Catching a robber on a digraph : instructions for use

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Abstract

In this report, we try to define the structure of cop win digraph, when only the robber must respect the orientation of the edges. As only a few articles talk about cops and robbers on directed graph, we try adapting the predefine structure in the non oriented game. Even if some examples have similarities, we cannot ensure that cop win digraphs have a specific structure.

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Contents

1 Introduction

Cops and robbers is a famous pursuit and evasion game define in the early 1980's. At its core, it was played on a graph, with two players, the first controlling a set of cops, and the second one robber. When the cops try to capture the robber by one staying at the same vertex, the robber win if and only if it will never be captured. The only rule was for both players to move accordingly their pawn to neighboring vertices. Such rules lead to a full of variants and many questions remain still unknown on this game.

Considering a graph G, it's interesting to know the minimal cops required to always capture a robber on this graph. Such a number is called the cop number. Finally, when it comes to graph with cop number 1, called also cop win graph, a characterisation of these graphs has already been proposed in [1]. For example, trees are always cop win and cycle of length at least 4 required two cops. However, cops and robbers plays on directed graph have received less attention and, nowadays, no structure has been defined for characterize cop win digraph. An algorithm for another version has been publised in [3].

That's why, we try, using the definition in the non oriented case, to propose our own ideas on the subject. Even if we haven't succeed in proving that cop win digraph could be defined, we've found some recurent attributes on several cop win digraphs that lead to think that there could be one. It's easy to see that in a cops and robbers game, a non connected graph is at least 2-cop win. That's why, in this paper, we will only consider digraph weakly and strongly connected.

2 Notation

we define some notations which will be useful for the rest of the report.

Notation 1 *oriented neighborhood : in a digraph D, we say that* $y \in N_+(x)$ *with* $y,x \in V(D)$ *if there is an edge oriented from x to y (* $x \rightarrow y$*). Mutually if there is an edge from y to x,* $y \in N_-(x)$*.* $N_+[x]$ *(resp.* $N_-[x]$*)* $\equiv N_+(x) \bigcup \{x\}$ *(resp.* $N_+(x) \bigcup \{x\}$*)*

Notation 2 *non oriented neighborhood : D a digraph,* $\forall\ x \in V(D)$ $N(x) = N_+(x) \bigcup N_-(x)$*.*

Notation 3 *symmetric edges : in a digraph D, a symmetric edges (x,y) is bidirectionnal so that* $y \in N_+(x)$ *and* $y \in N_-(x)$*. So a graph can be studied as a digraph with symmetric edges.*

Notation 4 *game's notation : we denote respectively the cop and the robber by* $\mathscr{C}$ *and* $\mathscr{R}$

Notation 5 *maximum and minimum degree of a graph : let G a graph, we denote respectively by* $\Delta(G)$ *and* $\delta(G)$ *the maximum and minimum degree in G.*

3 Rules

Let D a digraph, which is the game area. At the beginning, the player who controls the cop, put it on a vertex, then the second player does the same with its robber. At each round, the cop begins by moving or not in its neighborhood, then so does the robber by respecting the following rules :

- the cop has no forbidden vertex in its neighborhood : let v $\in$ V(D) be the position of the cop at round t. At round t+1, the cop may move in any vertex w $\in$ N[v].

- the robber moves only in the out-neighborhood of its position, or stays at the same vertex : let x $\in$ V(D) be the position of the robber at round t+1, then the robber may only move in any vertex y $\in$ N_+[x]

The cop wins if at some round, he is in the same vertex than the robber is. Otherwise, if it's impossible for the cop to meet the robber, the second player wins.

4 Definition

We present the results in the non oriented game :

Definition 1 *corner : a vertex u is a corner if there is some vertex v such that N[u]$\subseteq$ N[v]*

Lemma 1 *If G is a cop win graph, then G contains at least one corner*

Definition 2 *retract : Let H be an induced subgraph of G formed by deleting one vertex. We say that H is a retractof G if there is a homomorphism f from G onto H so that f is the identity on H.*

Theorem 1 *If H is a retract of G, then c(H) $\leq$c(G).*

Definition 3 *dismantable : a graph is dismantable if some sequence of deleting corners results in the graph K1.*

Theorem 2 *A graph is cop win if and only if it is dismantable.*

Then we use the definition of the non oriented case and adapt them to the game played on digraph :

Definition 4 *directed corner : a vertex u is a directed corner if there is some vertex v such that* $N_+[u] \subseteq$ *N[v]*

Definition 5 *dismantable : a graph is dismantable if it's K1 or there is a vertex u $\in$ V(G) so that u is a directed corner and all the connected components in G$\backslash$u are dismantable.*

We give some definitions of several graphs' families which are studied in the report and some notion of graph theory.

Definition 6 *girth : in a graph G, the girth is the length of its smaller cycle. In this report, the girth of a digraph D, will be the girth of its induced non oriented graph.*

Definition 7 *k-regular graph : a graph G is a k-regular graph with k$\geq$1 if and only if $\forall$ u $\in$ V(G) | N(v)|=k.*

A k-regular graph G , in the non oriented versus of the game, is cop win if and only if k=(n-1) with |V(G)|=n.

Definition 8 *planar graph : A graph is a planar graph if it can be drawn in a plane without any edges crossing.*

Definition 9 *outerplanar graph : a graph is an outerplanar graph if it can be embedded in a cycle such that all its vertices are in the cycle and its all edges are inside the area bounded by the cycle.*

Definition 10 *bipartite graph : A graph is a bipartite graph if it consists of 2 sets of vertices with edges only joining vertices between sets and not within a set.*

Definition 11 *triangulated graph : A graph is triangulated if every of its internal face is a triangle.*

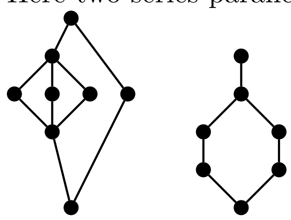
Definition 12 *series-parallel graph : A graph is series-parallel if it is constructed by a sequence of series and parallel compositions with a set of K2 as terminals. We define the series and parallel composition. Let's called S and T respectively the source and the sink of a series-parallel graph.*

- the parallel composition : we join two series-parallel graph A and B, by medging the sink of one with the source of the other.

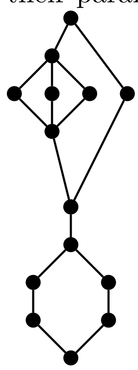
- the series composition : we join two series-parallel graph, A and B, by medging both their sink and their source.

We give an example of a series-parallel graph.

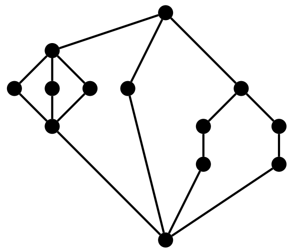
Here two series-parallel graph :



their parallel composition :



their series composition :



5 Cop win digraph's connexity

Proposition 3 *Given that a directed acyclic graph G, G is cop win.*

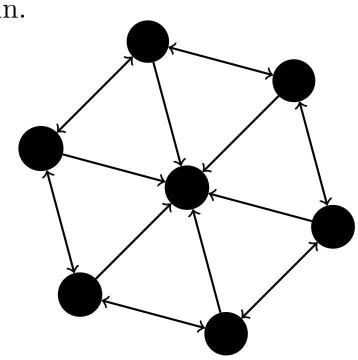
Proof 1 *At first, $\mathscr{C}$ is on a root of the DAG. Then, R chooses a branch of the DAG and there is no other way for him to go further on the branch not to be captured. As the DAG is finite, $\mathscr{R}$ will arrive at a corner $u \in V(G)$ so that $\mid N_+(u)\mid=0$ and after a finite number of round, $\mathscr{C}$ will arrive at the same vertex.*

Lemma 2 *Let G be a weakly connected digraph. If all the strongly connected components of G are cop win, then G is cop win*

Let D be the induced DAG, so that a vertex i of V(D) represents Gi, a strongly connected component of G. We play simultaneously on D and G, so that if the cop is on $i \in V(D)$, he plays on a vertex on Gi in G. As every strongly connected component is cop win, a robber can only stay a finite number of rounds in each component when the cop is also in it. By putting $\mathscr{C}$ on a root of D, $\mathscr{R}$ would have to follow the vertices on a branch. Then, as $\mathscr{R}$ cannot stay infinitely on the same vertex in D, he will arrive on a directed corner, that he would not be able to escape. As this vertex is associated to a cop win subgraph, $\mathscr{C}$ will capture $\mathscr{R}$.

Remark 1 *The upper lemma is not an equivalence.*

The following digraph D is cop win. The strategy of $\mathscr{C}$ consists in placing himself on the dominating vertex u, on the center of the wheel. But the strongly connected component D\{u} is a cycle of length 6, so it is not cop win.



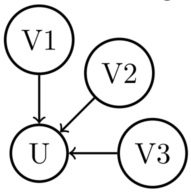
Proposition 4 *Let G a digraph, so that its induced non oriented graph is cop win in the non oriented version of the game. So G is cop win.*

Indeed, the orientation of the edges doesn't limitated the cop, so he can keep its strategy from the non oriented version.

Proposition 5 *For any connected graph G, we can apply an orientation of its edges so that we obtain D, cop win weakly connected digraph.*

We proceed by induction. K1 is already cop win. Suppose now there exists an integer n so that for every connected graph G, |V(G)|=n, we can orientate the edges of G, so that its digraph D is weakly connected and cop win.

Let G be a connected graph |V(G)|= n+1. Let u $\in$ G be a vertex so H=G\u is a connected graph. As V(H)=n, you can orientate its edges so H becomes a weakly connected digraph which is also cop win. You can orientate the edges (v,u) with v$\in$ H so u will be an end vertex.



We show that the digraph D obtained from G is cop win. $\mathscr{C}$ follows the strategy, he has for the directed graph obtained from H. If $\mathscr{R}$ remains in it, he will be captured. Otherwise, at some steps, $\mathscr{R}$ will go in U which is an end vertex. In any ways, $\mathscr{R}$ will be captured.

6 Cop win digraphs

Lemma 3 *If D is a cop win digraph, then D contains at least one directed corner.*

Proof 2 *Let D be a cop win digraph, and consider the second to last move of the cop. Let say $\mathscr{R}$ is on u $\in$ V(D) and $\mathscr{C}$ on v $\in$ V(D). The robber can choose to remain at its position so u$\in$N(v). He can also move in the out-neighborhood of u, so $N_+(u) \subseteq N[v]$. To conclude, $N_+[u] \subseteq N[v]$, so u is a directed corner.*

Theorem 6 *Let D be a digraph with non oriented girth at least 5. Then $c(G) \geq \delta+(D)$*

Remark 2 *This theorem is the adaptation to the non oriented case : let G be a digraph with girth at least 5. Then $c(G) \geq \delta(D)$*

Proof 3 *If $\delta+(D) = 0$, automatically, $c(G) \geq \delta+(D)$.*

We use d= $\delta+(D)$ -1 cop and show by the way that it is not sufficient to catch the robber for each round.

Let put the d cops and call $\mathscr{C}$ the cop's set of vertices. $\exists$ u $\in$ V(D) so that u $\notin \mathscr{C}$ and $\forall$ v in $\mathscr{C}$ (u,v) or (v,u) $\notin$ E(D).

Indeed $|V(D)| \geq \delta+(D)$ +1 so $\exists$ u $\in$ V(D) so that u $\notin \mathscr{C}$. Suppose now $\forall$ w $\notin \mathscr{C}$ $\exists$ v $\in \mathscr{C}$ (w,v) or (v,w) $\in$ E(D).

Let call $\mathscr{X} \equiv N_+(u) \bigcap \mathscr{C}$ and $\mathscr{Y} \equiv N_+(u) \setminus \mathscr{C}$.

$\mathscr{Y} \not\equiv \emptyset$ because $|\ N_+(u)| \geq$ d+1 .By hypothesis $\forall$ y $\in \mathscr{Y}$ $\exists$ x $\in \mathscr{C}$ so that y is joined to x.

If $\mathscr{X} \not\equiv \emptyset$ and x $\in \mathscr{X}$, then D would have a non oriented girth of length 3, which is contradictory to the initial properties of the digraph. So x $\notin \mathscr{C}$

If $\exists$ y1 and y2 $\in \mathscr{Y}$ so that $\exists$ x $\in \mathscr{C} \setminus \mathscr{X}$ with $\{y1,y2\} \subseteq N(x)$. Then D would have a girth of length at most 4, which is also a contradiction.

Because of these observations :

$$|\ \mathscr{Y}\ | + |\ \mathscr{X}\ | \leq d.$$

$$|\ \mathscr{Y}\ | + |\ \mathscr{X} \geq \delta+(D) = d+1.$$

Then d $\geq$ d+1 with d $\geq$ 0 , so we conclude that $\exists$ u $\notin \mathscr{C}$ and u is not joined to any vertex of $\mathscr{C}$. The robber has just to put himself on this vertex initially, so he will survive at the first round.

Now let $\mathscr{C}$t be the cops'set of vertices at round t and ut the position of the robber with the property ut is not joined to any vertex of $\mathscr{C}$t.

It's the turn of the cops. Let $\mathscr{C}$t+1 be their new positions. If ut is not joined to any vertex of $\mathscr{C}$t+1 then $u_{t+1} \equiv$ ut and the property is kept safe at round t+1.

Otherwise, because $\mid N_+(u_t)| \gneq d$ *then* $\exists\ v \in N_+(u_t) \setminus \mathscr{C}t{+}1$.

If $\exists\ x \in \mathscr{C}t{+}1$ $(x \not\equiv u_t)$ $v \in N(x)$, *then as D has girth at least 5,* $x \notin N_+(u)$. *So if every vertex of the outneighborhood of u is joined to some cops, we would have* $d \geq \delta{+}(D)$. *So* $\exists\ u_{t+1} \in N_+(u)$ *so that, in moving here, the robber will be safe.*

Conclusion : less than $\delta{+}(D)$ *cops is not enough to catch the robber.*

Theorem 7 *Every cop win digraph, without induced non oriented cycle of length 3, are dismantable.*

Proof 4 *Let G be a cop win digraph, without induced non oriented cycle of length 3. We will show by induction that there exists a sequence of deleting directed corners.*

As G is cop win, G admits a directed corner.

We will consider two cases : whether the directed corner is a cut vertex or not.

CASE 1 : u is a cut vertex.

Without loss of generality, we can suppose that u cuts the graph in two connected component $X \subseteq G$ *and* $Y \subseteq G$. *Let v be a vertex which dominates u, whith* $v \in X$. *We show that both* $X \bigcup \{u\}$ *and* $Y \bigcup \{u\}$ *are cop win. It's easy to see, that if the robber is on Y (resp. on X) and C goes on X (resp. Y), then the robber has just to play as if C was on U. So it's no use for the cop to go on X (resp. Y). As G is cop win, if R remains on X (resp. Y), he will be captured. So both* $X \bigcup \{u\}$ *and* $Y \bigcup \{u\}$ *are cop win. We obtain two digraphs with less vertices. So the property will be true. We can delete the vertices of X and Y in parallel, as u is already a directed corner and is the unique common vertices between X and Y.*

CASE 2 : u is not a cut vertex.

Suppose that $\forall\ u \in G$ *directed corner,* $G1{=}G\backslash\{u\}$ *doesn't contain any directed corner. So* $|V(G1)| \geq 2$. *Without loss of generality, we can suppose G1 connected. As G is cop win, if the robber has its move restricted in G1, the cop will also win. It implies that* $\exists\ v \in G1$ *so that* $\exists\ w \in G$, $N_+[v] \subseteq N[w]$. *As we suppose earlier that G1 has no directed corner, such a vertex v is unique and* $w \equiv u$. *If* $\mid N_+(u)| \geq 2$ *or* $\mid N_+(v)| \geq 2$, *then G will contain a non oriented cycle of length 3. So both u and v has outdegree at most 1. As a consequence v will have outdegree at most 1 also in G1 and will be a directed corner in it because it's not the only vertex of G1.*

Suppose now that G admits a sequence of deleting n corners, with n an integer $n \geq 1$, *so that* G_n *(the subgraph induced by the deletion), contains a directed corner. Suppose that* $\forall\ u$ *directed corner* $\in G_n$, $G_{n+1} = G_n \setminus u$ *has* *no directed corners. Without loss of generality we can suppose G_{n+1} connected (otherwise we repeat the same idea on each connected components). As G is cop win, $\exists$ v$\in$ G_{n+1} so that $\exists$ w $\in$ G, $N_+[v] \bigcap G_{n+1} \subseteq N[w]$. As G contains no non oriented cycle of length 3, $\mid N_+(v) \bigcap G_{n+1} \mid \leq 1$. So G_{n+1} contains at least one directed corner.*

Conclusion : we obtain a sequence of deleting corners, so G is dismantable.

Corollary 1 *Bipartitite cop win digraphs are dismantable.*

As bipartite graphs has no cycle of length odd, theirs non oriented induced subgraphs don't contain any cycle of length 3.

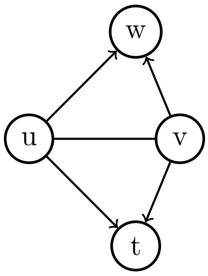
Before looking at some families of digraph, we show that in general some digraphs cop win are always dismantable under some conditions.

Theorem 8 *Let D be a digraph with no symmetric edges ($\not\leftrightarrow$), so that $\Delta(D) \leq 3$. Then D cop win $\Longrightarrow$ D dismantable.*

We show there exists a succession of deleting directed corners so D is dismantable.

At first, D is cop win so D has a directed corner. Let call it u. If D has several directed corners, then after deleting any of these directed corners, the sub-digraph contains also a directed corner. Nevertheless, if D has only one directed u, so v dominates u : N_+[u] $\subseteq$ N[v]

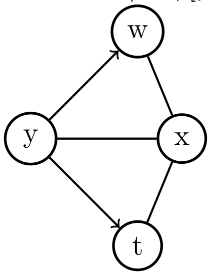
As D is cop win, if the robber is restricted in the subgraph D\{u} then he will be caught. So D\{u} has a directed corner x. If x $\not\equiv$ v, then, in D\x, we have u as a directed corner. But suppose this directed corner is v, so u and v codominates each other : if $\mid N_+$(u)\{v}$\mid \leq 1$ or the same for v, we can change the order of deletion so the sub digraph obtained will have a directed corner. Otherwise : $\mid N_+$(u)\{v}$\mid = 2$ and $\mid N_+$(v)\{u}$\mid = 2$.



Then, w and t are directed corner because their outdegree would be at most 1, so D\{u} contains also a directed corner t.

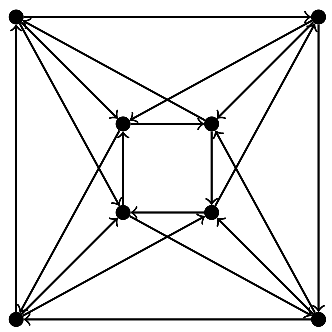
Moreover, let $\mathscr{P}$={u_1... u_n} be a succession of deleting directed corners in D. We can suppose D_n= D\ $\mathscr{P}$ is connected. Then as D is cop win, if the robber has its moves limited in D_n, he will be caught. So we have the following relation : $\mid D_n \mid$=1 (so D would be dismantable) or $\exists$ y $\in$ D_n so that $\exists$ x $\in$ D, N_+[y]$\bigcap$ D_n $\subseteq$ N[x].

If x∈ D_n or |N+[y]$\bigcap$ D_n | ≤ 1, then D_n has a directed corner. Otherwise if x∈ $\mathscr{P}$ and | N_+[y]$\bigcap$ D_n | $= 2$:



As Δ(D) ≤ 3, then w and t are directed corners in D_n. So we can delete each directed corners of D_n so D is dismantable by induction.

Remark 3 *we do not have an equivalence : Let D be a digraph with no symmetric edges ($\nleftrightarrow$), so that Δ(D)$\leq$ 3. Then D dismantable $\nRightarrow$ D cop win.*



Example 1 *In the following digraph, there is no directed corners. So the cop cannot limited the robber's move and catch him.*

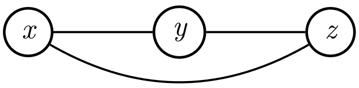
Theorem 9 *Outerplanar cop win digraphs are dismantable.*

Proof 5 *Let G be an outerplanar cop win digraph.*

If G has no non oriented cycle of length 3, then G is dismantable. Otherwise let y $\in$G be the vertex in the middle of a cycle of length 3.

We define a function $f:\ G \to G\backslash\{y\}$ so that $G\backslash\{y\}$ is a retract of G.

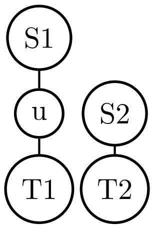
$f:\ y \to x$
$v \not\equiv y \to v$



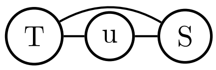
Remark 4 *This proof may be generalized to the family of series-parallel digraph which contains outerplanar digraph.*

Theorem 10 *Series-parallel cop win digraphs are dismantable.*

We need only look at the rules of composition in series-parallel digraphs.



by associating this two graphs into a series-composition, we obtain the cyle of length 3 that will be found in the digraph :



Proof 6 *u is a directed corner in D, because N[u]⊆N[S]. So, by repeating the same argument we defend for the outerplanar digraphs, we can delete the cyle of length 3 of the digraph, so it is always cop win. Without any cylce of length 3, we know eventually that the digraph is cop win.*

One question remains : we do not know wheter G cop win $\Longleftrightarrow$ G dismantable ? However, several digraphs, such as some grids, or some triangulated digraphs, have this characterisation.

Theorem 11 *Let G be a grid, so that if G has symmetric edges ($\leftrightarrow$), there are only in the horizontal or vertical direction. Then G is cop win and dismantable.*

Proof 7 *G is a grid cop win, so G has no non oriented cycle of length 3. As a consequence G is dismantable. Now we will show that if G has symmetric edges only in the horizontal or vertical direction, then G is cop win.*

We build a strategy for the cop to always capture the robber.

Without loss of generality, we suppose that symmetric edges are only in the horizontal direction, and let put the cop on the left top of the digraph. Then $\mathscr{R}$ chooses a vertex. At first, $\mathscr{C}$ will stay on the first vertical line, while its vertex is not aligned to the robber's one. As the grid is finite, it will always happen than $\mathscr{C}$ and $\mathscr{R}$ are aligned. It's the turn of the robber.

Then $\mathscr{C}$ follows this strategy :

- if $\mathscr{R}$ moves vertically, so does the cop in the same direction.

- if $\mathscr{R}$ moves horizontally or stays in the same vertex, the cop goes one vertex further on its right.

As there is no symmetric edges on the vertical lines, and the grid is bounded, $\mathscr{R}$ will be blocked at some times and will have to move horizontally *or stay in the same vertex. If $\mathscr{R}$ can move to the right, the distance between him and the cop will be kept. However, after a finite number of moves, the robber will be at best on the last vertical lines of the grid, so he could not go to the right anymore. At some stage, he won't be able to go the right or to move vertically anymore (because of the unidirection of vertical edges). So the distance between the cop and the robber will strictly decrease and the robber will be captured.*

Theorem 12 *Let D be a digraph with no symmetric edges, such as its non oriented induced graph is (n-2)-regular with $|V(D)|$=n. Then D is always cop win and dismantable.*

In the non oriented versus of the game, these graphs are not cop win.

Proof 8 *If n$\leq$ 4, it's easy to see D is cop win.*

Now, we denote by u_i and v_i, respectively, the vertex position of $\mathscr{C}$ and $\mathscr{R}$ at round i. Let's begin the game. At first, C chooses a vertex u_0, so $\mathscr{R}$ doesn't have a choice, in order not to be captured at the next round. Because $! \exists\ v_0 \in$ V(D) so that $v_0 \notin N[u_0]$.

First suppose $|\ N_+(v_0)|$=n-2. So if the robber moves from v_0, he will not be able to return in it, on another round, because D has no symmetric edges and $|N(v0)|$=n-2. If $\mathscr{C}$ goes on any vertex u_1 in $N(u_0)$, $\mathscr{R}$ will have to move from v_0. Then, on the next round, $\mathscr{C}$ returns on u_0 and dominates v_1 (the actual position of the robber).

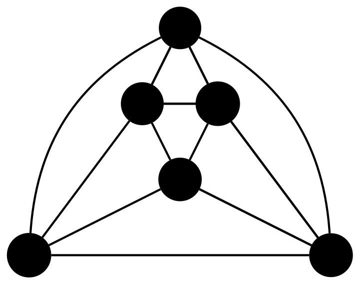
Otherwise $|\ N_+(v_0)| \leq$ n-3. In this case we show $\exists\ x \in N(u_0)$ so that $N_+(v_0) \subseteq N[x]$.

Let call X, the strict neigborhood of u_0 : $X \equiv N(u_0)$. $\forall\ y \in X$, $\{u_0,v_0\} \in N(y)$. As D is (n-2) regular, $|N(y) \bigcap X|$ = n-4. Then $\exists\ x \in X \setminus N_+(v_0)$. If x doesn't dominate v_0, so $\exists\ w \in N_+(v_0)$ so $w \notin N(x)$. But then, w is joined to all the vertices of $X \backslash \{x\}$. So in going in w at the next round, $\mathscr{C}$ will catch $\mathscr{R}$.

Conclusion : D is cop win.

Then, we prove that D is also dismantable. each vertex in X has its neighborhood in $X \bigcup \{u_0\}$ dominating by u0 itself. Also v_0 is a directed corner. So we can delete v_0, then each vertex in X, and we obtain the K1 graph composed only by $\{u_0\}$

We know that a cop win digraph with no non oriented cycle of length 3 is dismantable. What happens if we consider triangulated digraphs, so that there is only cycle of length 3 ? We do know such graphs wich are not cop win the original game :



We let the reader demonstrate that this upper graph is not cop win.

Theorem 13 *Let D be a triangulated digraph without any symmetric edges and $\Delta(G) \leq 4$. Then D is dismantable and cop win.*

Proof 9 *We begin by showing that D is dismantable. We proceed by induction.*

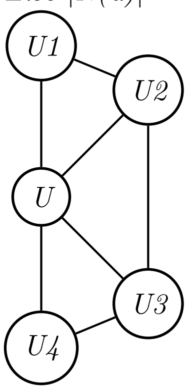
$K1$ is a triangulated digraph without any symmetric edges. Also $K1$ is dismantable.

Now we suppose that $\exists$ an integer n so that $\forall$ D triangulated digraph without any symmetric edges, so that $|V(D)|=n$ and $\Delta(D)=4$, D is dismantable.

Consider afterward a digraph D with such characteristics and $|V(D)| = n+1$ and $u \in V(D)$ a vertex on the external face of D :

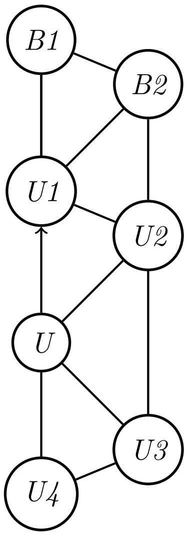
If $|N(u)| \leq 3$, then u is definitively a directed corner and $D \backslash u$ is still a triangulated digraph wich respects the properties so D is dismantable.

Else $|N(u)| = 4$.



If $N_+(U) \subseteq \{ \{U1, U2, U3, U4\}, \{U1, U2, U3\}, \{U1, U2\}, \{U1, U3\}, \{U1, U3, U4\}, \{U2, U3, U4\}, \{U2, U3\}, \{U2, U4\}, \{U3, U4\}, \{U1\}, \{U2\}, \{U3\}, \{U4\} \}$ then U is a directed corner,and $D \backslash U$ is still a triangulated digraph wich respects the properties so D is dismantable.

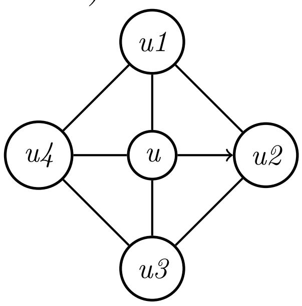
Else, without loss of generality we can suppose $U1$ (or $U4$) $\in N_+(U)$.



As D has no symmetric edges, $N_+(U1) \subseteq \{ B1, B2, U2\}$ so U1 is a directed corner. Because U1 is in the external face of D, $D \setminus U1$ is still a triangulated wich respects the properties so D is dismantable.

Now we propose a strategy for the cop to always capture the robber. We prove that if the cop always take the shortest path to the vertex position of the robber, he can decrease by one its distance after a certain number of rounds, so $\mathscr{R}$ will eventually be caught.

Suppose that at the beginning of a round, $\mathscr{C}$ is on a vertex v and $\mathscr{R}$ in on u, with $|N(u)| = 4$ (the cases when $|N(u)| \leq 3$ are dealt with by analogous arguments).



$\mathscr{C}$ chooses the shortest path from v to u. By this way, he can always reduce its distance from the robber. Indeed, we look at the robber's new vertex when dist($\mathscr{C}$, $\mathscr{R}$)$\geq$2 and it's the robber's turn to move.

- If $\mathscr{R}$ stays on u, then the cop reduces its distance by one.

- Else, without loss of generality, we can suppose $\mathscr{R}$ goes on u2.

Let $\mathscr{P}$ ={v, ...,u} the cop's trajectory. If $u_2 \in \mathscr{P}$ then the distance is reduced by one. Otherwise if $u_4 \in \mathscr{P}$ it is equivalent to go on u_1 or u_2 (because D is a triangulated digraph with $\Delta(D) \leq 4$, the predecessor of u_4 is a predecessor of u_1 or u_2). So the trajectory from v to u, is equivalent to the *trajectory from v to u_2 or u_1 and $\mathscr{C}$ can diminish its distance by one during this round.*

Then what's happen when dist($\mathscr{C}$, $\mathscr{R}$) =1 ?

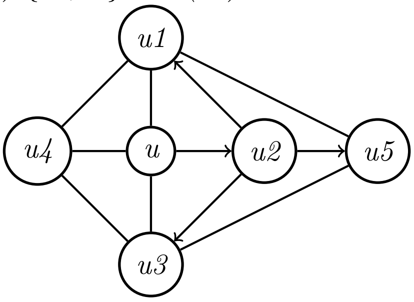
- If $\mathscr{R}$ stays on u, then the cop captures $\mathscr{R}$

- Else, without loss of generality, we can suppose $\mathscr{R}$ goes on u_2. If $\mathscr{C} \in \{u_1,u_2,u_3\}$, then the robber is captured, by the time the cop moves. Otherwise $\mathscr{C}$ is in u_4. We study the different configuration of the neighborhood of u_2 :

1°$N_+(u_2) \subseteq \{u_1,u_3\}$ C goes on u, so $\mathscr{R}$ vertex position is dominated.

2°$N_+(u_2) = \{u_1,u_3,u_5\}$

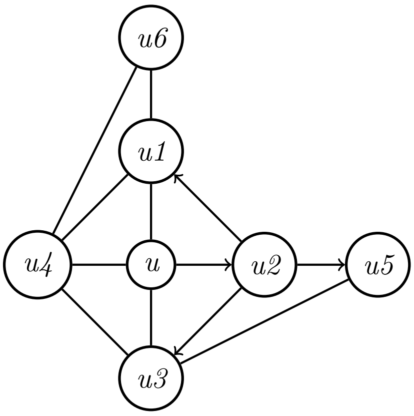
a) $\{u_1,u_3\} \subseteq N(u_5)$



As $\Delta(D) \geq 4$ and D is triangulated, u_5 has at most one neighbor different from $\{u_1,u_2,u_3\}$. However, u_1,u_2 and u_3 has already 4 neighbors so they can't be related to another one. And as D contains only triangle internal face, the new vertex would be joinded to u_1 or u_2. So $N(u_5) = \{u_1,u_2,u_3\}$

$\mathscr{C}$ goes on u_1, so R has no other choice but going in u_3. Then C goes in u_4, so $\mathscr{R}$ can only go (if it's possible with the direction of the edges) in u_5. Then $\mathscr{C}$ goes on u, then u_2, so $\mathscr{R}$ cannot escape from u_5..

b) $u_1 \notin N(u_5)$



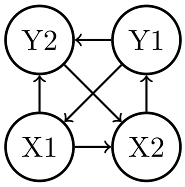
C goes in u_3 then R has no other choice but going in u_1 (if it's possible, otherwise he is caught). Then C returns in u_4 which dominates u_1 because Δ(D)=4.

Conclusion : D is cop win and dismantable.

7 The issue of symmetric edges

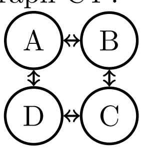
We present here a transformation in order to solve the issue of symmetric edges. Indeed there exists some transformation so the new digraph obtained will be cop win if and only if the precedent digraph is.

Proposition 14 *Let consider a symmetric edges in a digraph D : $x \leftrightarrow y$ with x,y $\in$ V(D). We build a new digraph $\mathscr{D}$ with no symmetric edges by repeating this method :*

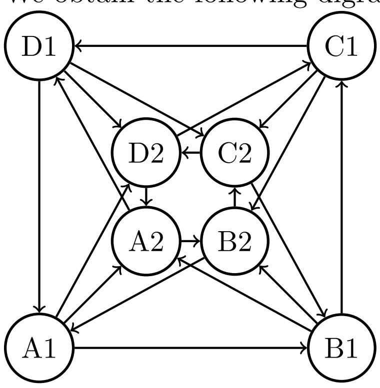


X1 and X2 are associated with the neighborhood of X, so are Y1,Y2.

So as to better understand, we give an example of the transforming digraph C4 :



We obtain the following digraph :



Lemma 4 *Let D be the digraph, and $\mathscr{D}$ be the transformed digraph of D by the upper function. D dismantable $\Longrightarrow$ $\mathscr{D}$ dismantable.*

Consider the deletion of directed corners in D : x,y so that N_{+}[x] $\subseteq$ N[y]

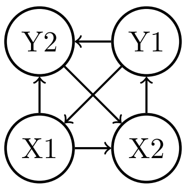
even if x has been duplicated (or one vertex in its neighborhood) into a subset of vertices $\{x_1,x_2\}$ in $\mathscr{D}$, $N_{+}[x_2] \equiv$ N[x], and $N_{+}[x_1]\backslash$N[x] without considering the vertices duplicated. So, even if y has also been duplicated, y_1 (or just y) will dominate x_2 and x_1 (or x if it still exists in $\mathscr{D}$). So we can remove x_1 and x_2, so the new digraph obtained from $\mathscr{D}$ is the transformed digraph of D$\backslash\{$x$\}$

Remark 5 *There is no equivalence, as we see with the upper example C4. C4 is not dismantable because there aren't any directed corners. Unfortunately we give an example of corner's deletion in its transformation :* $\{A2, D2, C2, B2, A1, B1, C1, D1\}$

Lemma 5 *Let* D *be the digraph, and* $\mathscr{D}$ *be the transformed digraph of* D *by the upper function.* D *cop win* $\iff \mathscr{D}$ *cop win.*

Proof 10 D *is an homomorphism of* $\mathscr{D}$ *: we associate any pair of duplicated vertices in* $\mathscr{D}$ *with its predecessor :* $\{x_1,x_2\} \rightarrow x$*. To sum up, if* $\mathscr{D}$ *is cop win, so is* D*.*

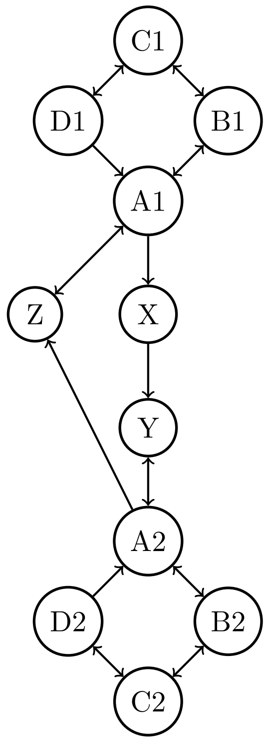
Moreover, suppose now D *is cop win. We create a strategy for the cop in* $\mathscr{D}$ *by always going in* x_1 *if he moves on* x *in* D*. Note that if* x *has not been modified in* $\mathscr{D}$ *then* $x_1 \equiv x$*. Suppose now* $\mathscr{C}$ *dominates* $\mathscr{R}$ *in* D *and let* $(x,y) \in V(G)$ *their position. We see that, even if* y *or* x *(or both) has been duplicated into a subset of two vertices,* x_1 *wich is the actual position of the cop, dominates* y_1 *and* y_2 *:*



8 Dismantibility

We will raise the issue of the notion of dismantability, in particular that some dismantable digraphs are not cop win.

Remark 6 *Whereas the notion of connected component is not included in the dismantable definition in the non oriented game, it's necessary when you play onto digraphs.*



We give an example : let G be the following digraph

First of all, we show that G loose its connexity during the deletion of directed corners.

At the beginning, the only directed corners are Y,X, and Z. Because A1 and A2 has Z and respectively X and Y as out-neighbors, for deleting A1 and A2 you have to delete Z,X, and Y first. But by this way, you will loose the connexity. Then in each component, you can delete in this order : [A1,D1,C1,B1] and [A2,D2,C2,B2]. So G is dismantable.

Then we prove, by checking every robber's initial position, that G is cop win. The main idea is that the robber has always a unique choice for its moves.

At the beginning, we put the cop on C1. Now we test each vertex different from C1 :

- $\mathscr{R}$ is on B1 or D1, so $\mathscr{R}$ is on the neighborhood of C1. As the cop begins each round, $\mathscr{R}$ will be captured.

- $\mathscr{R}$ is on A1. $\mathscr{C}$ moves in B1. Because the robber will be blocked in Z, $\mathscr{R}$ goes on X. Then he will not be faster enough to go out of the branch [X,Y,A1] before the cop arrives in A1. So $\mathscr{R}$ will be captured.

- $\mathscr{R}$ is on Z. $\mathscr{C}$ goes on B1 so $\mathscr{R}$ stays on Z and is captured.

- $\mathscr{R}$ is on v $\in$ {X,Y,A2,B2,C2,D2}. $\mathscr{C}$ goes sucessively on B1, Z, A2, B2, C2, B2, A2, Z, A1, B1, C1. After this moves, either the robber has already been captured, either he is on a vertex which has already been studied so the robber will be captured.

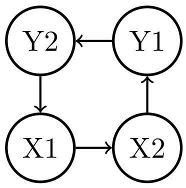
Remark 7 *We notice that if there wasn't a symmetric edges between A2 and Y, the graph won't lose its connexity during the deletion of directed corners. Indeed symmetric edges are irrevelant in the problem.*

A simple transformation to make abstraction of the symmetrics edges :

Lemma 6 $\forall$ *G digraph with symmetric edges,* $\exists$ G_1 *which doesn't contain any symmetric edges so that* G_1 *cop win* $\leftrightarrow$ *G cop win*

Let G be any digraph with symmetric edges and f the following transformation ont the vertices of G :

x $\in$ G such that $\exists$ y $\in$ G so that x $\longleftrightarrow$ y : we build two vertices x_1 and x_2 which conserves the same edges with the rest of the digraph and which have new relations with two vertex y_1 and y_2 like we see in the following



drawing : The other vertices are not modified.

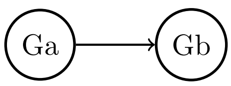
We will basically show that if G_1 is cop win, so is G. Indeed if we consider the induced non oriented grapg of G and G_1, we can say that G is an homomorphism of G_1 with vertices whith no symmetric edges which stay the same and X1 and X2 wich has X as their image by f-1.

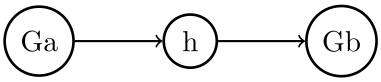
If G_1 is cop win, we imitate the cop's strategy by taking the vertex obtained by f-1 because the cop doesn't follow the direction of the edges..

The main issue is knowing if every dismantable digraph is cop win. Unfortunately, this is not true for a large number of dismantable digraphs.

Lemma 7 $\exists$ *D digraph, D dismantable* $\nRightarrow$ *cop win*

Let use a simple transformation. Let G be a digraph with no symmetric edges. We build a new digraph by putting in the middle of each edge a new vertex wich keeps the last orientation :





As a consequence, G is an homomorphism of H : we identify the new vertex h to Ga. So if H is cop win, the cop in G has just to copy the same strategy by identifying every vertex created in H by its in neighborhood (which is unique because G contains no symmetric edges).

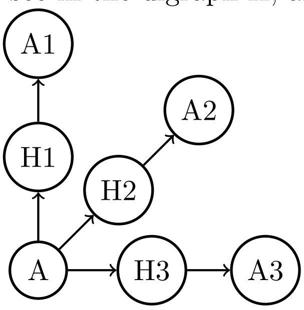
H cop win $\Longrightarrow$ G cop win.

Also we remark that the digraph obtained from this transformation will always be dismantable if the latest one doesn't contain any symmetric edges. We prove it by induction :

K1 is dismantable, and the image by the transformation from itself.

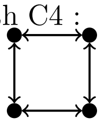
Suppose now that $\exists$ n so that $\forall$ D digraph with no symmetric edges, the transformed digraph H is dismantable.

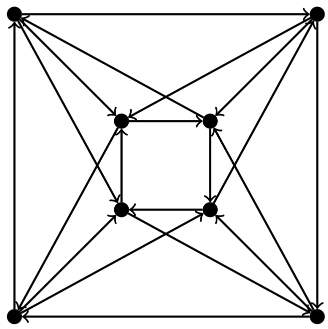
In a digraph D with $|V(D)|$ = n+1. $\exists$ a $\in$ D, which is not a cut edge and hi the family of vertices putting between a and its out neighborhood. Let see in the digraph H, after the operation onto the vertices :

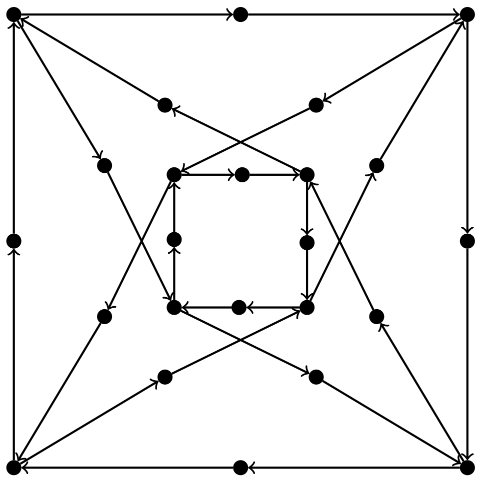


We have a sequence of directed corners [H1,H2,A,H3] so that we obtain the transformed digraph of another digraph of D\a, which is dismantable. So H is dismantable. Unfortunately, we already notice that we have some non cop win digraphs, with non symmetric edges. By this transformation, we build some dismantable digraphs non cop win.

For example, by creating a non cop win digraph with no symmetric edges with C4 :





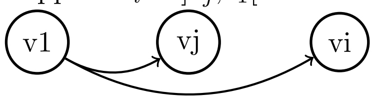


Lemma 8 *Let D be an outerplanar digraph with no symmetric edges. Then δ+(D) $\leq$ 1.*

We can order the set of vertices in D by calling them $\{v_1 ... \ v_n\}$.

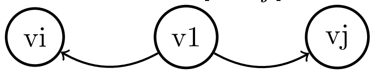
Suppose δ+(D)$\geq$2. Take v_1 with $\mid N_+(\text{u})\mid \geq 2$. We call v_i and v_j two of its outneighbors :

Suppose $v_i \in \,]v_j, v_1[$ like it's represented in the following drawing :



Without loss of generality, we can make the assumption : $\exists \, a_1 \in N_+(v_i) \bigcap \,]v_i, v_1]$ because $\mid N_+(v_i)\mid \geq 2$ so that, as D is outerplanar $N_+(a_1) \subseteq \,]v_j, v_1]$. Then recursively, we suppose a_1 has an out-neighbor in $[v_1, a_1[$ (respectively in $]a_1, v_1]$, so the size of the intervall which contains the outneighborhood of the vertex studied is strictly decreasing by at least one, so $\exists$ x $\in \,]v_1, v_i[\mid N_+(\text{x})\mid \leq$1.

Otherwise $v_i \in \,]v_1, v_j[$, like it's represented in the following drawing :



If $N_+(v_i)\subseteq]v_i,v_j]$ (as D is outerplanar so any edges crosses each other and doesn't contain symmetric edges v_i can't have a neighbor in $[v_j,v_1]$), then we modify the numerotation of D so vi becomes v_1 and we come back to the first hypothesis so δ+(D)$\leq$1.

Otherwise by repeating the same arguments in the subset $N_+(v_1)\bigcap]v_1,v_i[$ that those used for the first hypothesis, we demonstrate that $\exists$ x $\in]v_1,v_i[\mid N_+(\mathrm{x})| \leq 1$.

Theorem 15 *Let D be an outerplanar digraph with no symmetric edges. Then D is dismantable.*

Because of the upper lemma we know that δ+(D)$\leq$1. So D has a corner Let call it x. After its deletion, all the connected components of D$\setminus$\{x\} are outerplanar digraph with no symmetric edges. So by repeating the same argument, D is dismantable.

9 Conclusion

The question whether all cop win digraphs are dismantable remains unknow. but, several cop win digraphs' families are dismantable so we still tkink the implication does exist. Unfortunately, many dismantable digraphs are not cop win, which leads to think that a sub family of dismantable digraphs may also be cop win, so we would have had an equivalence.

We present here the main conjecture of this report which remain unresolved and we encourage the reader to think about these issue.

Conjecture 1 $\forall$ *digraph D, D cop win* $\Longrightarrow$ *D dismantable.*

Some sub questions are still revelant to improve our knowledge on the problem. We have seen that the resolution of the probleme is harder when the digraph has symmetric edges. Even if in the general case we can make abstraction of these by transforming the graph, into a new equivalent one for the cop win's property, we lose some basic characteristics such that planarity or minimum outdegree.

Conjecture 2 $\forall$ *planar digraph D, D cop win* $\Longrightarrow$ *D dismantable.*

Conjecture 3 $\forall$ *planar digraph D with no symmetric edges, D cop win* $\Longrightarrow$ *D dismantable.*

We have the same conjecture adapted to k-outerplanar digraph.

Then, we think that the definition of dismantibility must be reduced to a subset of the family, so D digraph cop win $\Longleftrightarrow$ D dismantable.

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