Master 2 Graphs

Final Exam, January 2020

2 hours

No documents are allowed. No computers, cellphones.

Instruction and comments: the points awarded for your answer will be based on the correctness of your answer as well as the clarity of the main steps in your reasoning. All proposed solutions must be proved. **All the sections are independent.**

1 Dijkstra. (4 points, 10 minutes)

Consider the following graph H with weighted edges depicted in Figure 1.

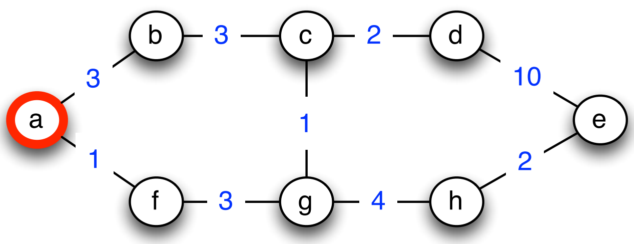

Figure 1: A weighted graph H. Blue integers denote the length of corresponding edges.

Question 1 *Give the definition of a* Shortest Path Tree *rooted in* a *in the graph* H.

Question 2 *Applying the Dijkstra's algorithm on* H, *compute a shortest path tree rooted in* a. *The first three steps of the algorithm must be detailed (***at most*** two lines per steps).*

In particular, indicate the order in which vertices are considered during the execution of the algorithm, and give the final solution.

2 Basics on planar graphs. (8 points, 50 minutes)

Recall that a graph is *planar* if and only if it admits an embedding (drawing) on the plane without crossing edges. Recall also that a tree is a connected acyclic graph.

Question 3 *Let* $T = (V, E)$ *be any tree. Prove that* T *is planar and that* $|V| = |E| + 1$.

Hint: induction is your friend

Question 4 *For each graph in Figure 2, say whether it is planar or not (justify your answers).*

Given a planar (without crossing) embedding of a graph $G = (V, E)$, its *dual* $G^* = (V^*, E^*)$ is the graph with one vertex per face of G and for every edge $e \in E$, there is an edge $e^* = \{f_1, f_2\} \in E^*$ where f_1 and f_2 are the faces (possibly $f_1 = f_2$) of G "touching" the edge e. Note that there is a one-to-one mapping between E and E^*. Given a spanning tree $T = (V, E')$ of G. Let $E'' \subseteq E^*$ be the set of edges of G^* that correspond to edges in E'. Let T^* be the subgraph of G^* induced by the edges in $E^* \setminus E''$. Recall that we proved that T^* is a spanning tree of G^* that is called the *interdigitating tree* of T in G^*.

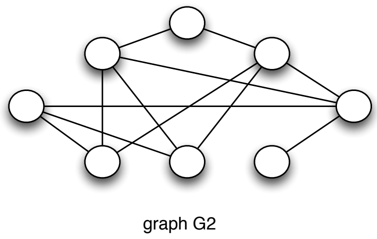

Figure 2: Three connected graphs

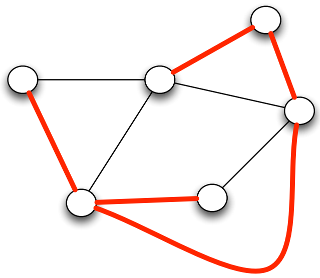

Figure 3: A planar embedding of a graph G and a spanning tree (with edges in bold red).

Question 5 *Draw the dual G^* of the planar graph G whose planar embedding is given in Figure 3. Then, draw the interdigitating tree of the spanning tree T of G depicted in red in Figure 3.*

Question 6 *Let $G = (V, E)$ be a connected planar graph together with a planar embedding. Let F be the set of faces of G. Let T be a spanning tree of T and T^* be its interdigitating tree in G^*. By noting that $|E| = |E(T)| + |E(T^*)|$, prove the Euler's formula: $|F| + |V| = |E| + 2$.*

Question 7 *Prove that, for any simple (no loop nor parallel edges) connected planar graph $G = (V, E)$ with $|V| > 2$, $|E| \leq 3|V| - 6$.*

Hint: note that each edge corresponds to at most two faces and each face contains at least three vertices (so at least three edges).

Question 8 *Deduce from previous question that any simple planar graph has a vertex of degree at most* 5.

Let $k \in \mathbb{N}^*$. A *proper k-coloring* of a graph $G = (V, E)$ is any function $c : V \to \{1, \cdots, k\}$ such that $c(u) \neq c(v)$ for all $uv \in E$.

Question 9 *Prove that any simple planar graph admits a proper* 6*-coloring.*

Question 10 *What is the minimum k such that any simple planar graph admits a proper k-coloring (no proof is asked here)? Why do we require the graph to be simple?*

3 Vertex Cover in planar graphs (Baker's technique). (8 points, 60 minutes)

Let $G = (V, E)$ be a graph. Recall that a *vertex cover* of G is a set $K \subseteq V$ such that $e \cap K \neq \emptyset$ for all $e \in E$ (i.e., K hits every edge of G). Recall that the *diameter* of G equals $\max_{u,v \in V} dist(u, v)$

where $dist(u, v)$ denotes the *distance* (length of a shortest path) between u and v.

We admit the following result.

Theorem 1 *Let G be a planar n-node graph with diameter D. A* minimum *vertex cover of G can be computed in time $2^{O(D)}n$.*

Notation: A schematic representation of the notations is depicted in Figure 4.

Let $G = (V, E)$ be a planar n-node graph and $r \in V$. Let us consider a BFS (Breadth First Search) of G rooted in r. For every $\ell \in \mathbb{N}$, let $Layer(\ell) = \{v \in V \mid dist(r, v) = \ell\}$ be the set of vertices at distance (exactly) ℓ from r. For instance, $Layer(0) = \{r\}$.

Let $k \in \mathbb{N}^*$. For every $0 \leq i \leq k$, let $G_{0,i}$ be the subgraph induced by the vertices in $\bigcup\limits_{0 \leq \ell \leq i} Layer(\ell)$. For every $j \in \mathbb{N}^*$ and every $0 \leq i \leq k$, let $G_{j,i}$ be the subgraph induced by the vertices in $\bigcup\limits_{k(j-1)+i \leq \ell \leq kj+i} Layer(\ell)$. Note that, for all $j \geq 0$, $V(G_{j,i}) \cap V(G_{j+1,i}) = Layer(kj+i)$.

Let $i \in \{0, \cdots, k\}$. For every $j \in \mathbb{N}^*$, let $G'_{j,i}$ be the graph obtained from $G_{j,i}$ by adding one vertex $r_{j,i}$ adjacent to all vertices of $Layer(k(j-1)+i)$ and one vertex $u_{j,i}$ adjacent to $r_{j,i}$. Moreover, $G'_{0,i} = G_{0,i}$.

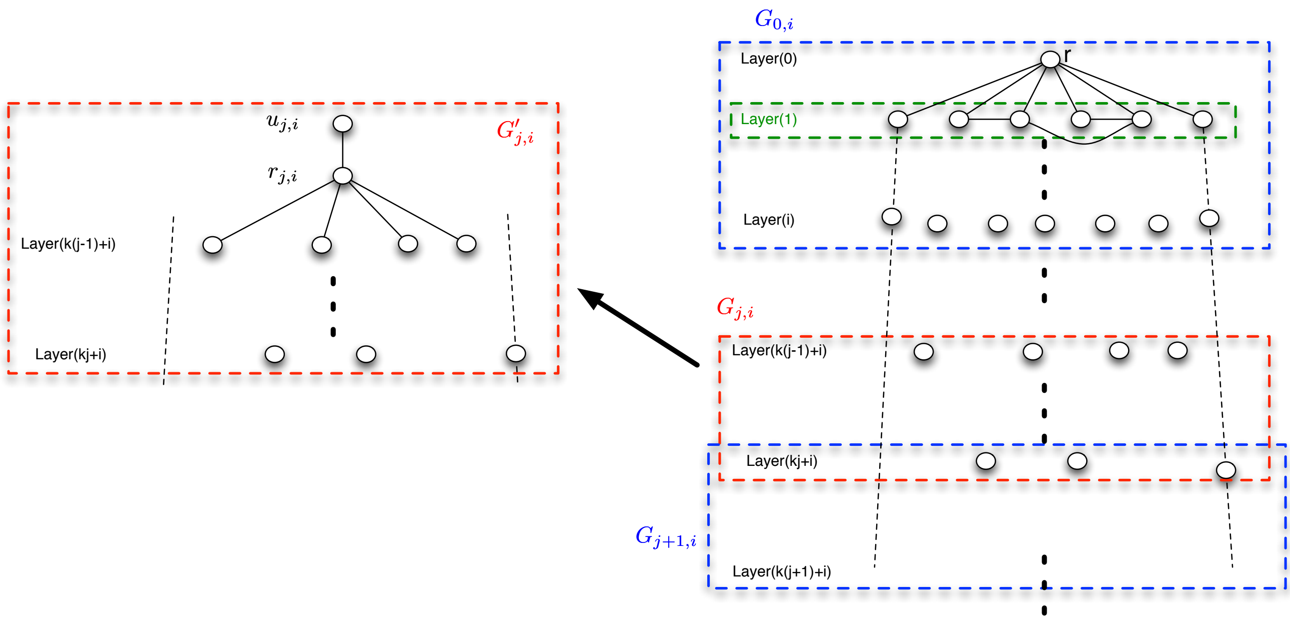

Figure 4: Schematic representation of the BFS of the graph G rooted in r (on the right) and of the notations.

Question 11 *Let $0 \leq i \leq k$ and $j \in \mathbb{N}$. Show that $G'_{j,i}$ has diameter at most $2(k+1)$.*

Question 12 *Let $0 \leq i \leq k$ and $j \in \mathbb{N}^*$.*

Show that any minimum vertex cover of $G'_{j,i}$ contains exactly one of $r_{j,i}$ and $u_{j,i}$.

Deduce that for any minimum vertex cover K of $G'_{j,i}$, then $K \setminus \{r_{j,i}, u_{j,i}\}$ is a minimum vertex cover of $G_{j,i}$.

Show that, for every $j \in \mathbb{N}$, a minimum vertex cover of $G_{j,i}$ can be computed in time $2^{O(k)}n$.

The remaining of this section consists of the analysis of the following Algorithm 1.

Algorithm 1

Require: A connected planar graph $G = (V, E)$ and $k \in \mathbb{N}^*$.

```
Let r ∈ V and do a BFS of G rooted in r.
for every 0 ≤ i ≤ k do
    for every 0 ≤ j ≤ n do
        Use Question 12 to compute a minimum vertex cover K_{j,i} of G_{j,i}.
    Let K_i = ⋃_{0≤j≤n} K_{j,i}.
Let 0 ≤ p ≤ k be such that |K_p| = min{|K_i| | 0 ≤ i ≤ k}.
return K_p
```

Question 13 *Show that Algorithm 1 computes a vertex cover of G. What is its time-complexity?*

Let OPT be a minimum vertex cover of G and, for each $0 \leq i \leq k$, let $OPT_i = OPT \cap \bigcup_{\ell \equiv i \mod (k+1)} Layer(\ell)$, i.e., OPT_i is the set of vertices of OPT at distance $i \mod (k+1)$ from r.

Question 14 *Show that there exists $0 \leq q \leq k$ such that $|OPT_q| \leq \frac{1}{k+1}|OPT|$.*

Let $0 \leq q \leq k$ be defined as in previous question, i.e., such that $|OPT_q| \leq \frac{1}{k+1}|OPT|$. For every $j \in \mathbb{N}$, let $K_{j,q}$ be a minimum vertex cover of $G_{j,q}$.

Question 15 *Show that $|OPT \cap V(G_{j,q})| \geq |K_{j,q}|$.*

Question 16 *Let $K_q = \bigcup_{0 \leq j \leq n} K_{j,q}$. Show that $|K_q| \leq |OPT| + |OPT_q|$.*

Question 17 *Deduce from previous questions that Algorithm 1 computes a vertex cover of G of size $(1 + \frac{1}{k+1})|OPT|$ in time $2^{O(k)} poly(n)$, where $|OPT|$ is the minimum size of a vertex cover of G.*

Such an algorithm is called an *EPTAS* (Efficient Polynomial-Time Approximation Scheme).

Question 18 *Explain why Algorithm 1 is an FPT (Fixed Parameter Tractable) algorithm with parameter k?*