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Internship report

Graph decompositions and treelength

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Abstract

The treelength of a graph is the largest distance between two vertices of a bag of a tree-decomposition, over all tree-decompositions of the graph. Thanks to a tree-decomposition with small length, we can resolve some problem as the Traveling Salesman Problem in polynomial time, or approximate the treewidth of a graph. However, deciding whether a graph has a treelength at most 2 is NP-hard. In planar graph, the complexity of computing the treelength is an open problem. For some sub-classes of planar graphs, the treelength is already known as for Outerplanar graph and trees. The next sub-class of planar graphs to be studied is naturally the one of series-parallel (SP) graphs. Our main results are an $\frac{3}{2}$-approximation algorithm, and the characterization of SP graphs with treelength at most 2 in terms of forbidden isometric subgraphs. To conclude, we also present some preliminary results for the characterization of SP graphs with treelength at most 3.

1 Introduction

Tree-decomposition of graphs are a powerful tool to decrease the necessary complexity of several problem as the Independent Set problem or the Gate Matrix Layout Problem...[1]

Roughly a tree-decomposition of a graph G is a representation of this graph into small pieces, called *bags*, which form together a tree with some properties. To a tree-decomposition, a measure is associated like the *width* which is the maximum number of vertices minus one in the bags and represents the idea "how far is this decomposition from a tree". The *treewidth* of a graph G, $tw(G)$, is the minimum width over all the tree-decomposition of G.

Examples. The treewidth of a tree (or path) is equal to 1 by definition, the treewidth of a cycle is equal to 2 [8], the treewidth of a *series-parallel* graph, a graph with two distinguished vertices called terminals, which is formed recursively by two simple composition operations, a parallel and a series composition, is also equal to 2. More precisely, a graph has treewidth 1 if and only if it is a tree, and has treewidth at most 2 if and only if it is a series-parallel graph.

Given a smallest tree-decomposition of a graph according to the width, we can resolve many problems much faster using dynamic programming. Courcelle's theorem [7] show that every graph property definable in the monadic second-order logic of graphs can be decided in linear time on graphs of bounded treewidth. In fact, a lot of graph property can be written in the monadic second-order logic as clique, colorabilty or independent vertex set.

Complexity. However, finding a tree-decomposition of a graph with minimum width is NP-hard, even NP-complete for bipartite graph. Given a fixed parameter k, there exists an algorithm which recognizes if we can construct a tree-decomposition of width k in time $O(n^{k+2})$ [2]. There exists also, for a fixed parameter k, a linear time algorithm which finds a tree-decomposition of a graph with width at most k, if it exists, in time $O(n \cdot k^{k^3})$ (exponentiel in k) [4]. The best known approximation algorithm to find the treewidth of general graphs is a $O(\sqrt{log(tw(G))})$-approximation algorithm [10]. A lot of work has been done on the computation of the treewidth of different classes as *outer planar* [8] (a graph with a planar drawing for which all vertices belong to the outer face of the drawing) and series-parallel graphs which have a linear time algorithm to compute a tree-decomposition of minimal width. It is proved for others classes that the complexity is polynomial as for *chordal* graphs, graphs in which all cycles of four or more vertices have an edge that is not part of the cycle but connects two vertices of the cycle...[3] For the particular case of planar graphs, the complexity is still open.

Treelength. The width is not the only measure related to a tree-decomposition. There is also the *length* representing the diameter of the bags or the *breadth* which correspond to the radius of the bags. The *treelength*, the minimum length over all the tree-decomposition of G, has also been study but a lot of work can still be done. The treelength of a tree (or path) is equal to 1, the treelength of a cycle of size k is equal to $\lceil \frac{k}{3} \rceil$. A subgraph H of G is *isometric* if, for every pair of vertices of H, the shortest distance between them in H or in G are equal. One interesting fact is that the presence of an isometric cycle of size k gives us a lower bound on the treelength of a graph G ($tl(G) \geq \lceil \frac{k}{3} \rceil$) [8]. It was shown that recognizing graphs with a treelength bounded by a fixed constant $k \geq 2$ is NP-complete [11] which is more difficult than recognizing graphs with a treewidth bounded by a fixed constant $k \geq 2$ since there exists an algorithm that do it in time $0(n \cdot 2^{2^3})$ [4]. Also, there exists a 3-approximation algorithm to compute a tree-decomposition of minimum length for general graph [8] which is a much better approximation than those for the treewidth.

Moreover, there is a relation between the treelength and the treewidth, the treelength $tl(G)$ of any graph G is at most $\lfloor \frac{is(G)}{2} \rfloor$ times its treewidth where $is(G)$ is the length of a longest isometric cycle in G and for any planar graph P, $tw(P) = O(tl(P))$ [6][8]. Then it is interesting to focus on the treelength that will probably helps us on the computation (exact or approximation) of the treewidth on certain classes of graphs.

These facts motivate us to study the treelength of planar graphs. Of course it is easier to begin by smaller classes of graphs. It is easy to construct a tree-decomposition of length 1 for a tree. For outerplanar graphs, the treelength has already been studied:

Theorem 1 *[8] Every outerplanar graph has treelength $\lceil \frac{k}{3} \rceil$, where k is the length of a maximum isometric cycle of the graph.*

Series-parallel graph. The next sub-classes of planar graphs is the sub-class of series-parallel graphs. For instance, this sub-class can be used as a representation of electronic or electric circuit. As we described before, they are graphs that can be constructed recursively by two simple composition operations, a parallel and a series composition. Some particular graphs of series-parallel graphs have already been studied. Precisely, *melon* graphs are a parallel composition of several paths $(P_1, \cdots, P_p)$ sorted such that P_p is a shortest path and P_1 is a longest path (see in Figure 1). Note that a shortest path P_p form with any other path, an isometric cycle and the union of a shortest and a longest path give us a maximum isometric cycle which gives a lower bound of the treelength of a graph. Finally, the union of two paths that are not a shortest path, is a cycle not isometric since there exists a shortest path between two vertices. If the difference of size between a longest and a shortest path is "big" then it is easy to construct a valid tree-decomposition of length $\lceil \frac{is(G)}{3} \rceil$ but when the difference is "small", the treelength increases to the length of a shortest path or even to $\lceil \frac{a_1+a_2}{3} \rceil$ where a_1 and a_2 is the size of the two longest paths. More precisely, Ducoffe, Nisse and Nivelle have proved:

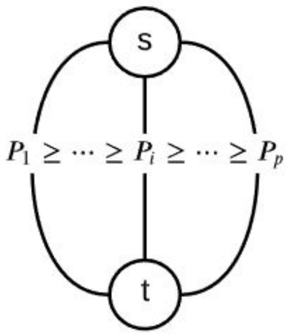

Figure 1: melon graphs

Theorem 2 *Let $G = (P_1, \cdots, P_p)$ be a melon graph with $|P_i| \geq |P_j|$, for all $1 \leq i \leq j \leq p$.*

- $t\ell(G) = \lceil \frac{|P_1|+|P_p|}{3} \rceil$ *if* $|P_p| \leq \lceil \frac{|P_1|+|P_p|}{3} \rceil$;
- $t\ell(G) = |P_p|$ *if* $\lceil \frac{|P_1|+|P_p|}{3} \rceil \leq |P_p| \leq \lceil \frac{|P_1|+|P_2|}{3} \rceil$, *and*
- $t\ell(G) = \lceil \frac{|P_1|+|P_2|}{3} \rceil$ *otherwise.*

Clearly, the work that has been done on melon graphs shows us that it seems hard to compute an optimal tree-decomposition for any series-parallel graph.

Our contributions. This paper presents the work we have done on series-parallel graph. We will begin by the definition of a tree-decomposition, a series-parallel graph, an ear decomposition and the treelength and recall some interesting properties in section 2. Then, in section 3, we present minimal forbidden isometric subgraphs (MFIS) for series-parallel graphs of treelength 2 and we proved that if a series-parallel graph G does not contain any of them, then we can construct a tree-decomposition of G of length 2 using *nested ear decomposition* which is a representation of series-parallel graphs into small parts such that every small parts are represented by *an ear* (i.e. a path) [9]. Finally, in Section 4, we will present an $\frac{3}{2}$-approximation algorithm using nested ear decomposition for computed the treelength of a series-parallel graph.

2 Definition and notation

We are going to search some properties about the tree-decomposition of minimal length of series-parallel graphs. To do that, we need some definitions:

A *graph* is an abstract model composed by vertices and edges between them, edges can be directed or undirected. A *path* is a finite sequence of distinct vertices where two consecutive vertices are adjacent. A *cycle* is a path where the first and the last vertex are the same. A *tree* is a connected acyclic undirected graph where a connected graph is a graph in which there is a path between every two vertices.

Definition 1 *[8] A tree-decomposition of a graph G is a tree $(T, \mathcal{X} = \{X_t | t \in V(T)\})$ whose nodes, called bags, are subsets of $V(G)$ such that:*

- $\bigcup_{t \in V(T)} X_t = V(G)$*;*
- $\forall \{u, v\} \in E(G)$*, there exists* $t \in V(T)$ *such that* $u, v \in X_t$*;*
- $\forall t, y, z \in V(T)$*, if* X_y *is on the path from* X_t *to* X_z *in* T *then* $X_t \cap X_z \subseteq X_y$*.*

A subset S is a *separator* of a graph G if $G \backslash S$ is not a connected graph. Moreover, if the tree-decomposition is *reduced*, which means that their isn't a bag contained in another one, the third condition implies that for any two adjacent bags X and Y of T, $X \cap Y$ is a separator of G.

Let us define some notations that will be used later. For a graph $G = (V, E)$ and $x, y \in V$, let $P_G(x, y)$ (or $P(x, y)$ if G is obvious) denote a shortest path from x to y in G. Let $dist_G(x, y)$ or $d_G(x, y)$ be the number of edges

of $P_G(x,y)$. Let $diam_G(X)$ be the maximum distance between two vertices of X in G (i.e. $diam_G(X) = max_{v_1,v_2 \in X} dist_G(v_1,v_2)$). Let the length of a tree-decomposition T of a graph G be defined such that:
$length(T) = max_{X \in V(T)} diam_G(X)$. The treelength of a graph G equals the minimum length over all tree-decompositions that G admits.

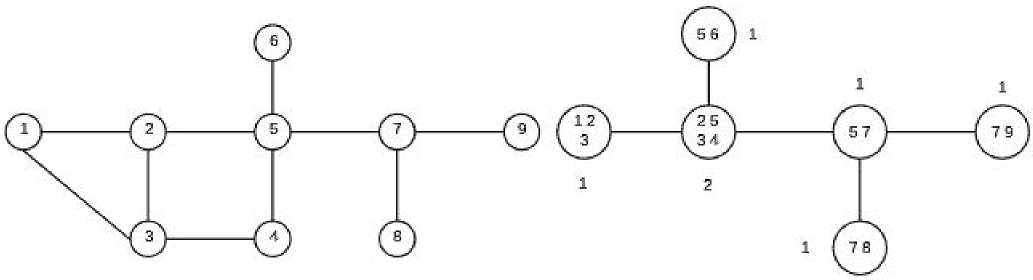

Figure 2: Example of $(T, \mathcal{X})$ (right), a tree-decomposition of minimum length for the graph G (left) with each bag's diameters (integer outside the bags)

For example, let us consider the bag X composed of the vertices 2, 3, 4 and 5 in Figure 2. $P_G(2,4)$ is $\{2,3,4\}$ or $\{2,5,4\}$, so the minimum distance between 2 and 4 is 2. It is the same for any pair of vertices not adjacent of X, which means that the diameter of X is 2. The maximum length of a bag in $(T, \mathcal{X}$ is 2, so the length of $(T, \mathcal{X}$ is equal to 2.

Lemma 1 *[8] The treelength of any cycle C_k (of length k) is $\lceil \frac{k}{3} \rceil$.*

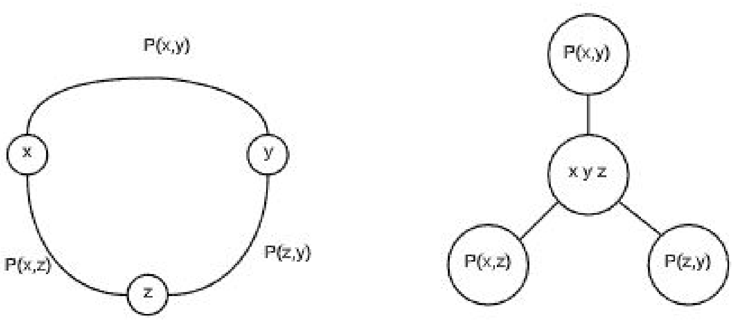

Figure 3: Example of $(T, \mathcal{X})$ (right), a tree-decomposition of minimum length $\lceil \frac{k}{3} \rceil$ for the graph C_k (left) with x, y and z which are pairwise at distance at most $\lceil \frac{k}{3} \rceil$.

Let $G = (V, E)$ be a graph and let H be a subgraph of G. The subgraph H is *isometric* in G if, for every $a, b \in V(H)$, $dist_G(a, b) = dist_H(a, b)$. Note that isometric cycles are a good lower bound for the treelength thanks to the previous and the following lemmas.

Lemma 2 *[8] For every graph G and every isometric subgraph H of G, we have $t\ell(G) \geq t\ell(H)$.*

Thanks to these lemmas, we can show that the treelength of G (in Figure 2) equals 2 because $\{2, 3, 4, 5\}$ is an isometric cycle C of G and that by Lemma 1, it's treelength is 2 and by Lemma 2, that $tl(G) \geq tl(C)$.

Let $is(G)$ be the size of a largest isometric cycle in G. From the previous lemmas, the following theorem can be proved.

Theorem 3 *[8] For every graph G, we have $t\ell(G) \geq \lceil \frac{is(G)}{3} \rceil$.*

Now, let us define formally what is a Series-parallel graph.

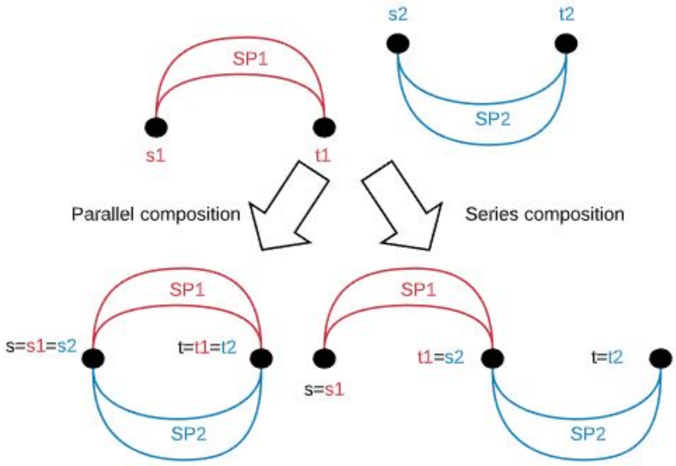

Figure 4: Series and parallel composition for two series-parallel graph

Definition 2 *[9] An undirected graph (G, s, t) is two-terminal series-parallel, with terminals s and t, if it can be recursively produced by a sequence of the following operations (see Figure 4):*

- *Create a new graph, consisting of a single edge with s and t as end-points (K_2).*
- *Parallel composition: Given two two-terminal series-parallel graph (X, s_X, t_X) and (Y, s_Y, t_Y), form a new graph $G = P(X, Y)$ which is the union of X and Y such that $s = s_X = s_Y$ and $t = t_X = t_Y$.*

- *Series composition: Given two two-terminal series-parallel graph* (X, s_X, t_X) *and* (Y, s_Y, t_Y) *form a new graph* $G = S(X, Y)$ *which is the union of* X *and* Y *such that* $s = s_X$, $t_X = s_Y$ *and* $t = t_Y$.

An undirected graph G *is series-parallel, if there exists two vertices* s *and* t *such that* (G, s, t) *is two-terminal series-parallel.*

In a tree T rooted to r, let a *child* of v be a vertex u such that $d(r, u) = d(r, v) + 1$ and $(u, v) \in E(T)$. Note that every series-parallel can be represented by a tree $\mathcal{T}$ (see Figure 5) where the root is the last composition used to formed G, the *leafs* (node with no child) are edges and each node, that is not a leaf, is the parallel or the series composition of its 2 children . Moreover, $\mathcal{T}$ can be computed in linear time [12].

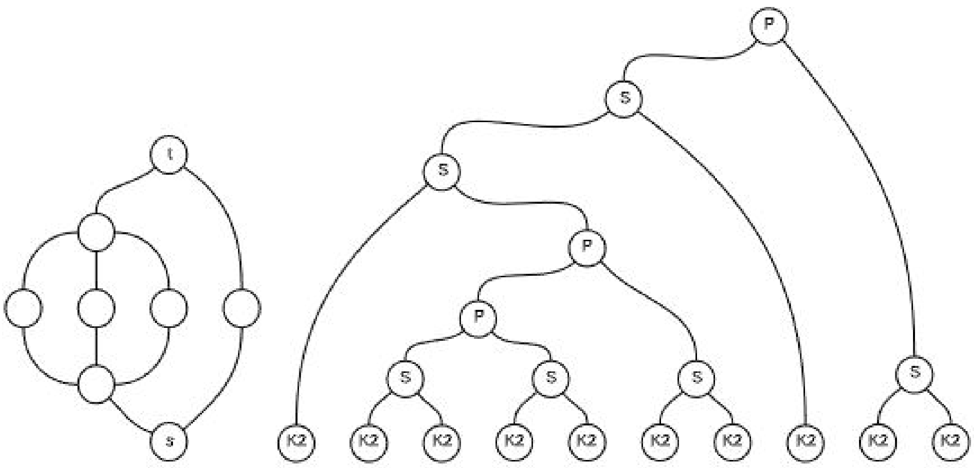

Figure 5: $\mathcal{T}$ (right) of an SP graph (left)

Claim 1 *Let* G *be a graph with parallel edges and loops and* G' *be obtained from* G *by removing all loops and the parallel edges (keeping one edge for each set of parallel edges). Then* $t\ell(G) = t\ell(G')$.

Proof. Let $(T', \mathcal{X}')$ be any tree-decomposition of G'. Let see that $(T', \mathcal{X}')$ is also a tree-decomposition of G because every edge e in $G \backslash G'$ has either an edge e' in G' that have the same endpoints (i.e. e and e' are parallel edges), and since the endpoints of e' are in a bag of $(T', \mathcal{X}')$, the endpoints of e are in a bag too, or the endpoints of e are the same (i.e. e is a loop) and then are contained in a bag of $(T', \mathcal{X}')$. Let $(T, \mathcal{X})$ be any tree-decomposition of G. Let see that $(T, \mathcal{X})$ is also a tree-decomposition of G' since G' is an isometric subgraph of G such that $V(G) = V(G')$. We can conclude that $tl(G) = tl(G')$. ■

By previous Claim, in what follows, we only consider simple graphs (without loops not parallel edges). A subset S is a *clique-separator* of a graph G if S is a separator and a clique (i.e. for every vertices x and y in S, $d(x, y) = 1$).

Theorem 4 *[8] Let G be any graph and S be a clique-separator. Let $\mathcal{C}$ be the set of the connected components of $G \setminus S$. Then, $t\ell(G) = \max_{C \in \mathcal{C}} t\ell(C \cup S)$.*

From previous theorem, we will focus on series-parallel graphs without clique-separators. Let a *biconnected* graph be a graph that has no separator of length 1. In what follows, the SP graphs are *biconnected* and have no edge separators.

Since isometric subgraphs and more precisely, isometric cycles are related to the treelength of a graph (see theorem 3) we will use the ear decompositions of a series-parallel graph to decompose wisely the graph into small parts and use it to construct a tree-decomposition by induction on the parts of the ear decomposition.

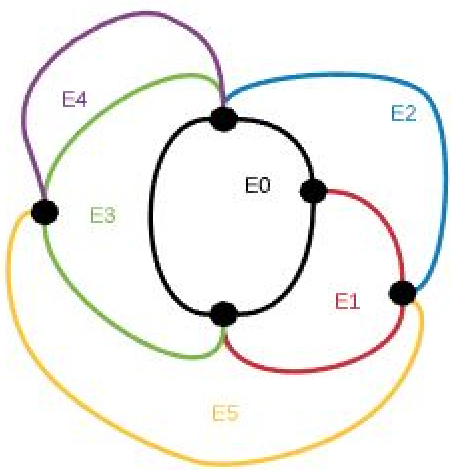

Figure 6: Example of an ear decomposition

Definition 3 *[5] An ear decomposition of an undirected graph G is defined to be a partition of the edges of G into a sequence of ears $E_0, E_1, \cdots, E_p$ such that E_0 is a cycle, and each E_i with $1 \leq i \leq p$ is a path in the graph such that $V(E_i) \cap V(G_{i-1}) = \{a_i, b_i\}$ where G_{i-1} is the subgraph induced by*

$\bigcup_{j \le i-1} V(E_j)$ *and* a_i *and* b_i *are the attachment vertices (i.e. endpoints) of* E_i*.*

Theorem 5 *[5] Every biconnected graph has an ear decomposition.*

Definition 4 *An ear decomposition is nested (see Figure 7):*

- *if the attachment vertices of* E_i *with* $i \ge 1$*,* a_i *and* b_i*, appear in a previous ear* E_j*, with* $j < i$ *(i.e.* $\exists j < i$ *such that* $a_i, b_i \in V(E_j)$*). Let say that* E_i *is attached to* E_{j_i} *where* j_i *is the smallest index such that the endpoints of* E_i *appear in* E_{j_i}*.*
- *If two ears* E_i *and* E'_i *are both attached to some ear* E_j*, then either* $P_i = P_{E_{j_i}}(a_i, b_i)$ *contains* $P'_i = P_{E_{j'_i}}(a'_i, b'_i)$*, or vice versa, or* P_i *and* P'_i *are disjoint.*

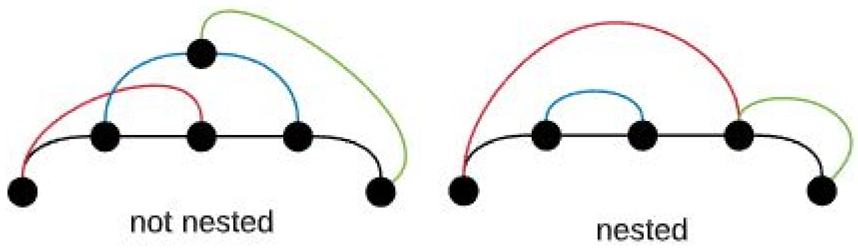

Figure 7: Example of nested and not nested ears

Theorem 6 *[9] Every biconnected series-parallel graph has a nested ear decomposition.*

Note that, if we have a maximal isometric cycle C, then we can compute a nested ear decomposition of a biconnected series-parallel graph G in polynomial time. Let $N_G(v)$ be the neighbors of v in G (i.e $N_G(v) = \{w \in V(G) | (v, w) \in E(G)\}$) and let $N_G(S)$ be the set of vertices in G adjacent to a vertex in S (i.e $N_G(S) = \bigcup_{v \in S} N_G(v) \setminus S$).

- Step E_0: $G_0 = G[V(C)]$
- Step E_i with $1 \le i \le p$: Let $C_1, \cdots C_k$ be the k connected component of $G \setminus G_{i-1}$. Let C^* be any component C_j for any $1 \le j \le k$ union $V(N_{G_{i-1} \cup C_j}(C_j))$. Note that $V(N_{G_{i-1} \cup C_j}(C_j)) = \{a_i, b_i\}$. Let P and P' be respectively a shortest path between a_i and v, and b_i and v for any $v \in C_j$ such that $P \cap P' = \{v\}$. Let $E_i = P \cup P'$. Finally, $G_i = G[V(G_{i-1}) \cup V(E_i)]$.

However, computing a maximum isometric cycle in a biconnected series-parallel graph can be done in polynomial time.

Lemma 3 *For any series-parallel graph G, a maximal isometric cycle of G and its length can be computed in polynomial time.*

Proof. For any series-parallel graph G, we can compute in linear time its tree $\mathcal{T}$ [12]. Let us note that the leafs of $\mathcal{E}$, are edges. For each leaf (L, s, t), $d(s,t) = 1$ and $is(L) = \infty$. Then, compute $d(s,t)$ and $is(N)$ recursively for each node (N, s, t) of $\mathcal{T}$, where (N, s, t) is formed by the parallel or the series composition of 2 two-terminals series-parallel graphs, (SP_1, s_1, t_1) and (SP_2, s_2, t_2):

- if $N = P(SP_1, SP_2)$, let see that $d(s,t) = max(d(s_1,t_1), d(s_2,t_2))$ and that a maximal isometric cycle is either contained in SP_1, in SP_2 or in the both. If it is contained in the both, then its length is equal to $d(s_1,t_1) + d(s_2,t_2)$ otherwise, there would be a shortest path between two vertices of the cycle. Therefore, $is(N) = max(is(SP_1), is(SP_2), d(s_1,t_1) + d(s_2,t_2))$.
- if $N = S(SP_1, SP_2)$, then $d(s,t) = d(s_1,t_1) + d(s_2,t_2)$ and $is(N) = max(is(SP_1), is(SP_2))$.

■

To decompose wisely the series-parallel graphs, we want an additional constraint on the ear decompositions. Let d_i be the distance between the attachment vertices of E_i in G_{i-1} for every $1 \leq i \leq p$ (i.e. $d_i = dist_{G_{i-1}}(a_i, b_i)$). Let us denote the length of an ear E_i by ℓ_i for every $0 \leq i \leq p$. We want an *increasing* nested ear decomposition, that is, $d_i \leq \ell_i$ for every $1 \leq i \leq p$.

Lemma 4 *For any biconnected series-parallel graph G, there is an increasing nested ear decomposition starting from a maximum isometric cycle.*

Proof. By Theorem 6, there exists a nested ear decomposition $\mathcal{E} = \bigcup_{0 \leq i \leq p}\{E_i\}$ starting with any cycle E_0. Then we can choose to begin with a maximal isometric cycle (i.e. $\ell_0 = is(G)$). Note that E_1 is an ear linked to E_0. Since E_0 is a maximum isometric cycle, we have that the shortest path between two vertices in E_0 is in E_0. So $dist_{E_0}(a_1, b_1) \leq \ell_1$. It's true for all the ear attached to E_0.
The goal is to modify a nested ear decomposition to an increasing nested ear decomposition. If $dist_{G_{i-1}}(a_i, b_i) \leq \ell_i$, then the desired property is respected. Suppose that $dist_{G_{i-1}}(a_i, b_i) > \ell_i$ and let $1 < j_i < i$ such that E_{j_i} is the ear containing the endpoints of E_i. Let $E'_{j_i} = (E_{j_i} \setminus P_{E_{j_i}}(a_i, b_i)) \cup E_i$ and $E'_i = P_{E_{j_i}}(a_i, b_i)$. Then $dist_{G_{i-1}}(a'_i, b'_i) \leq \ell'_i$. If there were some ear

contained in E_i, they are now contained in E'_{j_i} and if there were some ear contained in $P_{E_{j_i}}(a_i, b_i)$, they are now contained in E'_i. ■

Let see that, given a nested ear decomposition and by Lemma 4, we can find in polynomial time an increasing nested ear decomposition. If we have a maximal isometric cycle, we can also construct directly an increasing nested ear decomposition in polynomial time, the construction is the same as for a nested ear decomposition unless for the iteration where E_i is formed by a shortest path in C^* (any connected component C_j of $G \backslash G_{i-1}$ union $V(N_{G_{i-1} \cup C_j}(C_j)))$ between a_i and b_i.

3 Forbidden isometric subgraph for series-parallel graph of treelength 2

Given a graph G and a tree-decomposition $(T, \mathcal{X})$ of G. Let S be any subtree of T. Let G_S denote the subgraph of G induced by $\{v \in X_t \mid t \in S\}$.

Lemma 5 *Let G be any graph and C be any isometric cycle of length ℓ. In any tree-decomposition $(T, \mathcal{X})$ of G with length at most $\lceil \frac{\ell}{3} \rceil$, there exists a bag $X \in \mathcal{X}$ containing three vertices $a, b, c \in V(C)$ such that $\lceil \frac{\ell}{3} \rceil = dist(a, b) \geq dist(a, c) \geq \lfloor \frac{\ell}{3} \rfloor$ and $dist(a, c) \geq dist(c, b) \geq \lfloor \frac{\ell}{3} \rfloor - 1$.*

Proof. Let $(T, \mathcal{X})$ be any tree-decomposition of G of length at most $\lceil \frac{\ell}{3} \rceil$. Note that, by Theorem 3, $(T, \mathcal{X})$ has length exactly $\lceil \frac{\ell}{3} \rceil$. Since every edge must appear in some bag, there must be bags containing at least two vertices of C. For every $X \in \mathcal{X}$ with $|X \cap V(C)| \geq 2$, let $d(X) = \max_{u,v \in X \cap V(C)} dist(u, v)$. Let X be a bag maximizing $d(X)$ and $a, b \in X \cap V(C)$ with $dist(a, b) = d(X)$. Since $d(X) \leq length(X)$, then $dist(a, b) \leq \lceil \frac{\ell}{3} \rceil$. Let P be the path of C between a and b of length $\ell - dist(a, b)$, and let $c \in V(P)$ such that $0 \leq dist(a, c) - dist(b, c) \leq 1$. By definition, $dist(a, c), dist(b, c) \geq \lfloor \frac{\ell}{3} \rfloor$. If $c \in X$, then a, b, c and X satisfy the statement with $\lceil \frac{\ell}{3} \rceil = dist(a, b) \geq dist(a, c) \geq dist(b, c) = \lfloor \frac{\ell}{3} \rfloor$.

For purpose of contradiction, let us assume that no bag contains a, b and c. Let Y be a bag containing c (exists by the properties of a tree-decomposition) that is closest to X in T and let X' be a bag containing a and b that is closest to Y. Let $Z \notin \{X', Y\}$ be any bag on the path between X' and Y in T (or $Z = X' \cap Y$ if $X'Y \in E(T)$). Note that $c \notin Z$. Note also that at least one of a and b is not in Z (otherwise, it would contradict either the fact that X' is closest to Y or that no bag contains all a, b, c). Without lost of generality, let us assume that $a \notin Z$. By the properties of tree-decomposition, Z must separates a and c. Hence, there is a vertex u between a and c in P that belongs to Z. If $u \in X'$, then a, b and u are the required vertices. Indeed, by the maximality of $dis(a, b)$, $dist(a, b) \geq$

$dist(u,b)$ and so the shortest path between u and b in C goes through c. Hence $dist(u,b) > dist(b,c) \geq \lfloor \frac{\ell}{3} \rfloor$ and so $dist(u,b) = dist(a,b) = \lceil \frac{\ell}{3} \rceil$ and $dist(u,a) = \lfloor \frac{\ell}{3} \rfloor - 1$.

Let us now assume that $u \notin X'$. Let e be the edge incident to X' in the path between X' and Y in T. Let T_1 be the component of $T \setminus e$ containing X', and let T_2 be the other component (containing Y). Let $P' = (a = u_0, \cdots, u_k = c)$ be the subpath from a to c in P (note that $u \in P'$). Let $0 \leq i \leq k$ be the smallest integer such that $u_i \in G_{T_1} \setminus G_{T_2}$ and $u_{i+1} \in G_{T_2} \setminus G_{T_1}$. Such an integer exists since $a \in G_{T_1} \setminus G_{T_2}$ and $u \in G_{T_2} \setminus G_{T_1}$. This implies that the edge $u_i u_{i+1}$ cannot appear in any bag, a contradiction. ∎

Recall that a series-parallel graph SP of treelength 2 does not contain an isometric cycle of length at least 7 (i.e. $is(G) \leq 6$) because if it contains a such cycle, SP has treelength more than 2 by lemma 3. Therefore, we want to characterize every isometric subgraph that can't be contain in SP.

Definition 5 *A Dumbo graph is any graph built as follows. Start with a cycle $C_0 = (v_0, \cdots, v_5)$ of length 6, and add a path R of length (i.e number of edges) at least 3 between v_0 and v_2 and a path L of length at least three between v_3 and v_5. Such a graph is depicted in Figure 8.*

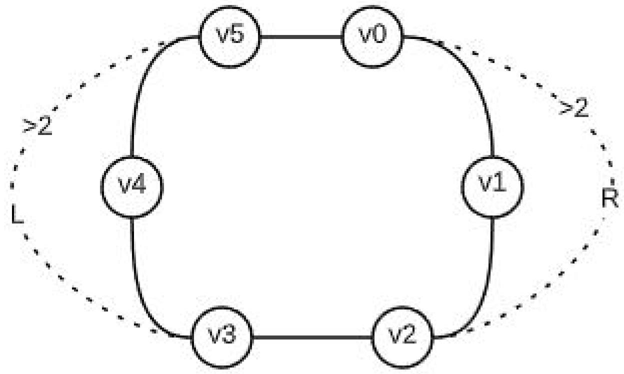

Figure 8: a Dumbo Graph of length 2

Note that a Dumbo graph D is a series-parallel graph since it can be construct by parallel and series composition with $s = v_0$ and $t = v_3$ (see Figure 9).

Lemma 6 *Let G be any series-parallel graph. If G contains a Dumbo graph as isometric subgraph, then $t\ell(G) > 2$.*

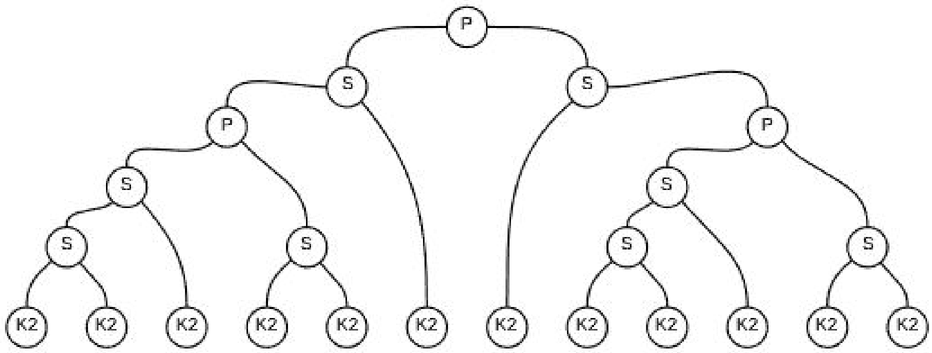

Figure 9: $\mathcal{T}$ (representation of its construction as a SP graph) of a Dumbo graph

Proof. Let $G = (V, E)$ be any series-parallel graph containing a Dumbo graph $D = (C_0, R, L)$ as isometric subgraph. For purpose of contradiction, let us assume that G admits a tree-decomposition $(T, \mathcal{X})$ of length at most 2. By Lemma 5, there must be a bag $X \in \mathcal{X}$ containing $\{v_0, v_2, v_4\}$ or $\{v_1, v_3, v_5\}$. By symmetry, let us assume that $\{v_0, v_2, v_4\} \subseteq X$. Let z be a vertex of $L \setminus \{v_5, v_3\}$ such that $|dist(z, v_5) - dist(z, v_3)| \leq 1$. Note that $dist(z, v_5), dist(z, v_3) \geq 1$ and $\max\{dist(z, v_5), dist(z, v_3)\} \geq 2$. Moreover, because G is series-parallel, every path from z to v_0, v_2 or v_4 goes through v_3 or v_5 (otherwise, there would be a K_4 minor). Note also that no bag contains $\{v_0, v_2, v_4, z\}$ since z is at distance at least 3 from some of v_0, v_2, v_4.

Let Y be a bag containing z that is closest to X, and let X' be the bag containing v_0, v_2, v_4 that is closest to Y. Let $Z' \notin \{X', Y\}$ be the neighbor of X' on the path between X' and Y in T and let $Z = X' \cap Z'$ (or $Z = X' \cap Y$ if $X'Y \in E(T)$). Note that $z \notin Z$. Note also that at least one of v_0, v_2 and v_4 is not in Z (otherwise, it would contradict either the fact that X' is closest to Y or that no bag contains all v_0, v_2, v_4 and z). Let $W = \{v_0, v_2, v_4\} \setminus Z$. By the properties of tree-decomposition, Z must separates every $w \in W$ from z. There are several cases to be considered depending on which vertex of v_0, v_2 and v_4 are not in Z:

- If at least v_2 belongs to Z, then $W \subseteq \{v_0, v_4\}$. Hence, there must be u in the z-v_5 subpath of L that is in Z if v_0 or v_4 are in W (i.e in every case) and there must be v in the z-v_3 subpath of L that is in Z if v_4 is in W. Since $z \notin Z$, $u \neq z$ and $d(u, v_3) \geq 2$ and then $d(u, v_2) \geq 3$. Therefore, there is no tree-decomposition of length 2 with at least v_2 in Z.

- If at least v_0 belongs to Z, then $W \subseteq \{v_2, v_4\}$. Hence, there must be v in the z-v_3 subpath of L that is in Z if v_2 or v_4 are in W (i.e in every case) and there must be u in the z-v_5 subpath of L that is in Z if v_4 is in W. Since $z \notin Z$, $v \neq z$ and $d(v, v_5) \geq 2$ and then $d(v, v_0) \geq 3$. Therefore, there is no tree-decomposition of length 2 with at least v_0 in Z.

- Finally, if at least v_4 belongs to Z, then $W \subseteq \{v_0, v_2\}$. Hence, there must be u in the z-v_5 subpath of L that is in Z if v_0 is in W and there must be v in the z-v_3 subpath of L that is in Z if v_2 is in W. Since $z \notin Z$, $v \neq z$, $u \neq z$, $d(u, v_3) \geq 2$ and $d(v, v_5) \geq 2$ and then $d(v, v_0) \geq 3$ and $d(u, v_2) \geq 3$. Therefore, there is no tree-decomposition of length 2 with at least v_4 in Z.

■

The following lemma will be used in the next one to deal with the case of ears of length 2.

Lemma 7 *Let G be any 2-connected series-parallel graph without clique-separator, with an increasing nested ear decomposition $\mathcal{E} = \bigcup_{0 \leq i \leq p}\{E_i\}$. Let $(T', \mathcal{X}')$ be a tree-decomposition, with length at least 2, of a subgraph G' of G induced by $E_0, \dots, E_j$ and let E_i such that $1 \leq j_i \leq j < i \leq p$ and $\ell_i = 2$. Then, there exists a tree-decomposition $(T, \mathcal{X})$ of $G' \cup E_i$ with same length and such that, for every $B' \in \mathcal{X}'$, there exists $B \in \mathcal{X}$ such that $B' \subseteq B$.*

Proof. Note that by hypothesis, both endpoints of E_i belong to G' since they belong to E_{j_i}. Let us first suppose that the endpoints of E_i are in a same bag B of $(T', \mathcal{X}')$. Then, the tree-decomposition obtained from $(T', \mathcal{X}')$ by adding a a bag $V(E_i)$ adjacent to B satisfies the statement of the Lemma.

Let us now consider the case where no bag of $(T', \mathcal{X}')$ contains the endpoints a_i and b_i of E_i. Let $X \in \mathcal{X}'$ and $Y \in \mathcal{X}'$ be such that $a_i \in X$, $b_i \in Y$ and the distance in T between two such bags is minimum.

Note that, because G has no edge-separator and because the ears are added in increasing order (i.e. $2 \leq d_i \leq \ell_i = 2$), a_i and b_i must have common neighbors in G'. Note also that, because G is series-parallel (in particular, the ears are nested) without clique separator, then every common neighbor w of a_i and b_i satisfies $N(w) = \{a_i, b_i\}$.

By the tree-decomposition properties, every bag W on the X-Y path in T' must separate $X \setminus Y$ from $Y \setminus X$. In particular, $N_{G'}(a_i) \cap N_{G'}(b_i) \subseteq W$. Similarly, $N_{G'}(a_i) \cap N_{G'}(b_i) \subseteq X$ and $N_{G'}(a_i) \cap N_{G'}(b_i) \subseteq Y$. Let v be the common neighbor of a_i and b_i in E_i. Then, adding v to every bag W on the X-Y path in T' (including X and Y) gives the desired decomposition. In particular, for every $v' \in W$, $dist_G(v', w) = dist_G(v', v)$ where w is any vertex in $N_{G'}(a_i) \cap N_{G'}(b_i)$, and so the obtained tree-decomposition has same length as $(T', \mathcal{X}')$. ■

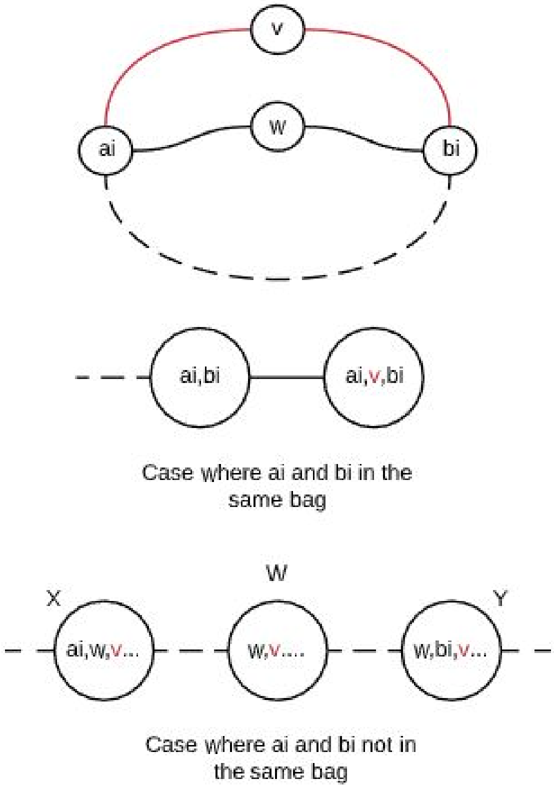

Figure 10: Lemma 7

Some notation will be used in the following proof, let H be a subgraph of G, a series-parallel graph with a nested ear decomposition $\mathcal{E} = \bigcup_{0 \leq i \leq p} \{E_i\}$, such that $H = \mathcal{E}' = \bigcup_{i' \leq j \leq i} \{E_j\}$ (i.e. H is a finite number of ear), let $Att(H) \subseteq V(H)$ be the set of vertices of H that are the attachment vertices of some ear E_k with $k > i$.

Lemma 8 *Let G be any (simple) series-parallel graph without clique-separator and with $is(G) \leq 6$. If G does not contain a Dumbo graph as isometric subgraph, then $t\ell(G) \leq 2$.*

Proof. Let us assume that G is not a chordal graph in which case the result is trivial. Hence, we may assume that $t\ell(G) \geq 2$.

Let G be any series-parallel graph without clique-separator, with $is(G) \leq 6$, and with no Dumbo graph as isometric subgraph. Let $\mathcal{E} = \{E_0\} \cup \{E_i \mid 0 < i \leq p\}$ be an increasing nested ear-decomposition of G with E_0 being a largest isometric cycle, i.e., $|V(E_0)| = is(G)$ and $d_i \leq \ell_i$ for every $1 \leq i \leq p$. For every $0 < i \leq p$, let a_i and b_i be the endpoints of E_i. Moreover, $\mathcal{E}$

contains no ear of length one since G is simple, series-parallel and without clique separator.

We will build a sequence $\mathcal{E}_1 \subset \mathcal{E}_2 \subset \ldots \subset \mathcal{E}_{p'} = \mathcal{E}$ such that $E_0 \in \mathcal{E}_1$ and for every $1 \leq i \leq p'$,

1. $G^i = G[\bigcup_{E \in \mathcal{E}_i} V(E)]$ is an isometric series-parallel subgraph of G with $\mathcal{E}_i$ as ear-decomposition;

2. No ear of length two is attached to G^i, i.e., every ear not yet in G^i with both endpoints in G^i has length at least 3;

3. G^i admits a tree-decomposition $(T^i, \mathcal{X}^i)$ of length 2, and

4. For every ear $E_j \in \mathcal{E} \setminus \mathcal{E}_i$ attached to G^i, there exists $t \in V(T^i)$ such that $\{a_j, b_j\} \subseteq X^i_t \in \mathcal{X}^i$, i.e., every ear not yet in G^i with both endpoints in G^i (so with length at least 3) has both its endpoints in some bag of $(T^i, \mathcal{X}^i)$.

The proof is by induction on $1 \leq i \leq p'$. Let us first build $\mathcal{E}_1$. There are several cases depending of the size of E_0.

- If $E_0 = (a, b, c, d)$ has length 4 (it cannot have length 3 since G is not chordal), recall that since G is series-parallel, the ears are nested, there cannot be an ear attached to a and c and an ear attached to b and d. Indeed, since G has no clique separator, no ear can be attached to two adjacent vertices and every ear attached to E_0 has a length 2 (else $is(G) > 4$). Then, up to symmetries, $Att(E_0) = \{a, c\}$ (if $Att(E_0) = \emptyset$, then $G = E_0$ and the result is trivial) (see Figure 11).

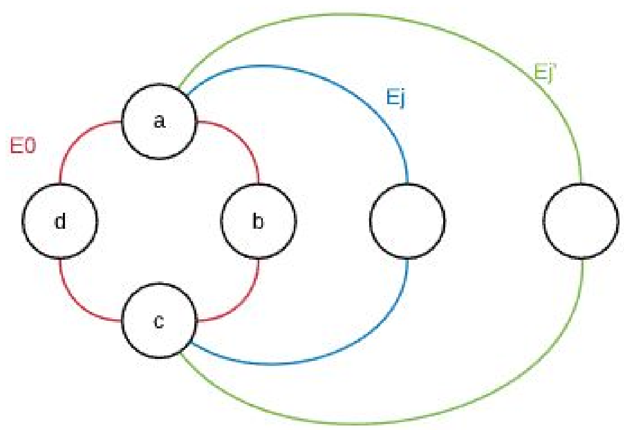

Figure 11: case where E_0 has length 4

Let $\mathcal{E}_1$ consist of E_0 and the set of all ears of length two attached to a and c. Then, $(T^1, \mathcal{X}^1)$ is the tree-decomposition with one "central" bag $\{a, b, c, d\}$ with one neighbouring bag E_j for every ear $E_j \in \mathcal{E}_1 \setminus \{E_0\}$ (see Figure 12). $(T^1, \mathcal{X}^1)$ is clearly a tree-decomposition of G^1 with length 2. Finally, because the ears are nested and there are no clique separators, every ear in $\mathcal{E} \backslash \mathcal{E}_1$ with attachment vertices in G^1 must have a and c as attachment vertices. If such an ear in $\mathcal{E} \backslash \mathcal{E}_1$ exists, it must have length at least 3 which would contradict the fact that E_0 is a largest isometric cycle. Hence, no such ear exists and $G^1 = G$.

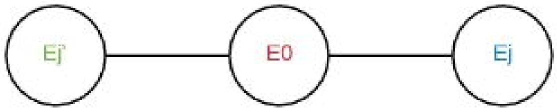

Figure 12: Tree-decomposition of G when E_0 has length 4

- If $E_0 = (a, b, c, d, e)$ has length 5 then, up to symmetries, $Att(E_0) \subseteq \{a, c, d\}$ (if $Att(E_0) = \emptyset$, then $G = E_0$ and the result is trivial). More precisely, ears can be attached to a and c or to a and d. Indeed, since G has no clique separator, no ear can be attached to two adjacent vertices (see Figure 13). Let $\mathcal{E}_1$ consist of E_0 and the set of all ears of

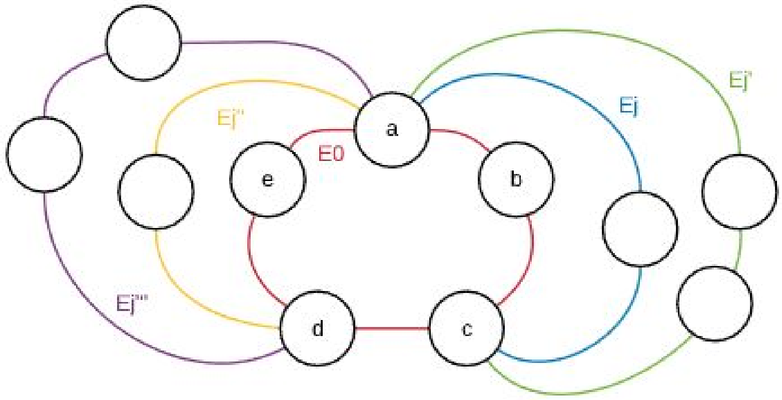

Figure 13: case where E_0 has length 5 (E_j and E''_j are contained in a bag since they have length 2. E'_j and E'''_j are not contained in a bag since they have length 3)

length two attached to E_0. Then, $(T^1, \mathcal{X}^1)$ is the tree-decomposition with one "central" bag $\{a, b, c, d, e\}$ with one neighbouring bag E_j for every ear $E_j \in \mathcal{E}_1 \setminus \{E_0\}$ (see Figure 14). $(T^1, \mathcal{X}^1)$ is clearly a tree-decomposition of G^1 with length 2. Finally, every ear in $\mathcal{E} \setminus \mathcal{E}_1$ attached to G^1 has its attachment vertices in E_0 because G is series-parallel and so, the ears are nested. More precisely, otherwise, since an ear cannot have adjacent attachment vertices (no clique separator), there would be an ear $E_j \in \mathcal{E} \setminus \mathcal{E}_1$ and one ear $E_k \in \mathcal{E}_1 \setminus \{E_0\}$ (w.l.o.g., say with attachment vertices a and c) with $a_j \in E_k \setminus \{a, c\}$ and $b_j \notin \{a, c\}$. This would imply that G contains a K_4 as minor, a contradiction.

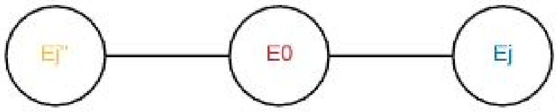

Figure 14: Tree-decomposition of $\mathcal{E}_1$ when E_0 has length 5 (E_j and E_j'' are contained in a bag since they have length 2. E_j' and E_j''' are not contained in a bag since they have length 3)

- Then, let us consider the case when $E_0 = (a, b, c, d, e, f)$ has length 6. If there is an ear attached to two vertices at distance 3, note that every such ear has length exactly 3 since E_0 is a largest isometric cycle. Moreover, all such ears have the same attachment vertices since the ears are nested (otherwise, there would be a K_4 minor). W.l.o.g., let

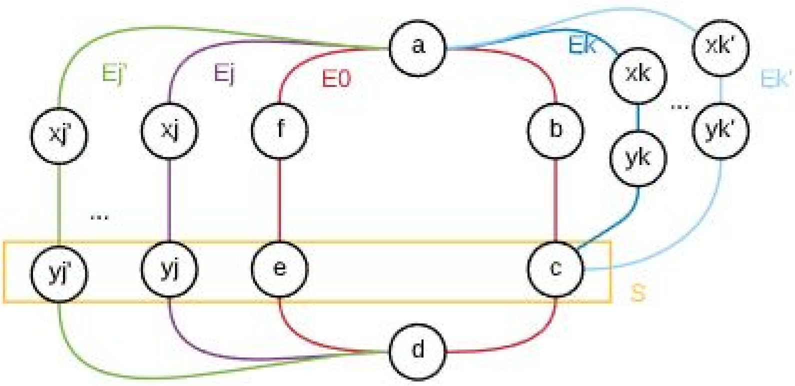

Figure 15: case where E_0 has length 6

a and d be the attachment vertices of all (if any) ears attached to vertices at distance 3 in E_0. Let $\mathcal{E}'_1$ consists of E_0 and all ears $E_j = (a_j = a, x_j, y_j, b_j = d)$ attached to a and d. Since G has no Dumbo graph as isometric subgraph, there are no two ears $X, Y \in \mathcal{E} \setminus \mathcal{E}'_1$ of length at least three such that X is attached to a and c (resp., e) or to a and y_j for some ear $E_j \in \mathcal{E}'_1$ and Y is attached to d and f (resp. b) or to d and x_k for some ear $E_k \in \mathcal{E}'_1$. Therefore, w.l.o.g., all ears of length at least 3 that are attached to $G' = G[\bigcup_{E \in \mathcal{E}'_1} V(E)]$ have a and some vertex in $B = \{c, e\} \cup \bigcup_{E_j \in E''_1} \{y_j\}$ as attachment vertices (see Figure 15). Let $(T', \mathcal{X}')$ be the tree-decomposition with one "central" bag $C = B \cup \{a\}$ with one neighbouring bag $\{a, x_j, y_j\}$ for every ear $E_j \in \mathcal{E}'_1 \setminus \{E_0\}$, one neighbouring bag $\{a, b, c\}$, one neighbouring bag $\{a, f, e\}$, and one neighbouring bag $\{d\} \cup B$. Then, $(T', \mathcal{X}')$ is clearly a tree-decomposition of G' of length 2 such that all ears of length at least 3 attached to G' have their attachment vertices in C. Finally, let F be the set of all ears of length 2 attached to G'. Let $\mathcal{E}_1 = \mathcal{E}'_1 \cup F$. By Lemma 7, from $(T', \mathcal{X}')$, we can obtain a tree-decomposition $(T^1, \mathcal{X}^1)$ of G^1 of length 2 such that every bag in $\mathcal{X}'$ is contained in some bag of $\mathcal{X}^1$ (see Figure 16).

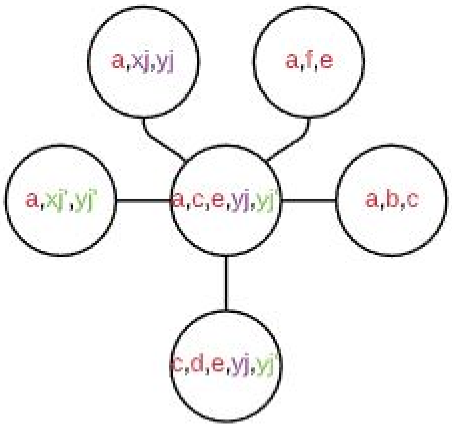

Figure 16: Tree-decomposition of $\mathcal{E}_1$ when E_0 has length 6

Finally, since G has no clique separator and is series-parallel (in particular the ears are nested), every ear attached to G^1 must have both its attachment vertices in a same bag of $(T^1, \mathcal{X}^1)$, and must have length at least 3 (since otherwise it would have been included in $\mathcal{E}_1$).

Now, let's prove by induction on $1 \leq i < p'$ that we can build an ear decomposition $\mathcal{E}_{i+1}$ from $\mathcal{E}_i$ with all the desired properties. Let E_j be any shortest ear not in $\mathcal{E}_i$ with attachment vertices $\{a_j, b_j\} \in V(G^i)$. Because

G has no clique separator and, by the induction hypothesis, G^i has a tree-decomposition $(T^i, \mathcal{X}^i)$ of length 2 with a bag containing a_j and b_j, note that $d_j = dist_G(a_j, b_j) = dist_{G^i}(a_j, b_j) = 2$. Moreover, because $is(G) = 6$ and that there is no ear of length two attached to G^i, the length ℓ_j of E_j is such that $3 \leq \ell_j \leq 4$. There are two cases depending of the length of E_j.

- If $E_j = (a_j, x, y, b_j)$ has length 3, then up to symmetries $Att(G^i \cup E_j) \cap V(E_j) \subseteq \{a_j, y, b_j\}$. Indeed, since G has no clique separator, no ear can be attached to two adjacent vertices and since all ears of $\mathcal{E}$ are nested, there isn't an ear attached to a_j and y and another one to x and b_j, or an ear attached to a vertex of $V(E_j)\backslash\{a_j, b_j\}$ and to a vertex of $V(G_i)\backslash\{a_j, b_j\}$ (see Figure 17). Let $\mathcal{E}'_{i+1}$ consist of $\mathcal{E}_i$ and E_j. Let $G' = G[\bigcup_{E \in \mathcal{E}'_{i+1}} V(E)]$ and $(T', \mathcal{X}')$ be the tree-decomposition build from $(T^i, \mathcal{X}^i)$ with a bag $B = \{a_j, x, y, b_j\}$ connected to a bag of $(T^i, \mathcal{X}^i)$ containing a_j and b_j. Then, $(T', \mathcal{X}')$ is clearly a tree-decomposition of G' of length 2. Finally, let F be the set of all ears of length 2 attached to G' (note that, because of the induction hypothesis and the fact that the initial ear decomposition is increasing, all such ears are attached to a_j and y). Let $\mathcal{E}_{i+1} = \mathcal{E}'_{i+1} \cup F$. By Lemma 7, from $(T', \mathcal{X}')$, we can obtain a tree-decomposition $(T^{i+1}, \mathcal{X}^{i+1})$ of G^{i+1} of length 2 such that every bag of $\mathcal{X}'$ is contained in some bag of $\mathcal{X}^{i+1}$ (see Figure 17). Clearly if there is an ear attached to the only middle vertex of an ear E_f of F then by definition of a nested ear decomposition, it's second endpoint is a vertex in E_f which contradicts the fact that G has no clique-separator. We can deduce that for every E_m attached to G^{i+1} there exists $t \in V(T^{i+1})$ such that $\{a_m, b_m\} \subseteq X_t^{i+1}$.

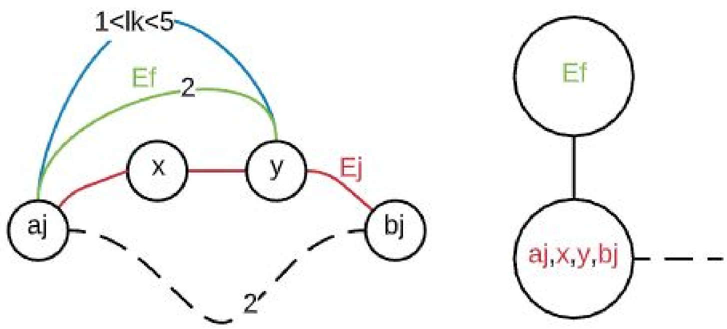

Figure 17: case where E_j has length 3

- Now, let us assume that $E_j = (a_j, x, y, z, b_j)$ has length 4. There are several cases depending of the vertices of E_j that are attachment vertices for other ears E_l in $\mathcal{E}\backslash(\mathcal{E}_i \cup \{E_j\})$ attached to E_j. Because G has no clique separator and $\mathcal{E}$ is an increasing nested ear decomposition, we have these following possibilities up to symmetries.

– If $Att(E_j) \subseteq \{a_j, y, b_j\}$ (see Figure 18), then let $\mathcal{E}'_{i+1}$ consist of $\mathcal{E}_i$ and E_j. Let $(T', \mathcal{X}')$ be the tree-decomposition of $G' = G[\bigcup_{E \in \mathcal{E}'_{i+1}} V(E)]$ built from $(T^i, \mathcal{X}^i)$ as follows. Let B be any bag of $(T^i, \mathcal{X}^i)$ containing both a_j and b_j (exists by the induction hypothesis). Let us add the bag $\{a_j, y, b_j\}$ adjacent to B and to the bags $\{a_j, x, y\}$ and $\{y, z, b_j\}$. Since $(T^i, \mathcal{X}^i)$ is a tree-decomposition of G^i of length 2, then $(T', \mathcal{X}')$ is also a tree-decomposition of G' of length 2. Let F be the set of ears of length 2 attached to E_j and let $\mathcal{E}_{i+1}$ consist of $\mathcal{E}'_{i+1}$ and F. By lemma 7, we can obtain from $(T', \mathcal{X}')$ a tree-decomposition $(T^{i+1}, \mathcal{X}^{i+1})$ of length 2 of G^{i+1}. Finally, $(T^{i+1}, \mathcal{X}^{i+1})$ satisfies the desired properties (in particular because G has no edge separator, every ear attached to G^{i+1} has its attachment vertices in a bag of $(T^{i+1}, \mathcal{X}^{i+1})$).

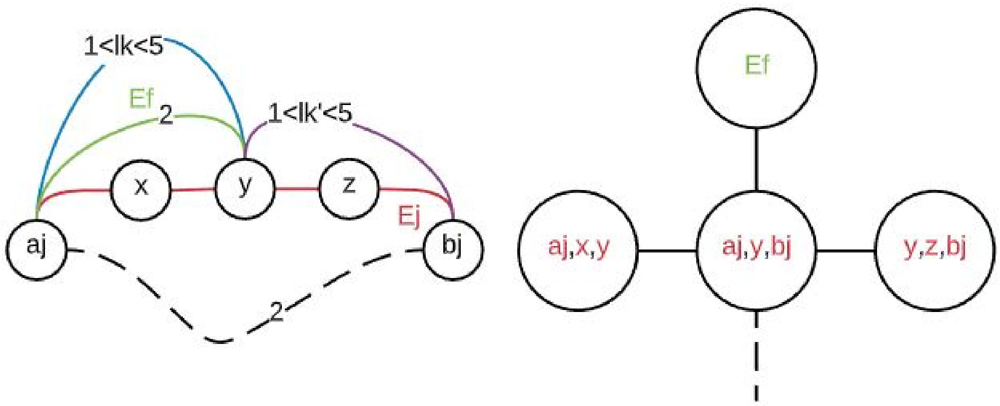

Figure 18: case where E_j has length 4 and $Att(E_j) \subseteq \{a_j, y, b_j\}$

– Now, let us assume, up to symmetry, that there exists an ear E' attached to a_j and z. Note that such an ear has length exactly 3 since $\mathcal{E}$ is an increasing nested ear decomposition and no isometric cycle has length more than 6. Let $\mathcal{E}'$ be the set of all ears $E_{j'} = (a_j = a_{j'}, x_{j'}, y_{j'}, b_{j'} = z) \notin \mathcal{E}_i$ of length 3 attached to a_j and z (in particular, E' is such an ear), and let $\mathcal{E}'_{i+1}$ consist of $\mathcal{E}_i \cup E_j \cup \mathcal{E}'$ (see Figure 19).

  Let us first show that no ear $E_q \in \mathcal{E} \setminus \mathcal{E}'_{i+1}$ of length at least 3 is attached to $x_{j'}$ and $b_{j'}$ for some j' such that $E_{j'} \in \mathcal{E}'$ (resp.

to x and z). For purpose of contradiction, let us assume that such an ear E_q exists. Recall that, by the induction hypothesis, a_j and b_j must belong to a same bag of $(T^i, \mathcal{X}^i)$ of length 2 and that, because there is no clique separator, $a_j, b_j \notin E(G)$. Hence, $dist_G(a_j, b_j) = dist_{G^i}(a_j, b_j) = 2$. Let E_ℓ be the first (i.e., with minimum ℓ) ear of G^i containing both a_j and b_j (such an ear must exist since E_j can only be attached to the vertices of some previous ear).

* If $E_\ell = E_0$, then, the subgraph induced by $V(E_0) \cup V(E_{j'}) \cup V(E_q)$ (resp. $V(E_0) \cup V(E_j) \cup V(E_q)$) is an isometric Dumbo graph, a contradiction.
* Otherwise (if $\ell \neq 0$), let a_ℓ and b_ℓ be the end points of E_ℓ, and let G^* be the subgraph induced by the vertices of the ears in $\{E_m \in \mathcal{E}^i \mid m < \ell\}$. Note that G^* is an isometric subgraph of G^i. W.l.o.g., $a_\ell \notin \{a_j, b_j\}$ (otherwise this would contradict that E_ℓ is the first ear in which both a_j and b_j appear). Let P be any shortest a_ℓ-b_ℓ path in G^*. Since a_ℓ and b_ℓ are not adjacent (otherwise there would be an edge separator in G), P has length at least 2. Then, the subgraph induced by $V(P) \cup V(E_\ell) \cup V(E_{j'}) \cup V(E_q)$ (resp. $V(P) \cup V(E_\ell) \cup V(E_j) \cup V(E_q)$) is an isometric Dumbo graph, a contradiction.

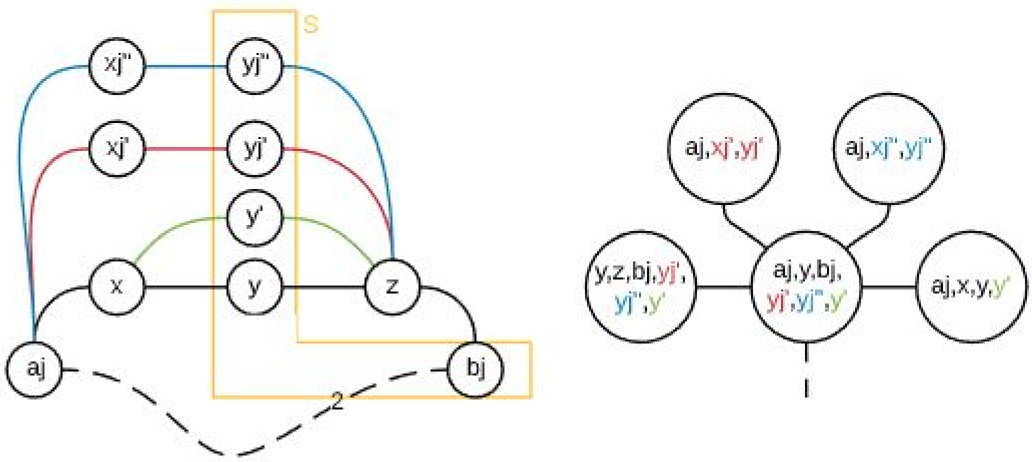

Figure 19: case where E_j has length 4 and there is at least one ear attached to a_j and z

Let B be any bag of $(T^i, \mathcal{X}^i)$ containing both a_j and b_j (exists by the induction hypothesis). Let $B' = \{a_j, b_j, y\} \bigcup_{j', E_{j'} \in \mathcal{E}'} \{y_{j'}\}$, let $B_{j'} = \{a_j, x_{j'}, y_{j'}\}$ for all j' such that $E_{j'} \in \mathcal{E}'$, let $B'' = \{b_j, z, y\} \bigcup_{j', E_{j'} \in \mathcal{E}'} \{y_{j'}\}$, and let $B_j = \{a_j, x, y\}$.

Let $(T', \mathcal{X}')$ be the tree-decomposition of $G' = G[\bigcup_{E \in \mathcal{E}'_{i+1}} V(E)]$ built from $(T^i, \mathcal{X}^i)$ by adding the bag B' adjacent to B, to B'', to B_j and to $B_{j'}$ for all j' such that $E_{j'} \in \mathcal{E}'$. It can be shown that $(T', \mathcal{X}')$ is a tree-decomposition of G', with length 2 and such that every ear of length at least 3 attached to G' has both its attachment vertices in some bag of $(T', \mathcal{X}')$. Let F be the set of ears of length 2 attached to some ear in $\mathcal{E}' \cup E_j$ and let $\mathcal{E}_{i+1}$ consist of $\mathcal{E}'_{i+1}$ union F. By lemma 7, we can obtain from $(T', \mathcal{X}')$ a tree-decomposition $(T^{i+1}, \mathcal{X}^{i+1})$ of length 2 of G^{i+1}.

Finally, $(T^{i+1}, \mathcal{X}^{i+1})$ satisfies the desired properties (in particular because G has no edge separator, every ear attached to G^{i+1} has its attachment vertices in a bag of $(T^{i+1}, \mathcal{X}^{i+1})$).

■

Above Lemmas lead to the following main result.

Theorem 7 *For any series-parallel graph G, $t\ell(G) \leq 2$ if and only if $is(G) \leq 6$ and G does not contain a Dumbo graph as isometric subgraph.*

Moreover, there is a polynomial algorithm that either computes a tree-decomposition of length at most 2 of G or exhibits a certificate that $t\ell(G) > 2$ (a large isometric cycle or an isometric Dumbo subgraph).

Proof. By Claim 1 and by theorem 4, we can consider only simple biconnected series-parallel graphs G' without edge separators. Then by Lemma 3 and Lemma 4, we can compute in polynomial time an increasing nested ear decomposition for G'. Finally by Lemma 8, we can compute in polynomial time a tree-decomposition of length at most 2 of G' or exhibits a large isometric cycle or an isometric Dumbo subgraph of G'. ■

4 Approximation algorithm

Theorem 8 *For any series-parallel graph G, we can compute in polynomial time a tree-decomposition $(T, \mathcal{X})$ of G such that $length(T) \leq \frac{3}{2} \cdot tl(G)$*

Proof. Let G be a series-parallel graph with largest isometric cycle of size $is(G)$. It follow from lemma 3 that $tl(G) \geq \lceil \frac{is(G)}{3} \rceil$. Let us see how to compute a tree-decomposition of length $\lceil \frac{is(G)}{2} \rceil$. Indeed, intuitively, every isometric cycle can be contained in a bag (i.e. $d(x, y) \leq \lfloor \frac{is(G)}{2} \rfloor$ for every x and y in an isometric cycle C in G). Let us consider an increasing nested ear decomposition $\mathcal{E}$ starting with a maximal isometric cycle E_0 for G. Let us build the decomposition as follows. Start with a bag containing E_0. Then, for every E_i with $1 \leq i \leq p$ connect a bag consisting of E_i to the bag containing E_{j_i} where $0 \leq j_i \leq i \leq p$ and j_i is the minimum index such that E_{j_i} contains a_i and b_i. Since $\mathcal{E}$ is a increasing nested ear decomposition, $\ell_i \geq d_i$ for every $1 \leq i \leq p$ and then $E_i \cup P_{G_{i-1}}(a_i, b_i)$ is an isometric cycle (i.e. $l(E_i) \leq \lfloor \frac{is(G)}{2} \rfloor$). Therefore $l(T) = \lfloor \frac{is(G)}{2} \rfloor \leq \lfloor \lceil \frac{is(G)}{3} \rceil \cdot \frac{3}{2} \rfloor \leq tl(G) \cdot \frac{3}{2}$. Since $\mathcal{E}$ and $is(G)$ can be computed in polynomial time (see Lemma 3), there is an $\frac{3}{2}$-Approximation algorithm for computing the treelength of a series-parallel graph. ∎

5 Conclusion

In this paper, we show how to compute a tree-decomposition of length $k = 2$ for a SP graph in polynomial time if it exists. We also show an $\frac{3}{2}$-approximation algorithm for computing the treelength of a series-parallel graph.

It would be interesting to generalize our characterization for any k. First, I am trying to generalize these results for $k = 3$. The first consequence of increasing k to 3 is to have more minimal forbidden isometric subgraphs. For example, let us see that there is some melon graphs that are forbidden as isometric subgraph. Precisely, melon graphs with only paths of length at least 4 with at least 2 paths of length 5 or at least 1 path of length 6 has a treelength at least 4. Another consequence arises when we are considering an ear E_j of length 4 in the induction. Indeed, if we construct $(T^i, \mathcal{X}^i)$ by adding a bag containing E_j adjacent to the bag containing a_j and b_j, there is not necessarily a tree-decomposition $(T, \mathcal{X})$ of length 3 that contains $(T^i, \mathcal{X}^i)$ as a sub-tree, even if the graph has treelength 3. Intuitively, the decision problem $tl(SP) \leq 3$ seems to be polynomial, but is really more difficult than $tl(SP) \leq 2$. Since the characterisation of minimal forbidden isometric subgraphs (MFIS) for SP graphs of length 3 become difficult (actually, I already identified more than 12 MFIS), we can try to define for each cycle $C \in G$, which vertex can be part of the tuple of 3 vertices that separate the cycle into pieces of length at most k. In fact, the goal is to characterise all the differents MFIS by some commons properties.

Our next goals are to study the computational complexity of computing the treelength in the class of series-parallel graphs. We hope that this complexity is polynomial, which will lead us to study the computational complexity of treelength in the class of planar graphs. Moreover, studying the class of planar graphs will allow us to use the relation between the treewidth and the treelength to approximate or compute one of the two measures.

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