs. That is, we would like to consider graphs that can be embedded (without crossing edges) in other surfaces than plane. Precisely, we consider torus (roughly, donuts or buoy...).

Question 11 (1 point) *Prove that K_5 can be embedded (drawn) in a torus (see it as a donuts) without crossing edges (i.e., draw K_5 on a torus without crossing edges).*

In what follows, we admit the following theorem:

Theorem 1 (Torus Euler's formula) *Given a connected planar graph $G = (V, E)$ with an embedding on a torus (without crossing edges), with f faces, then $|V| - |E| + f = 0$.*

Question 12 (1.25 point) *Show that a graph that can embedded on a torus (without crossing edges) has a vertex of degree at most 6.*

Question 13 (1.25 point) *Show that, for any graph G that can embedded on a torus (without crossing edges), $\chi(G) \leq 7$.*

3 Vertex Cover in planar graphs (Baker's technique). (9 points, 1 hour 40 minutes)

Reminder. Let $G = (V, E)$ be a graph. Recall that a *vertex cover* of G is a set $K \subseteq V$ such that $e \cap K \neq \emptyset$ for all $e \in E$ (i.e., K hits every edge of G). Recall that the *diameter* of G equals $\max_{u,v \in V} dist(u, v)$ where $dist(u, v)$ denotes the *distance* (length of a shortest path) between u and v.

We admit the following result:

Theorem 2 *Let G be a planar n-node graph with diameter D. A* minimum *vertex cover of G can be computed in time $2^{O(D)}n$.*

Notation: A schematic representation of the notations is depicted in Figure 1.

Let $G = (V, E)$ be a planar n-node graph and $r \in V$. Let us consider a BFS (Breadth First Search) of G rooted in r. For every $\ell \in \mathbb{N}$, let $Layer(\ell) = \{v \in V \mid dist(r, v) = \ell\}$ be the set of vertices at distance (exactly) ℓ from r. For instance, $Layer(0) = \{r\}$.

Let $k \in \mathbb{N}^*$. For every $0 \leq i \leq k$, let $G_{0,i}$ be the subgraph induced by the vertices in $\bigcup_{0 \leq \ell \leq i} Layer(\ell)$. For every $j \in \mathbb{N}^*$ and every $0 \leq i \leq k$, let $G_{j,i}$ be the subgraph induced by the vertices in $\bigcup_{k(j-1)+i \leq \ell \leq kj+i} Layer(\ell)$. Note that, for all $j \geq 0$, $V(G_{j,i}) \cap V(G_{j+1,i}) = Layer(kj+i)$.

Let $i \in \{0, \cdots, k\}$. For every $j \in \mathbb{N}^*$, let $G'_{j,i}$ be the graph obtained from $G_{j,i}$ by adding one vertex $r_{j,i}$ adjacent to all vertices of $Layer(k(j-1)+i)$ and one vertex $u_{j,i}$ adjacent to $r_{j,i}$. Moreover, $G'_{0,i} = G_{0,i}$.

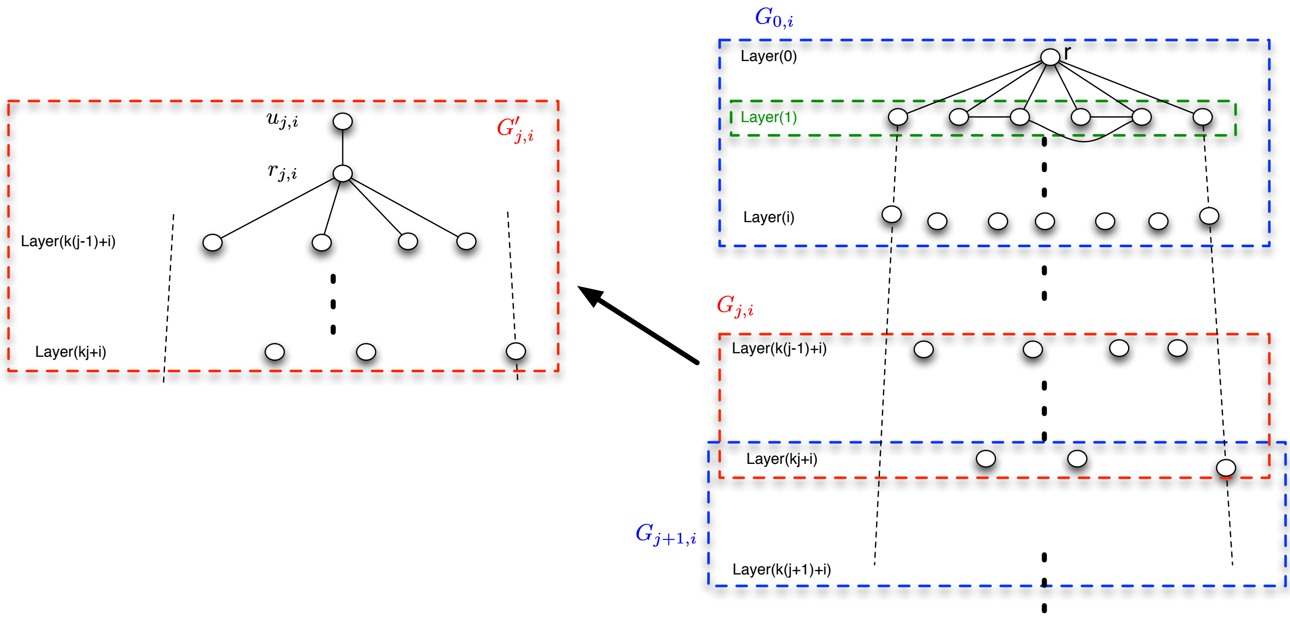

Figure 1: Schematic representation of the BFS of the graph G rooted in r (on the right) and of the notations.

Question 14 (0.5 point) *Let $0 \leq i \leq k$ and $j \in \mathbb{N}$. Show that $G'_{j,i}$ has diameter at most $2(k+1)$.*

Question 15 (2 points) *Let $0 \leq i \leq k$ and $j \in \mathbb{N}^*$.*

Show that any minimum vertex cover of $G'_{j,i}$ contains exactly one of $r_{j,i}$ and $u_{j,i}$.

Deduce that for any minimum vertex cover K of $G'_{j,i}$, then $K \setminus \{r_{j,i}, u_{j,i}\}$ is a minimum vertex cover of $G_{j,i}$.

Show that, for every $j \in \mathbb{N}$, a minimum vertex cover of $G_{j,i}$ can be computed in time $2^{O(k)}n$.

The remaining of this section consists of the analysis of the following Algorithm 2.

Question 16 (2 point) *Show that Algorithm 2 computes a vertex cover of G. What is its time-complexity?*

Let OPT be a minimum vertex cover of G and, for each $0 \leq i \leq k$, let $OPT_i = OPT \cap \bigcup_{\ell \equiv i \mod (k+1)} Layer(\ell)$, i.e., OPT_i is the set of the vertices of OPT at distance $i \mod (k+1)$ from r. Note that $\bigcup_{0 \leq i \leq k} OPT_i = OPT$ and that the sets OPT_i are pairwise disjoint, i.e., $(OPT_0, \cdots, OPT_k)$ is a partition of OPT.

Algorithm 2

Require: A connected planar graph $G = (V, E)$ and $k \in \mathbb{N}^*$.

Let $r \in V$ and do a BFS of G rooted in r.
for every $0 \leq i \leq k$ **do**
  for every $0 \leq j \leq n$ **do**
    Use Question 15 to compute a minimum vertex cover $K_{j,i}$ of $G_{j,i}$.
  Let $K_i = \bigcup_{0 \leq j \leq n} K_{j,i}$.
Let $0 \leq p \leq k$ be such that $|K_p| = \min\{|K_i| \mid 0 \leq i \leq k\}$.
return K_p

Question 17 (1 point) *Show that there exists* $0 \leq q \leq k$ *such that* $|OPT_q| \leq \frac{1}{k+1}|OPT|$.
hint: by contradiction

Let $0 \leq q \leq k$ be defined as in previous question, i.e., such that $|OPT_q| \leq \frac{1}{k+1}|OPT|$. For every $j \in \mathbb{N}$, let $K_{j,q}$ be a minimum vertex cover of $G_{j,q}$.

Question 18 (1 point) *Show that* $|OPT \cap V(G_{j,q})| \geq |K_{j,q}|$.

Question 19 (1.5 points) *Let* $K_q = \bigcup_{0 \leq j \leq n} K_{j,q}$. *Show that* $|K_q| \leq |OPT| + |OPT_q|$.

Question 20 (1 point) *Deduce from previous questions that Algorithm 2 computes a vertex cover of* G *of size at most* $(1 + \frac{1}{k+1})|OPT|$ *in time* $2^{O(k)}poly(n)$*, where* $|OPT|$ *is the minimum size of a vertex cover of* G.