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COPS AND ROBBER GAMES IN RADIUS CAPTURE AND FASTER ROBBER VARIANTS

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ABSTRACT

In the classical cop and robber game, two players, the cop C and the robber R, move alternatively along edges of a finite graph $G = (V, E)$. The cop captures the robber if both players are on the same vertex at the same instance. A graph G is called *cop win* if the cop always captures the robber after a finite number of steps. Nowakowski, Winkler(1983) and Quilliot (1983) characterized the cop-win graphs as graphs admitting a dismantling scheme. Chalopin et al. characterized the cop-win graphs in the game in which the cop and the robber move at different speeds s' and s, $s' < s$. Chalopin et al. also charaterized the bipartite graphs in the radius caputre variant in which the cop can capture the robber if their distance is not greater than one. Inheritting from the previous works, we investigate further two variants of cops and robber game, the faster robber and radius capture variants in some particular graphs such as square grid, k-chordal, planar, etc.

Key-words: Cops and Robber games, radius capture, faster robber

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Contents

1. Introduction

1.1. **Motivation.**

"Game of cops and robber" is the discrete version of "pursuit-evation games" which have vast applications in practice and importances in theory. There are several problems such as "search and rescue", "missile intercept", search for piece of information storing somewhere in network can be modelled by "pursuit-evation games". Applications of "Game of cops and robber" also can be found in research of Artificial Intelligence or Game Development. For instance, "ScotlandYard" based on "Games of cops and robber" is voted as Game of Year in 1983 [11].
In graph theory, research on "Game of cops and robber" helps us to find new properties of graphs, for instance, *dismantlable*, and also brings new points of view to some classical problems in graph theory such as separator, treewidth, chordal, etc. In scope of the internship, we perform research on different variants of "Game of cops and robber" that will help us understand more profoundly the classical version and give us more evidences and techniques to prove the Meyniel's conjecture[8].

1.2. **Rules of Cops and Robber game and variants.**

Cops and Robber is a pursuit-evation game with two players: C (Cops) and R (Robber) which play alternatively on a finite, connected, undirected graph $G = (V, E)$. Player C has a team of cops (at least 1 cop) who attempt to capture the robber. At the beginning of the game, C select some vertices and puts cops on these vertices. Then R puts the robber on a vertex. The cops and robber move along the edge paths of G. The players take turns starting with C. There are some variants of Cops and Robber game depending on the number of cops, how C and R can move the cops and robber at their turn and how a cop can "capture" the robber.

1.2.1. *Classical Cops and Robber game.* Winkler and Nowakowski [2] and Quilliot [3] defined the classical game which the number of cop is 1, and the cop C can "capture" the robber R if C can put himself on the vertex occupied by R. In *active* version, R must move whenever it is his turn. Differently, in *passive* version, R can remain stay at the vertex which he is occupying. *Within scope of this report, we consider only the* **passive version** *for all variants of cops and robber game.* The graph G is *cop win* if from any starting position of C and R, C can capture R after finite steps; otherwise, it is a *not cop win* graph. Now we consider the following examples for *cop win* and *not cop win* graphs.

Example 1. *A tree (Definition 7) T is a cop win graph.*

In tree, every vertex is a cut vertex. Hence, the vertex occupied by C partitions T in some connected components and one of those includes R. Suppose that X is the connected component including R. If C remains at his position, R is restricted in X. If R moves to vertex u along the unique path toward R, then R is restricted in the area $X' = X \setminus \{u\}$. So , after finite steps, C can capture the R.

Example 2. *A cycle (Definition 5) with length 4 is a not cop win graph.*

Suppose that the cycle $G = (V, E)$ has 4 vertices v_1, v_2, v_3, v_4 and (v_1, v_2), (v_2, v_3), (v_3, v_4), $(v_4, v_1) \in V(G)$. We remark that $dist(v_1, v_3) = dist(v_2, v_4) = 2$. Without loss of generality, suppose that C chooses v_1 to stay. Correspondingly, R will choose v_3 to stay. If C remains at his position, R also remains at his position. If C moves

to v_2 or v_4, R moves to v_4 or v_2 correspondingly. So R can maintain infinitively the distance ≥ 1 from C at any steps. Then C cannot capture R.
The set of *cop win* graphs in the classical game is denoted by CW.
Aigner and Fromme generalized [1] the game by putting a team of cops in the graph G. We denote by $cn(G)$ the minimum number of cops sufficient for C to win on the graph G.
The classical game are studied intensively through 30 years. Winkler and Nowakowski [2] and Quilliot [3] characterized CW as set of dismantlable graphs (Definition 25). Aigner and Fromme [1] proved that for any planar graph G, $cn(G) \leq 3$. It is well-known that $\forall n$, $\exists$ a n-vertex graph G such that $cn(G) = \Omega(sqrt(n))$ [6].
However, in the classical version, there exists a conjecture during more than 30 years.

Conjecture 1 (Meyniel)**.** [8] *For any n-vertex graph G,* $cn(G) = O(\sqrt{n})$.

1.2.2. *Radius Capture variant.*
The radius capture variant is introduced by Botano et al.[7] where a cop can capture the robber if distance between the cop and robber (radius of capture) is no greater than k $(k \geq 1, k \in N)$. And if the number cop is 1, the set of cop win graph in radius capture variant is denoted by $CWRC(k)$. We consider 2 following examples to illustrate the definition.

Example 3. *The cycle G (definition 5) with length 5 belongs to CWRC(1).*
Suppose that C puts himself on vertex u, R puts himself on vertex v. By definition, we have $dist(u, v) \leq 2$. Because C can move first, C moves to adjacent vertex w of u on the shortest path from u to v. Because $dist(w, v) \leq 1$, C capture R after one move.

Example 4. *The cycle G (Definition 5) with length 6 does not belong to CWRC(1).*

Suppose that the cycle $G = (V, E)$ has 6 vertices $v_1, v_2, v_3, v_4, v_5, v_6$ and (v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_5),(v_5, v_6),(v_6, v_1) $\in V(G)$. We remark that $dist(v_1, v_4) = dist(v_2, v_5) = dist(v_3, v_6) = 3$. Without loss of generality, suppose that C chooses v_1 to stay. Correspondingly, R will choose v_4 to stay. If C remains at his position, R also remains at his position. If C moves to v_2 or v_6, R moves to v_5 or v_3 correspondingly. So R can maintain infinitively the distance ≥ 2 from C at any steps. Then C cannot capture R.
Chalopin et al. [5] charaterized the bipartite graphs in $CWRC(1)$.

1.2.3. *Faster Robber variant.*
Let define the speed of the robber as s if at his turn, R can move to some vertex at most distance s from the vertex which it is occupying. By analog way, we define the speed of the cop as s'. If $s' > s$, in any graph G, C always can capture R by following the shortest path from the vertex occupied by C to the one occupied by R. We only consider the game if $s' \leq s$. This game is called the Faster Robber variant. The set of cop win graphs in this variant is denoted by $CWRC(s, s')$. By convention, $CWFR(s) \equiv CWFR(s, 1)$.
Recently, Chalopin et al. [5] characterized the graphs in $CWFR(s, s')$ and . Fomin et al. [9] also proved an interesting result for the $n \times n$ square grid G, $cn(G) = \Omega(\sqrt{log(n)})$. Alon et al. [6] proved that $\forall n$, $\exists$ a connected n-vertex graph G with

$cn(G) = \Omega(n^{s/s+1})$. They also generalized the Meyniel conjecture for any n-vertex graph G, $cn(G) = O(n^{s/s+1})$ [6].
In the report, we present our novel results in two variants described above. The report is structured as follows: Section 2 presents the notations and theorems used in the next sections. Section 3 and 4 are dedicated to the radius capture and faster robber variants respectively. Section 5 summarizes the results we gained and presents the future work.

2. Background on Graph Theory

In this section we review the basic definitions concerning graphs and some graph theorems used for proofs in this report.

2.1. Definitions.

Definition 1. *A* **graph** *G is a pair (V, E) of sets satisfying $E \subseteq V^2$, where V^2 denotes the set of all 2-element subsets of V. We also assume tacitly that $V \cap E = \emptyset$. The elements of V are the* **vertices** *of the graph G and the elements of E are its* **edges**.
A vertex v is *incident* with an edge e if $v \in e$. The two vertices incident with an edge are its *endvertices*.

Definition 2. *The* **degree** *of a vertex v, denoted by d(v), is the number of edges incident to the vertex.*

Definition 3. *A* **walk** *in G=(V,E) is a finite (non-empty) sequence $W = v_0e_1v_1e_2v_2...e_kv_k$ alternating vertices and edges such that, $\forall 0 \leq i \leq k, v_i \in V$, $\forall 1 \leq i \leq k, e_i \in E$, $\forall 1 \leq i \leq k, v_{i-1}$ and v_i are the endvertices of e_i.*
We denote W by $(v_0, v_k) - walk$..

Definition 4. *A* **path** *is a walk such that all vertices are distinct.*
A path connecting 2 vertices u, v in the graph G is denoted by $(u, v) - path$.

Definition 5. *A* **cycle** *is a walk such that all vertices are distinct except the start and terminus.*

Definition 6. *A graph G is* **connected** *if for any two vertices u, v, there exists a (u, v)-path in G.*

Definition 7. *A* **tree** *is a connected graph without any cycle.*

Definition 8. *The* **distance** *between two vertices u and v in graph G, denoted by dist(u,v), is the number of edges in a shortest path connecting them.*

Definition 9. *A subgraph H of a graph G is said to be* **induced** *if, for any pair of vertices x and y of H, (x,y) is an edge of H if and only if (x,y) is an edge of G. In other words, H is an induced subgraph of G if it has exactly the edges that appear in G over the same vertex set. If the vertex set of H is the subset S of V(G), then H can be written as G[S] and is said to be* **induced by S**.

Definition 10. *A graph is a* **bipartite graph** *if it consists of 2 sets of vertices with edges only joining vertices between sets and not within a set.*

Definition 11. *A graph is a* **k-chordal graph** *if every cycle of length greater than k has a chord.*
A chord is an edge joining two nonconsecutive vertices of a cycle. Equivalent to Definition 11, a k-chordal graph does not contain an induced cycle of length greater than k.

Definition 12. *A graph is a* **planar graph** *if it can be drawn in a plane without any edges crossing.*

Definition 13. *Let G be a planar graph and E be an planar embedding of G. An internal face of G and E is a subgraph $\{v_1, .., v_k\}$ induce a cycle C in G such that no edges nor vertices are in the finite part of the plane defined by C.*

By Jordan Curve Theorem [10], a plane is divided into 2 parts by the cycle C. One part is finite and the other is infinite.

Definition 14. *Let G be a planar graph and* E *be an planar embedding of G. An unbounded face F of a G and* E *be the infinite connected component of $R^2 \setminus$* E.

Definition 15. *A graph G is an* **outerplanar graph** *if all vertices of G belong to the unbounded face.*

Definition 16. *A graph G is* **k-outerplanar** *if for $k = 1$, G is outerplanar and for $k > 1$, G has a planar embedding such that if all vertices on the exterior face are deleted, the connected components of the remaining graph are all $(k-1)$ outerplanar.*
By definition, a graph is an outerplanar graph if it can be embedded in a cycle such that all its vertices are in the cycle and its all edges are inside the area bounded by the cycle.

Definition 17. *A graph is a* **triangulated graph** *if it is planar and the length of all internal faces is 3*

Definition 18. *A vertex v of a graph G is a* **cut vertex** *if G if $G \setminus v$ is not connected.*

Definition 19. *In the graph $G = (V, E)$, a vertex subset $S \subset V$ is a* **vertex separator** *for non adjacent vertices u and v if the removal the set S of V from G separates u and v into distinct connected components.*

Definition 20. *Two paths are* **independent** *if their internal vertices are distinct. In particular, two (s,t)-paths are independent if their common vertices are only s and t.*

Definition 21. *A graph is* **2-connected** *if it is connected and has no cut vertex.*

Definition 22. *Let u be a vertex in a graph $G = (V, E)$, k-vertex neighborhood of v, denoted by $N_k[u]$, is the set of vertices $v \in V$ such that $dist(u, v) \leq k$. And $N_k(u) = N_k[u] \setminus u$.*
By convention, $N_1[u] = N[u]$ and $N_1(u) = N(u)$.

Definition 23. *Let u be a vertex in a graph $G = (V, E)$, $L_k(u)$, is the set of vertices $v \in V$ such that $dist(u, v) = k$.*
By convention, $L_1(u) = L(u)$.

Definition 24. *Let G=(V,E) be an outerplanar graph embedded in a circle and u,v be 2 distinct vertices in V. $\overrightarrow{(u,v)}$ is denoted by the set of intermediate vertices in the arc (u,v) such that the direction from u to v is clockwise. $\overrightarrow{[u,v]} = \overrightarrow{(u,v)} \cup \{u,v\}$; $\overrightarrow{[u,v)} = \overrightarrow{(u,v)} \cup \{u\}$; $\overrightarrow{(u,v]} = \overrightarrow{(u,v)} \cup \{v\}$.*

Definition 25. *A graph is* **dismantlable** [2] [3] *if its vertices can be ordered $(v_1, v_2, ..., v_n)$ such that, $\forall i < n, \exists j > i$ with $N[v_i] \cap X_i \subseteq N[v_j]$, where $X_i = \{v_i, ..., v_n\}$.*

Definition 26. *A graph $G = (V,E)$ is* **(s,s')-dismantlable** [5] *if $V = (v_1, v_2, ..., v_n)$ such that $\forall i < n, \exists j > i$ with $N_s(v_i, G \setminus \{v_j\}) \cap X_i \subseteq N_{s'}(v_j)$, where $X_i = \{v_i, ..., v_n\}$.*

Definition 27. *A bipartite graph G is called dismantlable* [5] *if its vertices can be ordered $(v_1, v_2, ..., v_n)$ so that $(v_{n-1}, v_n$ is an edge of G and for each $v_i, i < n-1, \exists v_j, j > i$ such that $N(v_i) \cap X_i \subseteq N(v_j)$, where $X_i = \{v_i, ..., v_n\}$.*

Definition 28. *A* **configuration** *of the cops and robber game represents the state of game at some step, for instance, the positions of the cops and the robber at this step of game. A* **strategy** *defines which moves must be done by the cops, resp., the robber, given the current configuration and all proceeding moves.*

Definition 29. *A sequence of valid move or a trajectory for a cop (resp., the robber) is a potentially infinite sequence of vertices $(v_1, v_2, ..., v_p, ...) \in V$ such that for any $i \geq 1$, $dist(v_i, v_{i+1}) \leq s_C$(resp., $dist(v_i, v_{i+1}) \leq s_R$) where s_C, s_R are the speed of the cop and the robber respectively.*

2.2. Characterization of Cop Win graphs.

Theorem 1. [2] [3] *A graph $G \in CW$ iff G is dismantlable.*

Theorem 2. [5] *A graph $G \in CWFR(s, s')$ iff G is (s,s')-dismantlable.*

Theorem 3. [5] *A bipartite graph $G \in CWRC(1)$ iff G is dismantlable.*

Theorem 4 (Menger)**.** *Let G be a connected graph and u, v are two non-adjacent vertices. So the number of independent paths from u to v equals to the number of vertices in mimimum (u,v)-separator.*

Proposition 1. *[Folklore] Let G be a 1-connected graph, $\exists$ a tree T such that one-to-one mapping σ from V(T) to the set of blocks which are 2 connected components of G such that $(u, v) \in E(T)$ iff $|\sigma(u) \cap \sigma(v)| = 1$.*

3. Cop-win graphs in radius capture

In the radius capture variant, a cop can capture the robber in distance k, $(k \geq 1, k \in N)$ and the speed of the cop and the robber is 1. In this section, we present the novel results with square-grid, k-chordal and k-outerplanar graphs for the radius capture variant.

3.1. **Square-grid.**

Theorem 5. *[Folklore]Let G be a finite square-grid, then $G \in CWRC(1)$.*

Proof. Let $d > 1$ be the distance of the cop and the robber. We prove that C has a strategy to keep distance d decrease after the finite steps. Hence, d decreases to 1 and R is captured. Without loss of generality, suppose that at the initial configuration, C is in some vertex I and R is in the region bounded by East axis and South axis with origin from I as illustrated in Figure 1:

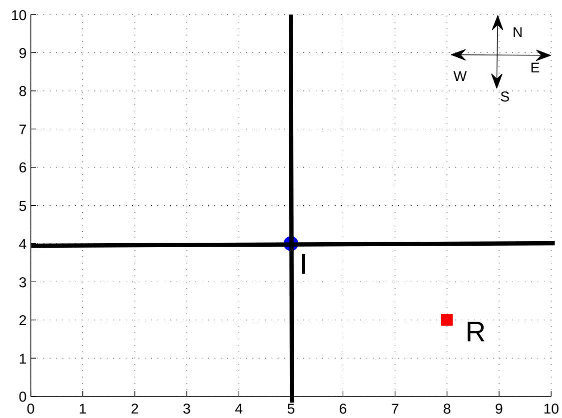

Figure 1. Initial Configuration

If we put the coordinates as illustrated in Figure 1, we denote $abs(C, R) = |abs(C) - abs(R)|$ and $ord(C, R) = |ord(C) - ord(R)|$. Initially, $d = abs(C, R) + ord(C, R) = abs(I, R) + ord(I, R)$.
Now we consider R's turn when C is on I.

(1) If R moves towards East or South, C will move correspondingly towards East or South. Then d remains unchanged but R goes closer to the border of the grid.

(2) If R remains at his position and $abs(I, R) \geq 1$, C moves towards East;otherwise, C moves towards South. Then d decreases by 1 after C's move.

(3) If R moves towards West, the stategy of C is based on $abs(I, R)$.

(a) If $abs(I, R) \geq 2$, C moves to East then $abs(C, R)$ decreases by 2 and $ord(C, R)$ remains unchanged. Then d decreases by 2.
(b) If $abs(I, R) = 1$, C moves towards South, then $abs(C, R)$ decreases by 1 and $ord(C, R)$ decreases by 1 also. Then d decrease by 2.
(c) If $abs(I, R) = 0$, then $ord(C, R)$ must be greater than 1, otherwise, R is captured. C will move towards South, then $abs(C, R) = 1$ and $ord(C, R)$ decreases by 1. Then d remain unchanged but C gets closer to the South border. Hence, such a move cannot be done infinitively. After the cop's move, R is currently in the region bounded by the West and South axes. By symmetry, we apply the analog strategy for C this case.

Also due to symmetry, we can define the analog strategy for C if R moves towards South.
Because G is finite, the moves of (1) and (3a) cannot be done infinitively. Hence, d decreases after a finite steps. Hence, d decreases such that $d = 1$ at some time, and the robber is captured at that time.

□

Remark 1. *There is the different view for the problem. Because G is a square-grid, so G is a bipartite graph. We will apply Theorem 3 to prove $G \in CWRC(1)$. Let G be $m \times n$ square grid. Let mark the name for the vertices of G by order form left to right, from top to bottom, we have*
$V(G) = \{\{v_1, v_2, ..., v_n\}\}, \{v_{n+1}, v_{n+2}, ..., v_{2n}\}, ..., \{v_{(m-1)n+1}, v_{mn}\}$.
By definition, we see that $(v_{mn}, v_{mn-1}) \in E(G)$ and for $(m-1)n + 1 \leq i < mn$, we have $N(v_i, G_i) \subseteq N(v_{i+1}, G_i)$. For v_{in} ,where $0 \leq i \leq m-1$, $N(v_{in}, G_{in}) \subseteq N(v_{(i+1)n}, G_{in})$.For v_{in+k}, where $0 \leq i \leq m-2$ and $k \leq i \leq n-1$, we have $N(v_{in+k}, G_{in+k}) \subseteq N(v_{(i+1)n+k+1}, G_{in+k})$. Hence, by Theorem 3, $G \in CWRC(1)$.

3.2. **k-chordal Graph.**

Theorem 6. *If G is k-chordal($k \geq 5$), then $G \in CWRC(k-4)$.*

Proof. The Figure 2 illustrates for the proof of Theorem 6.

Suppose that C and R on the vertices v and u respectively. Because the cop's turn is first, if $u \in N_{k-3}(v)$, the cop will capture the robber after one move. If $u \notin N_{k-3}(v)$, let X be the connected component including u in $G \setminus N_{k-3}(v)$. Let $S \subseteq N_{k-3}(v)$ be the set of vertices adjacent at least one vertex in X. By definition, $S \subseteq L_{k-3}(v)$, otherwise, if there exists some vertex $w \in S$ such that $dist(v, w) < k-3$, then for all vertex $x \in N(w)$, $dist(v, x) \leq k-3$ (contradictory to $w \in S$). Let $z \in N(v)$ satisfying for all vertex $y \in N(v)$, $|N_{k-4}(z) \cap S| \geq |N_{k-4}(y) \cap S|$. We will prove that $S \subseteq N_{k-3}(z)$. So, C can move from v to z and avoid R move out of X.
If $S \setminus N_{k-3}(z) = \emptyset$, $S \subseteq N_{k-3}(z)$ by definition; otherwise, consider some vertex $t \in S \setminus N_{k-3}(z)$, there must exist a vertex $y \in N(v), y \neq z$ such that $t \in N_{k-4}(y)$. Because $|N_{k-4}(y) \cap S| \leq |N_{k-4}(z) \cap S|$ and $t \in N_{k-4}(y) \setminus N_{k-3}(z)$, there must exist a vertex $s \in N_{k-4}(z) \setminus N_{k-4}(y)$. Let $P0$ be the shortest path from t to s in X, $P1$ be the shortest path from z to s, $P2$ be the shortest path from y to t. Because $t \notin N_{k-3}(z)$, then $V(P1) \cap V(P2) = \emptyset$.
Consider the cycle $cyc = \{v, z, P1, s, P0, t, P2, y, v\}$, we see that its length $\geq 1 +$

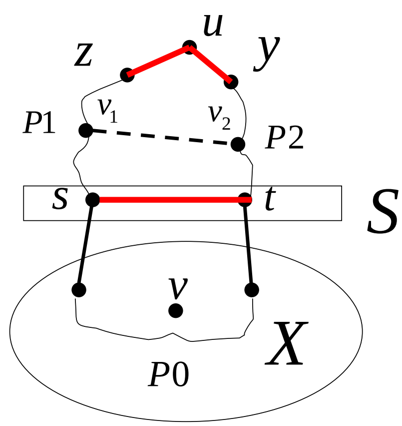

FIGURE 2. Illustration of Theorem's Proof in Radius Capture k

$(k-4)+1+(k-4)+1 = 2k-5 \geq k$. Because $P0, P1, P2$ are the shortest paths, they are 1-connected. By definition, there is no edge connecting some vertex in P0 and some vertex in $P1 \cup P2$. If there exists an edge between some vertex v_1 in $P1 \cup \{z, s\}$ and some vertex v_2 in $P2 \cup \{y, t\}$, suppose $v1 \in L_l(v)$ and $v_2 \in L_m(v)$ where $1 \leq l, m \leq k-3$.
If $l \geq k$, then $d(z,t) \leq d(z, v1) + d(v1, v2) + d(v2, t) = l - 1 + 1 + k - 3 - m \leq k-3$. Hence $t \in N_{k-3}(z)$ (contradiction).
If $l < k$, $\forall k, x \in (N_{k-4}(z) \cap S, d(y,x) \leq d(y, v2) + d(v2, v1) + d(v1, x) = m - 1 + 1 + k - 3 - l \leq k - 4$, then $x \in N_{k-4}(y)$. Because $t \in N_{k-4}(y) \setminus N_{k-3}(z)$, then $(N_{k-4}(z) \cap S) \subset (N_{k-4}(y) \cap S)$. So $|N_{k-4}(z) \cap S| < |N_{k-4}(y) \cap S|$ (contradiction).
So length of $cyc \geq k$ but it does not have any egde connecting 2 non-consecutive vertices of cyc (contradictory to k-chordal hypothesis).
Therefore, we have $S \subseteq N_{k-3}(z)$.
Let X' be the connected component including R in $G \backslash N_{k-4}(z)$. Obviously, $X' \subseteq X$. Take a vertex $w \in N_{k-4}(z) \cap S$, there must exist a vertex $x \in X$ adjacent to w. Hence, $x \in N_1(w) \cap X$; hence, $x \in N_{k-3}(z)$; hence, $x \notin X'$; hence, $X' \subset X$. Because size of X is reduced after each move of the cop, then after finite steps, the robber is captured. □

Theorem 7. *If G is k-chordal ($k \geq 5$) and a cop can capture the robber with radius capture 1, then $cn(G) \leq k-4$.*

Proof. Let $v \in V$ be any vertex and place all cops at v. Then, the robber chooses a vertex. Now, at some step, assume that the cops are occupying $\{v_1, ..., v_n\}$ which induce a chordless path, $n \leq k-4$, and it is the turn of the cops. Let $N = \bigcup_{i \leq n} N_2[v_i]$, if the robber occupies a vertex in N, he is captured during the next move. Otherwise, let $X \neq \emptyset$ be the connected component of $G \setminus N$ occupied by the robber. Let $S \subseteq N$ be the set of vertices adjacent to some vertex in X. By definition, $S \subseteq \bigcup_{i \leq n} L_2(v_i)$. Now there are 3 cases to be considered:
(i) If $S \subseteq \bigcup_{i=2}^{n} N_2(v_i)$ or $S \subseteq \bigcup_{i=1}^{n-1} N_2(v_i)$, we can "remove" a cop from v_1 or v_{k-4} and the cops occupy a shorter chordless path while the robber is restricted to X.

(ii) If $n < k-4$ and $S \not\subseteq \bigcup_{i=1}^{n-1} L_2(v_i)$ and $S \not\subseteq \bigcup_{i=2}^{n} L_2(v_i)$, tet t be any vertex in $(L_2(v_n) \cap S) \setminus \bigcup_{i=1}^{n-1} L_2(v_i)$ and v_{n+1} be any vertex in $N(v_n) \cap N(t)$. By definition, $v_{n+1} \notin \bigcup_{i=1}^{n-1} N(v_i)$ so that $v_1, v_2, .., v_{n+1}$ induce a chordless path. We can "add" a "new cop" at v_{n+1}. Let $N' = \bigcup_{i \leq n+1} N_2[v_i]$, and X' be the connected component of $G \setminus N'$ occupied by the robber, then because $N(t) \cap X \neq \emptyset$, hence $L_2(v_{n+1}) \cap X \neq \emptyset$, $X' \subset X$.
(iii) If $n = k-4$ and $S \not\subseteq \bigcup_{i=1}^{k-5} L_2(v_i)$ and $S \not\subseteq \bigcup_{i=2}^{k-4} L_2(v_i)$. We will prove that there must exist a vertex $z \in N(v_1)$ such that $S \subseteq \bigcup_{1}^{k-5} L_2(v_i) \cup N_2(z)$ or $z \in N(v_{k-4})$ such that $S \subseteq \bigcup_{2}^{k-4} L_2(v_i) \cup N_2(z)$ and $\{z, v_1, ..., v_{k-5}\}$ or $\{v_2, ..., v_{k-4}, z\}$ is chordless. Therefore, $k-4$ cops can move from $\{v_1, ..., v_{k-4}\}$ to $\{z, v_1, ..., v_{k-5}\}$ or $\{v_2, ..., v_{k-4}, z\}$. The Figure 3 illustrates for case (iii).

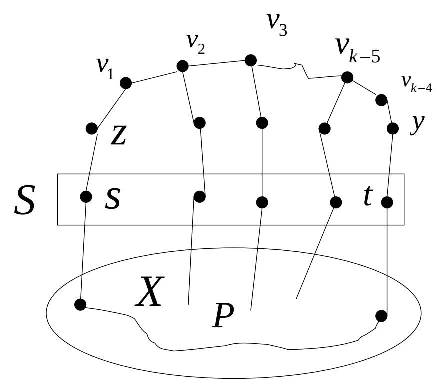

FIGURE 3. Illustration of case (iii)

Without loss of generality, suppose that there exists a vertex $z \in N(v_1) \setminus \bigcup_{i=2}^{k-4} N(v_i))$ such that $|N(z) \cap S|$ is maximum among all vertices in $(N(v_1) \setminus \bigcup_{i=2}^{k-4} N(v_i)) \cup (N(v_{k-4}) \setminus \bigcup_{i=1}^{k-5} N(v_i))$. By definition, there is no egde between z and v_i, $i = 2, .., k-4$. Next, denote $N' = (\bigcup_{i=1}^{k-5} L_2(v_i)) \cup N_2(z)$, we need to prove $S \subseteq N'$. Considering some vertex $t \in L_2(v_{k-4}) \cap S \setminus \bigcup_{i=1}^{k-5} L_2(v_i)$. We need to prove that $t \in N_2(z)$; hence, $S \subseteq N'$. By definition, there is no edge between t and $v_i, i = 1, 2, ..., k-4$. If $t \notin N(z)$, we will prove that $t \in L_2(z)$. By definition, there must exist some vertex $y \in N(v_{k-4}) \setminus (\bigcup_{i=1}^{k-5} N(v_i) \cup N(z))$ such that $t \in N(y)$. By definition also, there is no edge between y and $v_1, v_2, ..., v_{k-5}$ and z. Because $|N_1(y) \cap S| \leq |N_1(z) \cap S|$, there must exist a vertex $s \in N_1(z) \cap S$ such that $s \notin N_1(y)$. Because X is connected, there exist a path P in X connecting s and t. We have the cycle $(s, z, v_1, v_2, .., v_{k-4}, y, t, P, s)$ with length $> k$. Hence,there must be an edge between s and t. Hence, $t \in L_2(z)$. So, the cops can move from $\{v_1, .., v_{k-4}\}$ to $\{z, v_1, ..., v_{k-5}\}$. Because there is no egde between z and $\{v_1, ..., v_{k-5}\}$, $\{z, v_1, ..., v_{k-5}\}$ is a chordless. Let $N' = \bigcup_{i=1}^{k-5} N_2[v_i] \cup N_2[z]$, and X' is the connected component of $G \setminus N'$ occupied by the robber. Because there is a vertex $w \in X \cap N_2(z)$, then $X' \subset X$.
We see that in case (ii) and (iii), after the cops' move, the connected component including the robber is strictly shrinked. In case (i), after a finite steps "remove"

the cops without increasing the connected component including the robber, we can change to case (ii) or (iii) to "add" some cops, then the connected component including the robber is strictly shrinked. Therefore, after a finite steps, the cops can capture the robber. □

3.3. **Outerplanar Graph.**

First, we give the proof for the lemma below which is applied to prove all theorem in this section.

Lemma 1 (Folklore)**.** *Let $G = (V, E)$ be a 2-connected outerplanar graph, then V can be order as $(v_1, v_2, ..., v_n)$ such that V is embedded in a cycle, $(v_i, v_{i+1}) \in E$ where $i = 1, 2, .., n$, $v_{n+1} = v_1$ and all other edges are "inside" the area bounded by the cycle $(v_1, v_2, ..., v_n, v_1)$.*

Proof. Suppose that $\{v_1, v_2, ..., v_n\}$ are embedded in a circle by the clockwise direction such that $\overrightarrow{(v_i, v_{i+1})} = \emptyset$ (*) where $v_{n+1} = v_1$. We will prove that $(v_i, v_{i+1}) \in E$ and all other edges are "inside" the the area bounded by the cycle $(v_1, v_2, ..., v_n, v_1)$. By contradiction, without loss of generality, suppose that $(v_1, v_2) \notin E$. Because G is 2-connected, by Menger's Theorem, there must exist 2 independent paths from v_1 to v_2. Suppose that v_i and v_j are 2 vertices in 2 independent paths. By property of independent paths, v_i, v_j are distinct. By (*) and symmetry, without loss of generality, suppose that $j < i$, then $v_j \in \overrightarrow{(v_2, v_i)}$. Hence, the path from v_1 to v_j cross the path from v_2 to v_i (contradictory to property of planar graph). So $\forall i \in \{1, 2, .., n\}, (v_i, v_{i+1}) \in E(G)$ where $v_{n+1} = v_1$. By the property of outerplanar, all the other edges of $E(G)$ are "inside" the area bounded by the cycle $(v_1, v_2, ..., v_n, v_1)$. The Figure 4 illustrate for the proof of Lemma 1. □

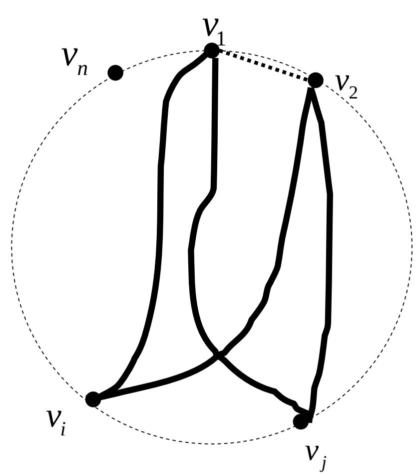

FIGURE 4. Illustration of Lemma 1

Corollary 1. *This Lemma 1 can be applied for each outerface of k-outerplanar graph.*

Theorem 8. *Let G be a connected outerplanar graph. $G \in CWRC(1)$ if and only if length of every internal face of $G < 6$.*

Before giving the proof of Theorem 8, we will consider the following lemma:

Lemma 2. *Let G be an outerplanar graph and F be an internal face of G. $V(F)$ is ordered as $(v_1, v_2, ..., v_k)$ such that v_i, v_{i+1} are adjacent ($v_{k+1} = v_1$) . Let v_i, v_j be 2 non adjacent vertices in $V(F)$, then $\{v_{i-1}, v_{i+1}\}$ are (v_i, v_j)-separator where $v_0 = v_k, v_1 = v_{k+1}$.*

Proof. We prove by contradiction, if $\{v_{i-1}, v_{i+1}\}$ is not the separator of v_i and v_j , there must exist a a path $P = (v_i, ..., y, ..., v_j)$ such that $v_{i-1}, v_{i+1} \notin V(P)$ and $\exists y \in V(P) \setminus \{v_{i-1}, v_{i+1}\}$. Without loss of generalization, 4 vertices $v_i, v_{i-1}, v_{i+1}, v_j$ can be drawn as illustrated in Figure 5

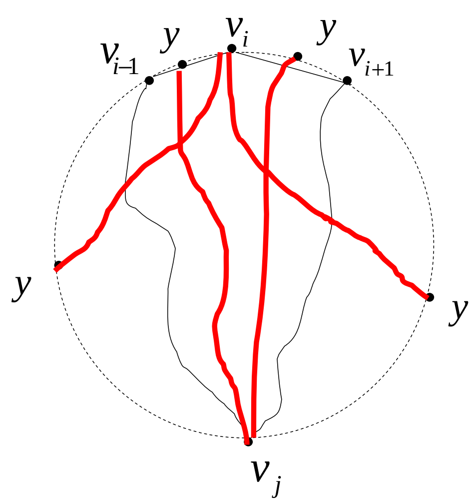

FIGURE 5. Illustration of Lemma 2

By definition of outerplanar graph, 5 vertices $v_i, v_{i-1}, v_{i+1}, v_j, y$ can be embedded in a circle. So $y \in \overrightarrow{(v_i, v_{i+1})} \cup \overrightarrow{(v_{i+1}, v_j)} \cup \overrightarrow{(v_{i+1}, v_j)} \cup \overrightarrow{(v_j, v_{i-1})}$. By symmetry, suppose that $y \in \overrightarrow{(v_i, v_{i+1})} \cup \overrightarrow{(v_{i+1}, v_j)}$. If $y \in \overrightarrow{(v_i, v_{i+1})}$, the path from v_i to v_j crosses (v_i, v_{i+1}). If $y \in \overrightarrow{(v_{i+1}, v_j)}$, the path from v_i to y will cross the path from v_{i+1} to v_j. So it is a contradiction. $\square$

Proof. $\Rightarrow$: We prove the Theorem 8 by contraposition. We need to prove that if G admits an internal face > 5 , $G \notin CWRC(1)$. Let F be the largest internal face in G and $\{v_1, v_2, ...v_k\} = V(F)$ such that $(v_i, v_{i+1}) \in E(G)$ where $i = 0, 1, .., k$ and $v_{k+1} = v_1$, $k \geq 6$. By definition of the outerplanar graph and internal face F, $(v_i, v_j) \notin E(G)$ if $|i - j| \notin \{1, k - 1\}$. Without loss of generalization, suppose that at the initial configuration, C is on a vertex in $\overrightarrow{[v_1, v_2)}$. R can be put at vertex v_4 correspondingly in the initial configuration. Because length of F ≥ 6, hence $dist(v_1, v_4) = 3$. If C is on v_1, $dist(C, R) = 3$; otherwise, suppose C is on some vertex $u \in \overrightarrow{(v_1, v_2)}$. By planar property, $\{v_1, v_2\}$ is a (u, v_4)separator,

then $dist(C,R) \geq 1 + min\{dist(v_1,v_4), dist(v_2,v_4)\} = 3$. Because $\{v_1,v_2\}$ is a (u,v_4)separator and by Lemma 2, $\{v_k,v_2\}$ is a (v_1,v_4)-separator, then $\{v_k,v_2\}$ is also a (u,v_4)-separator. Hence $\{v_k,v_2\}$ is also a (C,R)-separator. R escapes C as follow the strategy:
R stays at v_4 and waits for C get closer to R. C has to move to v_2 or v_k. If C moves to v_2, R moves to v_5. If C moves to v_k, R moves to v_3. So $dist(C,R)$ always ≥ 2. Therefore, the robber can escape the cop infinitively.
$\Leftarrow$: We need to prove that if G is outerplanar graph and all internal faces ≤ 5, $G \in CWRC(1)$.
Case 1: G is 2-connected.
By applying the Lemma 1, we order the set of vertices $V = \{v_1, v_2, ..., v_n\}$ embedded in a cycle such that $(v_i, v_{i+1}) \in E$ where $i = 1, 2, .., n$, $v_{n+1} = v_1$. Without loss of generality, suppose that C and R are on v_1 and v_i respectively where $1 \leq i \leq n$. If $dist(v_1,v_i) \leq 2$, R will be captured after one move. If $dist(v_1,v_i) \geq 3$,then $3 \leq i \leq n-2$. Let v_s be some vertex adjacent to v_1 such that $1 < s < i-1$ and $i-s$ is minimum. There exists v_s because v_2 is a vertex adjacent to v_1 and $1 < 2 < i$. Then let v_x be adjacent to v_s such that $s < x < i$ and $i-x$ is minimum. There exists v_x because v_{s+1} is a vertex adjacent to v_s and $v_{s+1} \neq v_i$, so $s < s+1 < i$. By the analog way, let v_t be some vertex adjacent to v_1 such that $i < t < n-1$ and $t-i$ is minimum and v_y be some vertex adjacent to v_t such that $i < y < t$ and $y-i$ is minimum. The Figure 6 illustrates the position of vertices $v_1, v_i, v_s, v_t, v_x, v_y$.

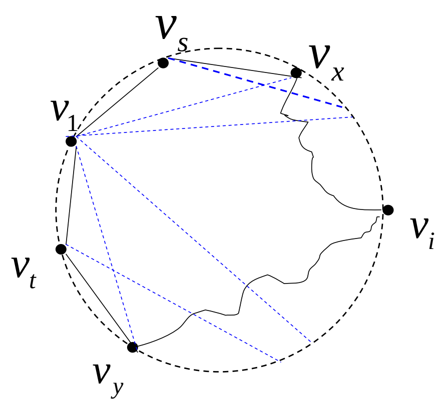

FIGURE 6. Positions of $v_1, v_i, v_s, v_t, v_x, v_y$

Let X be the connected component including R in $G \setminus N_2[v_1]$. We will prove that $V(X) \subseteq S = \{v_{x+1}, v_{x+2}, ..., v_{y-1}\}$. So R is restricted in S if C stays at v_1. By contradiction, suppose that there is some vertex $v_k \in X \backslash S$. Because $v_k \notin N_2[v1] \cup S$, then $v_k \notin \{v_1, v_s, v_t, v_x, v_y\} \cup S$. Hence, every path from v_k to v_i crosses the path $(v_x, v_s, v_1, v_t, v_y)$ (contradiction).
Now we will prove that there exists a vertex $z \in N(v_1)$ such that if C move to z, R is restricted in the set $S' \subset S$. By definition, $\forall k, s \leq k \leq t, v_k \notin N(v_1)$; $, x < k \leq i, v_k \notin N(v_s)$; $\forall k, i \leq k < y, v_k \notin N(v_t)$.

- If $\exists k, i < k < y$ such that $v_k \in N(v_s)$, then $z \equiv v_s$ because R is restricted in $S' = \{v_{x+2}, ..., v_{k-1}\} \subset S$.

- By the analog proof, if $\exists k, x < k < i$ such that $v_k \in N(v_t)$, then $z \equiv v_t$.
- If $\nexists k, i < k < y$ such that $v_k \in N(v_s)$ and $\nexists k, x < k < i$ such that $v_k \in N(v_t)$, then $N(v_s) \cap S = \emptyset$ and $N(v_t) \cap S = \emptyset$. Let P be the shortest path from v_x to v_y in S.
  - If $(v_s, v_t), (v_s, v_y), (v_x, v_t), (v_x, v_y) \notin E(G)$, then $V(P) \neq \emptyset$. Hence, $(v_1, v_s, v_x, P, v_y, v_t, v_1)$ is an internal face > 5 (contradiction).
  - If $(v_s, v_y) \in V(G)$, let $S' = \{v_{x+2}, ..., v_{y-1}\} \subset S$ then $z \equiv v_s$.
  - If $(v_t, v_x) \in V(G)$ and $z \equiv v_t$, the proof is analog.
  - If $(v_x, v_y) \in V(G)$, let $S' = \{v_{x+2}, ..., v_{y-1}\} \subset S$ and $z \equiv v_s$. Let X' be the connected component including R in $G \backslash N_2(z)$. If there exists some vertex $v_k \in X' \setminus S'$. By definition, $v_k \notin \{v_s, v_x, v_{x+1}, v_t\} \cup S'$. Hence, any path from v_k to any vertex in S' crosses (v_x, v_y) (contradiction). Therefore, $X' \subseteq S'$. So R is restricted in area $S' \subset S$.

Because the size of $V(S)$ strictly decreases after the cop's moves, therefore, after finite steps, $|V(S)| = 0$, it implies $|V(X)| = 0$, then the robber is captured.
Case 2: G is 1-connected but 2-connected.
By Proposition 1 , there must exist a tree T satisfying the conditions Proposition 1. Suppose that at initial configuration, the cop is in the 2-connected component $C_1 \in V(T)$. Let $U \subseteq V(C_1)$ be a set of cut vertices. Let u be some vertex in U, then $Comp(u)$ is denoted by the biggest connected component including the vertex u such that $V(Comp(u)) \cap V(C_1) = \{u\}$.
If the robber has a valid sequence of moves $S_r = \{r_1, r_2, ..., r_p, ...\}$ in G, we can define a retract sequence of moves $S'_r = \{r'_1, r'_2, ..., r'_p, ...\}$ in C_1 where $r'_i = r_i$ if $r_i \in V(C_1) \setminus U$ and $r'_i = u$ if $r_i \in V(Comp(u))$. We need to verify S'_r valid by proving $\forall i \geq 1, dist(r'_i, r'_{i+1}) \leq 1$.

- If $r_i, r_{i+1} \in V(C_1)$, $d(r'_i, r'_{i+1}) = d(r_i, r_{i+1}) \leq 1$.
- If $r_i \in V(C_1), r_{i+1} \notin V(C_1)$, because $dist(r_i, r_{i+1}) = 1$, then $r_{i+1} \in Comp(r_i)$. Hence, $dist(r'_i, r'_{i+1}) = dist(r_i, r_i) = 0$.
- If $r_i \notin V(C_1), r_{i+1} \in V(C_1)$. By analog proof, $dist(r'_i, r'_{i+1}) = 0$.
- If $r_i, r_{i+t} \notin V(C_1)$, because $dist(r_i, r_{i+1}) = 1$,there must exist a cut vertex v such that $r_i, r_{i+1} \in Comp(v)$. Hence, $dist(r'_i, r'_{i+1}) = dist(v, v) = 0$.

Because C_1 is 2-connected, by case 1, the cop has a winning strategy σ in C_1. C will play the strategy Σ in G based on σ in C_1. First, C plays strategy σ by considering S_r in G as S'_r in C_1. If by σ, C can capture R at $v \in V(C_1) \setminus U$, C can capture R in G at that step. If by σ, C capture R at $u \in U$, hence at that move, $R \in Comp(u)$ and $C \in N(u)$. Now C will play Σ by moving to u. Let denote C_2 be a 2-connected component such that C_2 is a isomorphic subgraph of $Comp(u)$, $C_2 \in V(T)$ and $u \in V(C_2)$. By induction, we can define a strategy σ_2 for C playing in C_2. We remark that if the cop using the strategy as presented in case 1 for 2-connected outerplanar, the robber never moves to the vertex occupied by the cop at initial configuration. Hence R cannot move to u in C_2; hence, R cannot move to C_1 if C plays the Σ strategy. So because T is a tree of 2-connected components, by induction, the cop can capture the robber after finite steps. $\square$

Theorem 9. *Let G be a k-outerplanar ($k \geq 1$) graph and $\mathcal{E}$ be a planar embedding satisfying the following properties:*
i) No internal face with length > 5
ii) If F_i is the i^{th} outerface, then F_i is 2-connected $\forall 1 \leq i \leq k$.
iii) $\forall v \in V(F_i), N(v) \cap V(F_{i+1}) \neq \emptyset$ where $1 \leq i \leq k-1$
iv) $\forall v \in F_i$ where $2 \leq i \leq k-1, d(v) \leq 4$
v) $\forall v \in F_k, d(v) \leq 3$,
vi) $|V(F_i)| < |V(F_{i+1}|$ where $1 \leq i \leq k-1$,
then $G \in CWRC(1)$.

Before giving the proof of Theorem 9, we prove some lemmas as follows:

Lemma 3. *Let G, $\mathcal{E}$ be k-outerplanar and planar embedding satisfying the conditions in Theorem 9. If $u \in V(F_i)$, then $|N(u) \cap V(F_{i-1})| \leq 1$ where $2 \leq i \leq k$.*

Proof. If $i = k$, because $d(u) \leq 3$, then $|N(u)| \leq 3$. Because F_k is 2-connected and $v, w \in N(u) \cap V(F_k)$. Hence $|N(u) \cap V(F_{k-1})| \leq 1$. If $2 \leq i < k$, because $d(u) \leq 4$, then $|N(u)| \leq 4$. By definition, there are 3 distinct vertices y, v, w in $N(u)$ such that $y \in N(u) \cap V(F_{i+1})$ and $v, w \in N(u) \cap V(F_i)$. Hence $|N(u) \cap V(F_{k-1})| \leq 1$. □

Corollary 2. *$\nexists u \in V(F_k)$ such that $V(F_{k-1}) \subseteq N(u)$.*

Lemma 4. *Let G, $\mathcal{E}$ be k-outerplanar and planar embedding satisfying the conditions in Theorem 9. If u, w are 2 adjacent vertices in F_k , then $(N(u) \cup N(w)) \cap V(F_{k-1}) \neq \emptyset$.*

Proof. By contradiction, suppose that there exist 2 adjacent vertices in $V(F_k)$ such that $(N(u) \cup N(w)) \cap V(F_{k-1}) = \emptyset$. Because F_k is 2-connected, without loss of generalization, by corollary 1, suppose that $V(F_k)$ is ordered as $(u \equiv y_1, v \equiv y_2, ..., y_{n_k})$ satisfying the condition of Lemma 1. By corollary 2, suppose that y_i, y_j are the vertices in $V(F_k)$ such that $N(y_i) \cap V(F_{k-1}) \neq \emptyset$ and $N(y_j) \cap V(F_{k-1}) \neq \emptyset$ and i is minimum and j is maximum $(2 \leq i \leq j \leq n)$. Let $s \in N(y_i) \cap V(F_{k-1})$ and $t \in N(y_j) \cap V(F_{k-1})$ such that $|\overrightarrow{(t, s)}|$ is minimum. Let P_1 be shortest path from y_j to y_i such that $V(P_1) \subseteq \overrightarrow{(y_j, y_i)}$. Let P_2 be the shortest path from t to s such that $V(P_2) \subseteq \overrightarrow{(t, s)}$. Because there is no edge crossing between vertex in P_1 and P_2, we have an internal face $(s, P_2, t, y_j, P_1, y_i)$. Because $d(y_i), d(y_j) \leq 3$, and $N(y_i) \cap V(F_{k-1}) \neq \emptyset$, there is no edge between y_i and $\{y_j, y_{j+1}, ..., y_{i-2}\}$. Hence, $y_{i-1} \in V(P_1)$. By analog proof, $y_{j+1} \in V(P_1)$. There are at least 6 distinct vertices as $y_i, y_{i-1}, y_{j+1}, y_j, s, t$ in this internal face . It is a contradiction.
The Figure 4 illustrates for the proof above.

□

Proof. Now we will prove Theorem 9 by induction. For the base case: $k = 1$, the claim is true by applying Theorem 8 with G admitting no internal face ≥ 5. Now suppose that the claim is true with some k, we will prove it is also true with $k+1$. By induction hypothesis, G is a $(k+1)$-outerplanar. Let G' be k-outerplanar graph obtained from G by removing the vertices on the $(k+1)$-outerface. By induction hypothesis, $G' \in CWRC(1)$. By the condition (iii), $\forall u \in F_k$, there exists a vertex $v \in F_{k+1}$ such that $(u, v) \in E(G)$. Let $S_r = (r_1, r_2, ..., r_p, ...)$ be a arbitrary valid sequence of moves of the robber in G.
Let $f_i(v)$ be a function mapping some vertex in F_{i+1} to some vertex in F_i, where

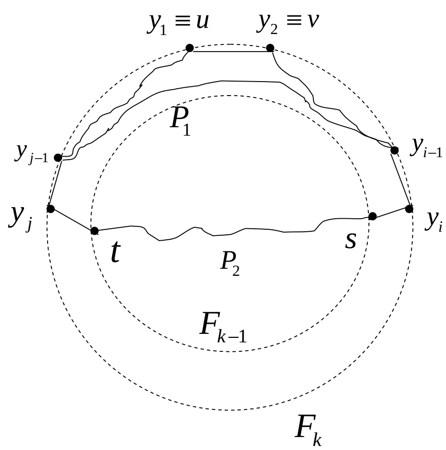

FIGURE 7. Illustration of Lemma 4's Proof

$1 \leq i \leq k$. So $\forall v \in F_{i+1}, f_i(v) = N(v) \cap V(F_i)$. Let $S'_r = (r'_1, r'_2, ..., r'_p, ...)$ be a sequence of moves of the robber in G' such that for r_1,

$$r'_1 = \begin{cases} r_1 & \text{if } r_1 \in F_l \text{ and } 1 \leq l \leq k, \\ f_k(r_1) & \text{if } r_1 \in F_{k+1} \text{ and } f_k(r_1) \neq \emptyset, \\ u \in F_k & \text{where u is defined below.} \end{cases}$$

Without loss of generality, by corollary 1, we can suppose that $V(F_{k+1})$ can be order as $(v_1, v_2, ..., v_p)$ where $r_1 \equiv v_1$. Let $v_x \in V(F_{k+1})$ such that $f_k(v_x) \neq \emptyset$ and x is minimum. So $u = f_k(v_x)$. And for $i \geq 2$,

$$r'_i = \begin{cases} r_i & \text{if } r_i \in F_l \text{ and } 1 \leq l \leq k, \\ f_k(r_i) & \text{if } r_i \in F_{k+1} \text{ and } f_k(r_i) \neq \emptyset, \\ r'_{i-1} & \text{,otherwise.} \end{cases}$$

We will prove that S'_r is valid sequence of moves of the robber in G' by showing that $dist(r'_i, r'_{i-1}) \leq 1 \forall i \in N$.
By contradiction, suppose that there exists some i such that $dist(r'_i, r'_{i-1}) \geq 2$. Because $dist(r_i, r_{i-1}) = 1 \neq dist(r'_i, r'_{i-1})$, so at least one vertex of r_i , r_{i-1} must be in $V(F_{k+1})$.
If $r_i \in V(F_{k+1})$ and $r_{i-1} \notin V(F_{k+1})$, we have $r_{i-1} \equiv r'_{i-1}$ and $r'_i = f(r_i) = r'_{i-1}$, then $dist(r'_{i-1}, r'_i) = 0$ (Contradiction).
The proof for $r_{i-1} \in V(F_{k+1})$ and $r_i \notin V(F_{k+1})$ is analog.
If $r_{i-1}, r_i \in V(F_{k+1})$, there are 3 paths going from r'_i to r'_{i-1} as $(r'_i, r_i, r_{i-1}, r_{i-2}..., r_j, r'_{i-1})$ where $r_{i-1}, r_{i-2}..., r_j \in V(F_{k+1})$,$\overrightarrow{(r'_i, r'_{i-1})}$, $\overrightarrow{(r'_{i-1}, r'_i)}$. Without loss of generality, suppose that $\overrightarrow{(r'_i, r'_{i-1})}$ is inside the area bounded by $(r'_i, r_i, r_{i-1}, r_{i-2}..., r_j, r'_{i-1})$ and $\overrightarrow{(r'_{i-1}, r'_i)}$. Because r'_i, r'_{i-1} is not adjacent, there must exist a vertex $u \in \overrightarrow{(r'_i, r'_{i-1})}$. By definition of S'_r, there is no edge between u and the vertices $r_i, r_{i-1}, r_{i-2}..., r_j$. So if there exists a vertex $v \in V(F_{k+1})$ such that $v \in N(u)$, (u, v) must cross $(r'_i, r_i, r_{i-1}, r_{i-2}..., r_j, r'_{i-1})$ or $\overrightarrow{(r'_{i-1}, r'_i)}$. It is contradictory to planar property. So

we have S'_r is valid moves of the robber.
By induction, suppose that Σ is a winning strategy for cop in G' with $S'_c = (c'_1, c'_2, ..., c'_p, ...)$, the corresponding valid sequence of moves of the cop playing Σ. We define a winning strategy σ for C with valid sequence of moves in G with $S_c = (c_1, c_2, ..., c_p, ...)$. Let $c_i = c'_i$ until the cop captures the robber in G', suppose at the step q. If $r_{q-1} \notin F_{k+1}$, then $r_{q-1} \equiv r'_{q-1}$, thus $d(c_q, r_{q-1}) = d(c'_q, r'_{q-1}) = 1$, then R is captured. If $r_{q-1} \in F_{k+1}$ and $dist(r_{q-1}, c_q) = 1$, then R is also captured at step q, otherwise, R can move to r_q at some vertex in F_{k+1}. At step $q+1$ when R is at r_q, C moves to $c_{q+1} \equiv r'_{q-1}$. We will prove that $dist(c_{q+1}, r_q) \leq 2$.
By contradiction, suppose $dist(c_{q+1}, r_q) \geq 3$, thus $dist(c_{q+1}, r_{q-1}) \geq 2$. Because $dist(r'_{q-1}, r_{q-1}) = dist(c_{q+1}, r_{q-1}) \geq 2$, then $N(r_{q-1}) \cap V(F_k) = \emptyset$, otherwise $r'_{q-1} = N(r_{q-1}) \cap V(F_k)$ then $dist(r'_{q-1}, r_{q-1}) = 1$. By Lemma 4 and $N(r_{q-1}) \cap V(F_k) = \emptyset$, there exists some vertex $u \in V(F_k) \cap N(r_q)$. Because $dist(c_{q+1}, r_q) \geq 3$, then $dist(c_{q+1}, u) \geq 2$. There must exist some vertex $y \in \overrightarrow{(c_{q+1}, u)}$ and some vertex $z \in \overrightarrow{(u, c_{q+1})}$. Let consider 3 paths from $r'_{q-1} = c_{q+1}$ to u as $\overrightarrow{(c_{q+1}, u)}$, $\overrightarrow{(u, c_{q+1})}$ and $P = (r'_{q-1} = c_{q+1}, r_j, ..., r_{q-1}, r_q)$. Without loss of generality, suppose that $\overrightarrow{(u, c_{q+1})}$ is inside the area X bounded by $\overrightarrow{(c_{q+1}, u)}$ and P. Be definition of S'_r and corollary 2, there is no edge between z and the vertices $\{r_j, ..., r_{q-1}, r_q\}$. And because z is bounded by X, therefore, $N(z) \cap V(F_{k+1}) = \emptyset$ (Contradiction).
Now we have $dist(c_{q+1}, r_q) \leq 2$. If $dist(c_{q+1}, r_q) = 1$, the robber is already captured, otherwise, $dist(c_{q+1}, r_q) = 2$, and it is the robber's turn. We consider 2 cases as follows:
i) If there exists some vertex $u \in V(F_k) \cap N(c_{q+1}) \cap N(r_q)$,then by Lemma 3, $\{u\} = V(F_k) \cap N(r_q)$. So R cannot move to u or stay at r_q, R has to move to $r_{q+1} \in V(F_{k+1}) \cap N(r_q)$. So C can move to c_{q+2} at $u \in F_k$ and maintain $dist(C, R) = 2$. By Lemma 3 and $|V(F_{k+1}) > V(F_k)|$, $x \in V(F_{k+1})$ such that $N(x) \cap V(F_k) = \emptyset$. Let the cop continues to use this strategy, the robber must move to some vertex x such that $N(x) \cap V(F_k) = \emptyset$ and $dist(C, R) = 2$. And the configuration is analog to the case ii.
ii)If $\emptyset = V(F_k) \cap N(c_{q+1}) \cap N(r_q)$. Because $dist(c_{q+1}, r_q) = 2$, $w \in N(c_{q+1}) \cap N(r_q)$, then w must be in $V(F_{k+1})$. We will prove that $N(r_q) \cap V(F_k) = \emptyset$. By contradiction, suppose $N(r_q) \cap V(F_k) = u$. Because $u \notin N(c_{q+1})$, $dist(c_{q+1}, u) \geq 2$. Let consider 3 paths from c_{q+1} to u as (c_{q+1}, w, r_q, u), $\overrightarrow{(c_{q+1}, u)}$, $\overrightarrow{(u, c_{q+1})}$. Suppose that $\overrightarrow{(c_{q+1}, u)}$ is inside the area S bounded by (c_{q+1}, w, r_q, u) and $\overrightarrow{(u, c_{q+1})}$. Let $v \in \overrightarrow{(c_{q+1}, u)}$. By Lemma 3, there is no edge between v and w. Because v is inside S, so $N(v) \cap V(F_{k+1}) = \emptyset$. It is contradictory to the condition (iii).
Now, because the robber can not stay at r_q or move to w, thus, he has to move to some vertex $y \in F_{k+1}$. By Lemma 4, there exists a vertex z in $F_k \cap N(y)$. If $z \in N(c_{q+1})$, the cop will capture the robber by moving to z, otherwise, there is a internal face of G as $\{c_{q+1}, w, r_q, y, z, P\}$ where P is $\overrightarrow{(z, c_{q+1})}$ or $\overrightarrow{(c_{q+1}, z)}$. Length of this internal face is > 5. It is a contradiction.

□

If we relax the constraints in Theorem 9 by removing the condition (v) or (vi), the claim is not true. We will consider the couter-examples as follows:

Counter example 1 *A counter-example if we relax the condition (v).*

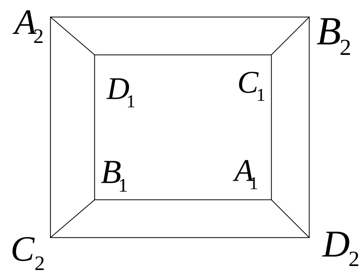

FIGURE 8. Illustration of Counter Example 1

Consider a graph G as illustrated in Figure 8. Without loss of generality, suppose that C,R occupies the vertex A_2, A_1 correspondingly. In G, there are 4 pairs of vertices $(A_1, A_2), (B_1, B_2), (C_1, C_2), (D_1, D_2)$ with distance 3. So if C remains at A_2, R remains at A_1 respectively. If C moves to some vertex in $\{B_2, C_1, D_2\}$, R can moves to the corresponding vertex as B_1, C_2, D_1. Hence, $dist(C, R)$ always ≥ 2 and R can escape C infinitively.
Counter example 2 *A counter-example if we relax the condition (v).*

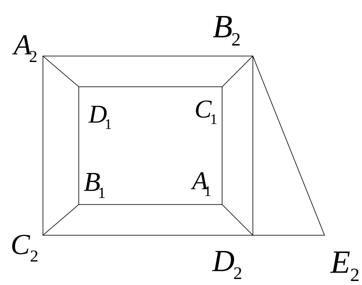

FIGURE 9. Illustration of Counter Example 2

Consider a graph G as illustrated in Figure 9. We observe that $V(F_2) = \{A_2, B_2, C_2, D_2, E_2\}$ and $V(F_1) = \{A_1, B_1, C_1, D_1\}$. So $|V(F_2)| > |V(F_1)|$. $d(B_2) = d(D_2) = 4 > 3$ does not satisfy the condition (v). We also observe that $\forall v \in \{A_2, C_2, D_2\} \cup V(F_1), dist(B_2, v) = dist(E_2, v)$. So if we consider $\{B_2, E_2\}$ as one vertex X for C, R can escape from C infinitively by applying the strategy in the counter example 1.

Theorem 10. *Let G be a 2-outerplanar and triangulated graph and F_1,F_2 be the internal and external outerplanar face respectively. If F_1 and F_2 are 2-connected, then $G \in CWRC(1)$.*

Lemma 5. *There must exist at least 2 distinct vertices u, v in F_1 such that $N(u) \cap V(F_2) \neq \emptyset$ and $N(v) \cap V(F_2) \neq \emptyset$.*

Proof. Because G is connected, then there must exist at least 1 vertex u such that $N(u) \cap V(F_2) \neq \emptyset$. By contradiction, suppose that $\nexists v \neq u, v \in V(F_1)$ such that $N(v) \cap V(F_2) \neq \emptyset$. By corollary 1, $V(F_2)$ can be ordered as $(v_1, v_2, ..., v_{n_2})$ such that $\forall 1 \leq i \leq n_2, (v_i, v_{i+1}) \in E(F_2)$ where $v_{n_2+1} = v_1$. Without loss of generality, suppose that $N(u) \cap V(F_2) = \{v_{i_1}, v_{i_2}, ..., v_{i_k}\}$ where $1 \leq v_{i_1} < v_{i_2} < ... < v_{i_k} \leq n_2$. By planar property, F_1 must be restricted in area bounded by $u, v_{i_j}, \overrightarrow{(v_{i_j}, v_{i_{j+1}})}, v_{i_{j+1}}$ where $1 \leq j \leq k$. Therefore, $V(F_1), v_{i_j}, \overrightarrow{(v_{i_j}, v_{i_{j+1}})}, v_{i_{j+1}}$ make an internal whose length greater than 3 (contradiction). This Figure 10 illustrates the Lemma 5. $\square$

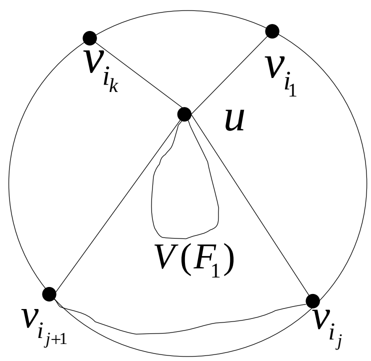

FIGURE 10. Illustration of Lemma 5

Lemma 6. *$\forall u \in V(F_1)$, then $N(u) \cap V(F_2) \neq \emptyset$*

Proof. By contradiction, suppose that $\exists u \in V(F_1)$ such that $N(u) \cap V(F_2) = \emptyset$. By corollary 1, without loss of generality, we can order $V(F_1) = \{u \equiv v_1, v_2, ..., v_k\}$ such that $(v_i, v_{i+1}) \in E(F_1)$ where $v_{k+1} = v_1$. By Lemma 5, let v_i, v_j be the vertices such that $N(v_i) \cap V(F_2) \neq \emptyset$ (resp., $N(v_j) \cap V(F_2) \neq \emptyset$) and i is miminum (resp., j is maximum).
Let y and w are two vertices such that $y \in N(v_i) \cap V(F_2)$, $w \in N(v_j) \cap V(F_2)$ and $dist(y, w)$ is minimum, and suppose the shortest path from y to w is P. By planar hypothesis, (v_i, y) and (v_j, w) do not cross each other. By contradiction hypothesis, there is no edge between some vertex in P with the vertices in $\overrightarrow{(v_i, v_j)}$. By definition of w and y, there must be no edge between v_i and v_j with any vertex in P. By applying Lemma 1 for the outerface $F1$, any edge between two arbitrary vertices in $\{u = v_1, v_2, ..., v_i\} \cup \{v_j, v_{j+1}, ..., v_k\}$ does not cross the internal face $(u, v_2, ..., v_i, y, P, w, v_j, v_{j+1}, ..., v_k, u)$. Hence this internal face includes four distinct vertices u,v_i,v_j,y, it is contradictory to the triangulated property. This Figure 11 illustrates the Lemma 6. $\square$

Lemma 7. *If the vertices of $V(F_1)$ are embedded in a circle and ordered as $(v_1, v_2, ..., v_k)$ such that $(v_i, v_{i+1}) \in E(G)$ where $v_{k+1} \equiv v_1$ and $\exists i \neq j$ such that $v_i, v_j \in N(u)$ where $u \in V(F_2)$, $\overrightarrow{[v_i, v_j]} \subseteq N(u)$ or $\overrightarrow{[v_j, v_i]} \subseteq N(u)$.*

Proof. If $j = i-1$ or $j = i+1$, it is a trivial case because $\overrightarrow{(v_{i-1}, v_i)} = \overrightarrow{(v_i, v_{i+1})} = \emptyset$. If $j \notin \{i-1, i+1\}$, we prove by contradiction, suppose that there exist 2 vertices y and w in F_1, such that $w \in \overrightarrow{(v_i, v_j)}$ and $y \in \overrightarrow{(v_j, v_i)}$ and $w, y \notin N(u)$. Consider 3

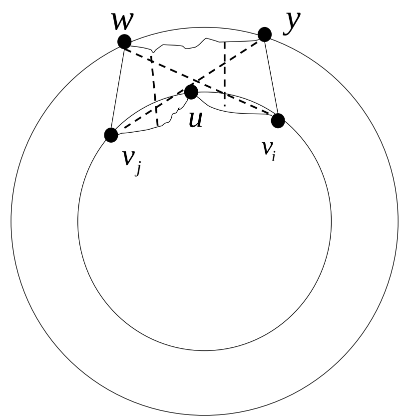

Figure 11. Illustration of Lemma 6

independent paths from v_j to v_i as $P_1 = (v_j, u, v_i)$, $P_2 = (v_j, v_{j+1}, ..., y, .., v_{i-1}, v_i)$, $P_3 = (v_j, v_{j-1}, ..., w, ..., v_{i+1}, v_i)$. Because $u \notin F_1$, there are 2 cases P_2 is inside the area bounded by (P_1, P_3) or P_3 is inside the area bounded by (P_1, P_2). Without loss of generality, suppose that P_2 is inside the area bounded by (P_1, P_3), hence, u is inside the cycle (P_1, P_3). Hence, there is no edge between u and some vertex in F_2, it is contradictory to Lemma 6. This Figure 12 illustrates the Lemma 7. □

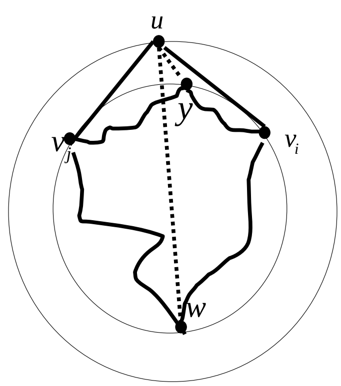

Figure 12. Illustration of Lemma 7

Proof. If R moves only in F_1, by Theorem 8, C has a strategy to capture R after a finite steps by moving only in F_1. If R has some moves in F_2, let consider the configuration such that at the first time after some finite moves, C in F_1 and R in F_2 and it is C's turn. If $dist(C, R) \leq 2$, C can capture R after one move; otherwise, without loss of generality, suppose that C (resp.,R) is at v_1 (resp., y_1) where $V(F_1)$ is ordered as $(v_1, v_2, ..., v_{n_1})$ (resp., $V(F_2)$ is ordered as $(y_1, y_2, ..., y_{n_2})$) such that $\forall 1 \leq i \leq n_1, (v_i, v_{i+1}) \in E(F_1)$ where $v_{n_1+1} = v_1$ (resp., $\forall 1 \leq i \leq n_2, (y_i, y_{i+1}) \in E(F_2)$ where $y_{n_2+1} = y_1$). Let $y_s \in N(v_2) \cap V(F_2)$ and $y_t \in N(v_{n_1}) \cap V(F_2)$. By symmetry, without loss of generality, suppose that $s \geq t$ and $R \in \overrightarrow{(y_s, y_t)}$. Let

$S = \{u | u \in \overrightarrow{[y_s, y_1]}; \emptyset \neq N(v_1) \cap N(u) \cap V(F_1)\}$. We have $y_s \in S$ and with $u \in S$, $dist(v_1, u) \leq 2$. Let $T = \{u | u \in \overrightarrow{[y_1, y_t]}; \emptyset \neq N(v_1) \cap N(u) \cap V(F_1)\}$. We have $y_t \in T$ and with $u \in T$, $dist(v_1, u) \leq 2$. Let $y_h \in S$ and $y_l \in T$ where $(1 \leq l \leq h \leq n_2)$ such that $|\overrightarrow{(y_h, y_l)}|$ is minimum. By order of $V(F_2)$, we have $y_1 \in \overrightarrow{(y_h, y_l)}$. Let v_i be some vertex such that $v_i \in N(v_1) \cap N(y_h) \cap V(F_1)$ and i is maximum. Let v_j be some vertex $v_j \in N(v_1) \cap N(y_l) \cap V(F_1)$ and j is minimum. By planar property, 2 paths (v_1, v_i, y_h) and (v_1, v_j, y_l) do not cross each other and $dist(C, R) = dist(v_1, y_1) \geq 3$, then $2 \leq i < j \leq n$.
Let $N_1 = \overrightarrow{[v_i, v_j]}$, $N_2 = \overrightarrow{[y_h, y_l]}$. Let $N = N_1 \cup N_2$. If C remains stay at v_1, R is restricted in N. We will prove that there exist $z \in N(v_1) \cap V(F_1)$ and $N' \subset N$ so that if C moves to z and stays at z, R is restricted in N' .
We will prove that $\forall v_k, i < k < j$, $(v_1, v_k) \notin E(G)$. By Lemma 6, $N(v_k) \cap N_2 \neq \emptyset$. If $(v_1, v_k) \in E$, let $y_w \in N(v_k) \cap V(F_2)$. By planar property, $y_w \in [y_h, y_l]$. It is contradictory to condition $|\overrightarrow{[y_h, y_l]}|$ minimum. Hence, because of triangulated property, 3 vertices (v_1, v_i, v_j) must be in a triangulated face, then $(v_i, v_j) \in E(G)$. Let z be v_i. Because $y_h \in N(v_i)$ and $y_l \in N_2(v_i)$, so R is still restricted in N when C moves to z. Let $N'_1 = \overrightarrow{[v_{i+1}, v_j]}$, $N'_1 = \overrightarrow{[y_{h+1}, y_l]}$ and $N' = N'_1 \cap N'_2$. Because $y_{h+1}, y_l, v_{i+1}, v_j \in N_2(z)$, then R is restricted in $N' \subset N$. So after a finite steps, C can capture R. This Figure 13 illustrates the proof of Theorem 10. $\square$

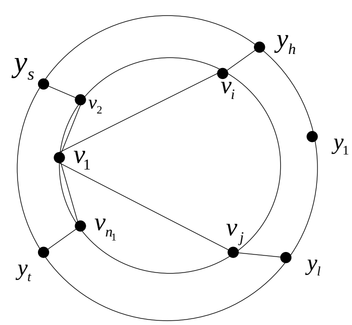

FIGURE 13. Illustration of Theorem 10

Theorem 10 is not true if G is 3-outerplanar. We will consider the following counter-example as illustrated in Figure 14 . We denote F_1, F_2, F_3 by 1^{st},2^{nd},3^{rd} outerfaces. Let consider F_3 and F_2, we have 6 pairs of vertices with distance 3 as (A_3, A_2), (B_3, B_2), (C_3, C_2), (D_3, D_2), (M_2, O_2), (N_2, P_2).
Case 1: If C chooses some vertex $u \in V(F_3) \cup V(F_2)$, R chooses some vertex v respectively such that (u, v) is one of 6 pairs.
Case 1.1: If C only moves within $V(F_3) \cup V(F_2)$. By symmetry, suppose that C stays at a vertex in $\{A_3, C_2, M_2\}$, then R stays at the corresponding vertex $\{A_2, C_3, O_2\}$.
(a): If C stays at A_3, R stays at A_2 respectively. C can move to some vertex as $A_3, B_3, D_3, C_2, M_2, P_2$, then R move to the corresponding vertex as $A_2, B_2, D_2, C_3, O_2, N_2$. So $dist(C, R) = 2$ after the cop's move and $dist(C, R)$ remains 3 after the robber's move.

(b): If C stays at C_2, R stays at C_3 respectively. C can move to some vertex as $C_2, M_2, D_2, B_2, P_2, A_3$, then R moves to the corresponding vertex as $C_3, O_2, D_3, B_3, N_2, O_2$. So $dist(C, R) = 2$ after the cop's move and $dist(C, R)$ remains 3 after the robber's move.
(c): If C stays at M_2, R stays at O_2 respectively. C can move to some vertex as M_2, C_2, D_2. Then R moves to the corresponding vertex as O_2, C_3, D_3. So $dist(C, R) = 2$ after the cop's move and $dist(C, R)$ remains 3 after the robber's move.
Hence, if C moves within $V(F_3) \cup V(F_2)$, R has a strategy to move such that $dist(C, R)$ always ≥ 2.
Case 1.2: If C moves to $V(F_1)$. So at the previous turn of C, C is at some vertex in $V(F_2)$, without loss of generality, suppose that C is at A_2. It implies that R is at A_3 at this step. C can only move to some vertex in $\{E_1, F_1, G_1\}$. So $dist(C, R) \geq 3$ and the configuration of C and R changes to case 2.
Case 2: If C chooses some vertex $u \in V(F_1)$, R chooses some vertex $v \in V(F_3)$ such that $dist(u, v) \geq 3$. We remark that each vertex in
$V(F_1) = \{A_1, B_1, C_1, D_1, E_1, F_1, G_1, H_1\}$ having at most 2 vertices in $V(F_3)$ such that distance ≤ 3 (*).
Case 2.1: If C moves only within $V(F_1)$. Because $R \in V(F_3)$, if $dist(C, R) \geq 3$, R remains at his position, otherwise, if $dist(C, R) = 2$, by (*) and each vertex in $V(F_3)$ having 2 adjacent vertices in $V(F_3)$, R can move to another vertex in $V(F_3)$ such that $dist(C, R) \geq 3$.
Case 2.2: If at some step, C moves to some vertex in $V(F_2)$. We remark that this vertex must be in $\{A_2, B_2, C_2, D_2\}$ and $dist(C, R) \geq 2$. If $dist(C, R) \geq 3$, C, R must occupy two vertices in $\{(A_3, A_2), (B_3, B_2), (C_3, C_2), (D_3, D_2)\}$ then R remains at his position at this step. If $dist(C, R) = 2$, without loss of generality, suppose that C is at A_2. So R must be at B_3 or D_3, then R moves to A_3 at this step. Hence, the configuration of C and R changes to Case 1.

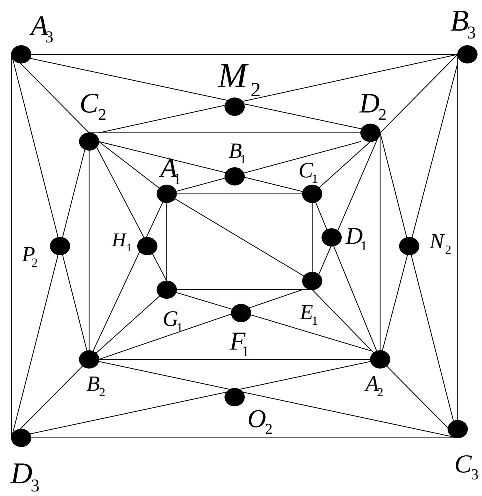

FIGURE 14. Counter Example of 3-outerplanar and triangulated graph

By analysis of case 1 and case 2, it shows that R can escape from C infinitively. $\square$

4. Faster robber

In this variant, the robber has a speed s and a cop has a speed $s' \leq s$. A cop can capture the robber if the cop can move to the vertex occupied by the robber.

4.1. **Faster robber in G^k.**

Theorem 11. *Let $G = (V, E)$ be a graph and k be an positive integer. Let $F^k = \{(u, v) \in V \times V | d(u, v) = k\}$ and $G^k = (V, E \cup F^k)$. Then $G \in CWFR(k, k)$ iff $G^k \in CW$.*

Proof. If $k = 1$, it is trivial. If $k \geq 2$, by Theorem 2, $G \in CWFR(k, k)$ iff V can be ordered as $\{v_1, v_2, ..., v_n\}$ such that $\forall i, \exists j > i$ such that $N_k(v_i, G \setminus v_j) \cap X_i \subseteq N_k(v_j)$ in G where $X_i = \{v_i, v_{i+1}, ..., v_n\}$ (*).
We denote $N^k[u] = N[u]$ in G^k and $N^k(u) = N(u)$ in G^k. By Theorem 1, $G^k \in CW$ iff V can be ordered as $\{v_1, v_2, ..., v_n\}$ such that $\forall i, \exists j > i$ such that $N^k[v_i] \cap X_i \subseteq N^k[v_j]$ (**).
Now we need to prove that if the order of vertices in V satisfies (*) , it also satisfies (**) and vice verse.
$\Rightarrow$: Let $V = (v_1, ..., v_n)$ satisfy (*). We need to prove $\forall 1 \leq i < n, \exists j > i$ such that $N^k[v_i] \cap X_i \subseteq N^k[v_j]$. Let $y \in X_{i+1}$ be the vertex satisfying $N_k(v_i, G \setminus y) \cap X_i \subseteq N_k(y)$. We will prove that we can take $v_j = y$. Consider some vertex $z \in N^k[v_i] \cap X_i$, we need to prove that $z \in N^k[y]$. By definition, $d(v_i, z) \leq k$ in G. There are 3 cases:
(i) If $z = y$, $z \in N^k[y]$.
(ii) If $z \neq y$ and y is not in a shortest path from v_i to z in G, then $z \in N_k(v_i, G \setminus y) \cap X_i \subseteq N_k(y)$. Hence, $z \in N^k[y]$.
(iii) If $z \neq y$ and y is in shortest paths from v_i to z in G, then, $d(y, z) < k$ in G. Hence, $z \in N^k[y]$.
So, $\exists v_j = y$ such that $j > i$ and $N^k[v_i] \cap X_i \subseteq N^k[v_j]$.
$\Leftarrow$: Let $V = (v_1, ..., v_n)$ satisfy (**). We need to prove $\forall 1 \leq i < n, \exists j > i$ such that $N_k(v_i, G \setminus v_j) \cap X_i \subseteq N_k(v_j)$. Let $y \in X_{i+1}$ be the vertex sastifying $N^k[v_i] \cap X_i \subseteq N^k(v_j)$. We will prove that we can take $v_j = y$. Consider some vertex $z \in N_k(v_i, G \setminus v_j)$, we need to prove that $z \in N_k(v_j)$ in G. Because $d(v_i, z) \leq k$ in G and $z \in X_i$, therefore, $z \in N^k[v_i] \cap X_i \subseteq N^k(v_j)$. Because $z \neq v_j$, $z \in N_k(v_j)$. So $\exists j > i$ such that $N_k(v_i, G \setminus v_j) \cap X_i \subseteq N_k(v_j)$. □

Conjecture 2. *Let $G = (V, E)$ be a graph. Let $F = \{(u, v) | d(u, v) = 2, N(u) \subseteq N(v)\}$ and $G' = (V, E \cup F)$. The question is if $G' \in CW(1)$, then $G \in CWRC(1)$.*

4.2. **Faster Robber in Square-Grid.**

This section presents the theoretical result of "Game of Cops and Faster Robber in Square-Grid" which the speed of cops is 1 and the speed of robber is 2. This version of games is investigated by Fomin et al. [9] with the proof that the sufficient number of cops to capture the robber in the $n \times n$ square grid is $\Omega(\sqrt{logn})$. In this section, we will prove that the upper bound of the cop number is $\lceil n - 1/4 \rceil + 4$.

Theorem 12. *Let n be an integer greater than 4 and G be a $n \times n$ square grid, then $cn(G) \leq \lceil (n - 1)/4 \rceil + 4$.*

Let denote i^{th} cop by C^i. For a square gird $m \times n$, let $row(i)$(resp.,$col(j)$) be the set of vertices in the i^{th} row (resp., the j^{th} column). The vertex v in $row(i) \cap col(j)$ has $abs(v) = j$, $ord(v) = i$. Before giving the proof of this theorem, we consider the following lemmas:

Lemma 8. *Let G(m,n) be a square grid $m \times n$ $(m, n \geq 1)$. If h cops (with speed = 1) are sufficient to capture the robber (with speed $s \in N$ and $s \geq 2$), then h cops are sufficient to capture the robber in a square grid G'(m',n') $(m \geq m' \geq 1$; $n \geq n' \geq 1)$.*

Proof. We will prove the lemma is true for a square grid $G'(m, n-1)$. By induction and symmetry, the claim is true for grid $G'(m', n')$ where $m', n' \in N$, $m \geq m' \geq 1$, $n \geq n' \geq 1$. We have $G' = G \setminus col(n-1)$. Let $S_r = (r_1, r_2, ..., r_p, ...)$ be the valid sequence of moves of the robber in G'. Obviously, S_r also is the valid sequence of moves of the robber in G. Suppose that the cops play the winning strategy σ with the sequence of moves of C^k in G as $S_{c^k} = \{c_1^k, c_2^k, ..., c_p^k, ...\}$ where $1 \leq k \leq h$. In G', the cops play the strategy Σ with the valid retract sequence of moves as $S'_{c^k} = (c_1^{k\prime}, c_2^{k\prime}, ..., c_p^{k\prime}, ...)$ where $1 \leq k \leq h$ where

$$c_i^{k\prime} = \begin{cases} c_i^k & \text{if } abs(c_i^{k\prime}) < n-1 \\ (n-2, ord(c_i^k)) & \text{,otherwise.} \end{cases}$$

We will prove that S'_{c^k} is valid sequence of moves by showing that $dist(c_i^{k\prime}, {c_{i+1}^k}') \leq 1, \forall i \in N$. For convenience, we will write c_i, c'_i instead of $c_i^k, c_i^{k\prime}$.
If $c'_i \equiv c_i, c'_{i+1} \equiv c_{i+1}$, it is trivial.
If $c'_i \neq c_i, c'_{i+1} \neq c_{i+1}$, then $c'_i = (n-2, ord(c_i))$ and $c'_{i+1} = (n-2, ord(c_{i+1}))$. Then $dist(c'_i, c'_{i+1}) = |ord(c_{i+1}) - ord(c_i)| \leq dist(c_i, c_{i+1}) \leq 1$.
If $c'_i \neq c_i, c'_{i+1} \equiv c_{i+1}$, then $c'_i = (n-2, ord(c_i)), c_i = (n-1, ord(c_i))$ and $abs(c_{i+1}) \leq n-2$. Because $dist(c_i, c_{i+1}) = |abs(c_{i+1}) - abs(c_i)| + |ord(c_{i+1}) - ord(c_i)| \leq 1$ and $|abs(c_{i+1}) - abs(c_i)| = |(n-2) - (n-1)| = 1$, then $ord(c_{i+1}) = ord(c_i)$. Therefore $c'_i \equiv c'_{i+1}$, then $dist(c'_i, c'_{i+1}) = 0 < 1$.
If $c'_i \equiv c_i, c'_{i+1} \neq c_{i+1}$, the prove is analog. Because σ is the winning strategy, suppose the robber is captured by the cop C^k at some step q in G. Because $abs(r_q) = abs(c_1^i) < n-1$, so $c_q^{i\prime} \equiv c_q^i$. Hence, the robber also is captured by the cop C^k at the step q in G'. □

Let $\lceil (n-1)/4 \rceil = k$. Hence, $n \in \{4k-2, 4k-1, 4k, 4k+1\}$. We only need to prove the theorem with $n = 4k+1$, then by Lemma 8, it is true for the other cases. Before going in theorem's proof, we consider Lemma 9.

Lemma 9. *Let G be a square grid $m \times (4k+1)$ where $m \geq 4$, $k \geq 1$. In the rows l and $l+1$, $(k+1)$ cops are positioned as follows:*
C_{2i} in $(8i, l)$ for $0 \leq i \leq \lfloor k/2 \rfloor$
C_{2i+1} in $(8i+4, l+1)$ for $0 \leq i \leq \lfloor (k-1)/2 \rfloor - 1$
If the robber is at some arbitrary vertex of row $l+2$ or $l+3$ and never moves to row $l+4$ and cops moving first, the cops will have a strategy to avoid the robber move to the row l.

Proof. The initial configuration (the configuration O) is shown in Figure 15. The circles represent the cops while the square represents the robber. In the next figures, the circles always represents the cops.
By definition, there exists a number i such that $abs(C_i) \leq abs(R) \leq abs(C_{i+1})$.
Case 1: $0 < i < k-1$. By symmetry, without loss of generality , we assume that i is even. Hence, $2 \leq i \leq k-2$. The first move of the cops depends on $abs(R)$, which leads 3 typical configurations of the cops and the robber. For convenience, we assume that the cops will not capture the robber at the first move.

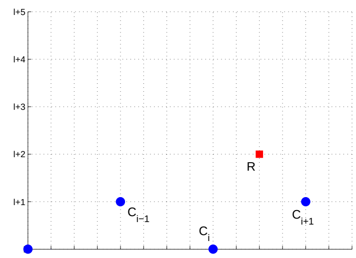

FIGURE 15. The initial configuration O

(i) If $abs(C_i) = abs(R)$, the robber can occupy the square points as illustrated in Figure 16. The cops C_i remain at his position while the cops C_j shift to the right if $0 \leq j < i$ and shift to the left if $k \geq j > i$. Hence, the configuration of the cops and the robber changes to the configuration A as illustrated in Figure 16 and this is the robber's turn.

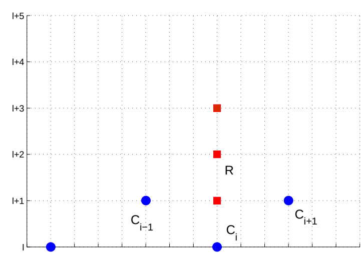

FIGURE 16. The configuration A

(ii) If $abs(C_i) < abs(R) < abs(C_{i+1})$, the robber can occupy the square points as illustrated in Figure 17. The cops C_j will shift to the right if $0 \leq j \leq i$ and to the left if $i + 1 \leq j \leq k$. Hence, the configuration of the cops and the robber changes to the configuration $B1$ as illustrated in Figure 17 and this is the robber's turn.

By symmetry, we consider the configuration $B2$ as illustrated in Figure 18 :

(iii) If $abs(C_{i+1}) = abs(R)$, the robber can occupy the square points as illustrated in Figure 19. The cop C_i remain at his position while the cops C_j shift to the right if $0 \leq j < i$ and to the left if $k \geq j > i$. Hence, the configuration of the cops and the robber changes to the configuration C as illustrated in Figure 19 and this is the robber's turn.

After the first move of the cops, we consider the robber's move.

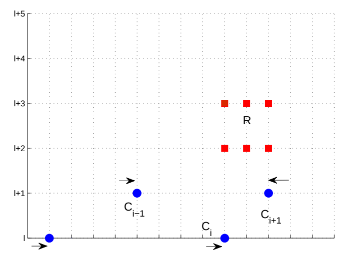

FIGURE 17. The configuration B1

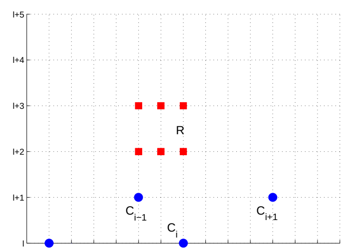

FIGURE 18. The configuration B2

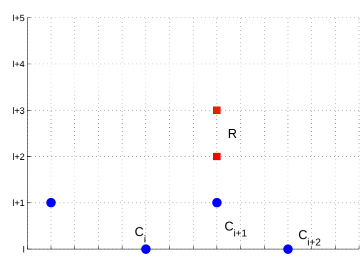

FIGURE 19. The configuration C

Configuration A
In the configuration A, all possible positions for the robber's move are the square or triangle points as illustrated in Figure 20.
Let consider 5 cases:

A.i. If the robber only moves within the square points, the cops remain at their positions. Hence, at the robber's turn, the configuration of the cops and the robber is still the configuration A.

A.ii. If the robber moves to the $\triangleleft$ points, the cop C_i shifts 1-vertex to the left while the other cops remain at their positions. Hence, the configuration of the cops

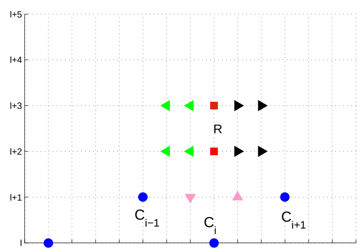

FIGURE 20. All possible positions for the robber's move from the configuration A

and the robber changes to the configuration $B2$.

A.iii. If the robber moves to the $\triangleright$ points, the cop C_i shifts 1-vertex to the right while the other cops remain at their positions. Hence, the configuration of the cops and the robber changes to the configuration $B1$.

A.iv. If the robber moves to the $\triangledown$ point as $(abs(C_i)-1, l+1)$, then the cops C_j with $j \geq i$ shift 1-vertex to the left while the other cops remain at their positions. Hence, the configuration of the cops and the robber changes to the new configuration $D1$ as illustrated in Figure 21

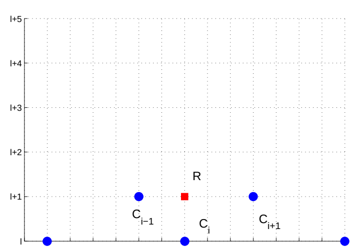

FIGURE 21. The configuration D1

We remark that in the $D1$ configuration, the cop C_j is shifted 1-vertex to the right if $0 \leq j < i$ and 2-vertex to the left if $i+1 \leq j \leq k$.

A.v. If the robber moves to the $\triangle$ point as $(abs(C_i)+1, l+1)$, then the cops C_j with $j \leq i$ shift 1-vertex to the right while the other cops remain at their positions. Hence, the configuration of the cops and the robber changes to the new configuration $D2$ as illustrated in Figure 22
We remark that in the $D2$ configuration, C_j is shifted 2-vertex to the right if $0 \leq j < i$ and 1-vertex to the left if $i+1 \leq j \leq k$.

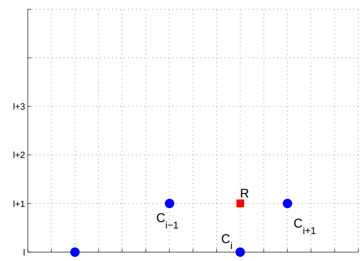

FIGURE 22. The configuration D2

Configuration D1

Let consider the configuration $D1$. It is the robber's turn and all possible positions for the robber's move are the triangle points as illustrated in Figure 23

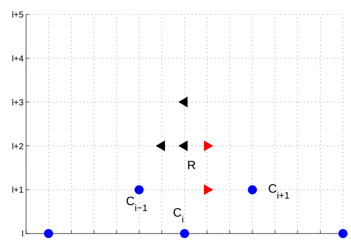

FIGURE 23. All possible positions for the robber's move in the configuration D1

D1.i. If the robber moves to the $\triangleleft$ points, the cops C_j with $j > i$ shift 1-vertex to the right while the other cops remain at their positions. The configuration of the cops and the robber changes to the configuration $B2$.

D1.ii. If the robber moves to the $\triangleright$ point, the cops C_j with $j \geq i$ shift 1-vertex to the right while the other cops remain at their positions. The configuration of the cops and the robber changes to the configuration A.

Configuration D2

Let consider the configuration of the cops and robber be $D2$. It is the robber's turn and all possible positions for the robber's move are the triangle points as illustrated in Figure 24

D2.i. If the robber moves to the $\triangleleft$ points, the cops C_j with $j \leq i$ shift 1-vertex to the right while the other cops remain at their positions. The configuration of the cops and the robber changes to the configuration A.

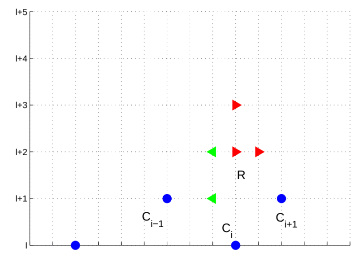

FIGURE 24. All possible positions for the robber's move in the configuration D2

D2.ii. If the robber moves to the $\triangleright$ points, the cops C_j with $j < i$ shift 1-vertex to the right while the other cops remain at their positions. The configuration of the cops and the robber changes to the configuration $B1$.

Configuration C.
Let consider the configuration C. It is the robber's turn and all possible positions for the robber's move are the square and triangle points as illustrated in Figure 25

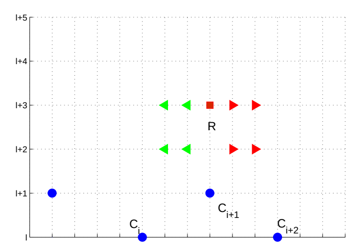

FIGURE 25. All possible positions for the robber's move in the configuration C

C.i. If the robber moves to the square point, all the cops remain at their positions, hence the configuration of the cops and the robber still is the configuration C.

C.ii. If the robber moves to the $\triangleleft$ points, all the cops remain at their positions except the cop C_{i+1} shifting 1-vertex to the left. Then the configuration of the cops and the robber changes to the configuration $B1$.

C.iii. If the robber moves to the $\triangleright$ points, all the cops remain at their positions except the cop C_{i+1} shifting 1-vertex to the right. Then the configuration of the cops and the robber changes to B2 (by considering C_{i+2} as C_i) if $i + 2 < k$. If $i = k - 2$ (k is even), the configuration of the cops and the robber changes to $B2 - border$ which we will discuss later.

Configuration B1.

Let consider the configuration $B1$. It is the robber's turn and all possible positions for the robber's move are the square and triangle points as illustrated in Figure 26

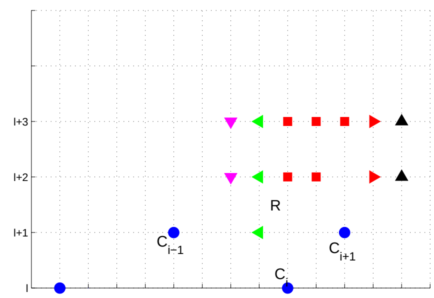

FIGURE 26. All possible positions for the robber's move in the configuration B1

B1.i. If the robber moves to the square points, all the cops remain at their positions. Hence, the configuration of the cops and the robber still is the configuration $B1$.

B1.ii. If the robber moves to the $\triangleleft$ points, all the cops remain at their positions except the cop C_i shifting 1-vertex to the left. Then the configuration of the cops and the robber changes to the configuration A.

B1.iii. If the robber moves to the $\triangleright$ points, all the cops remain at their positions except the cop C_{i+1} shifting 1-vertex to the right. Then the configuration of the cops and the robber changes to the configuration C.

B1.iv. If the robber move to the $\triangledown$ points, all the cops shift 1-vertex to the right except C_i shifting 1-vertex to the left. The configuration of the cops and the robber change to the configuration $B2'$ as illustrated in Figure 27.

We remark that in the configuration $B2'$, the cops C_j with $0 \leq j < i$ are shifted 2-vertex to the right while the cops C_j with $i \leq j \leq k$ remain at the same positions in comparison with their positions in the initial configuration O. It means that the configuration $B2'$ is equivalent to the configuration $B2$ shifting 1-vertex to the right.

B1.v. If the robber move to the $\triangle$ points, all the cops shift 1-vertex to the left except C_i shifting 1-vertex to the right. If $i < k-2$, the configuration of the cops and the robber changes to the configuration $B2''$ as illustrated in Figure 28, otherwise, it is the configuration $B2'' - border$ which we will discuss later .

We remark that in the configuration $B2''$, the cops C_j with $i+1 < j \leq k$ are shifted 2-vertex to the left while the cops C_j with $0 \leq j \leq i+1$ remain at their positions

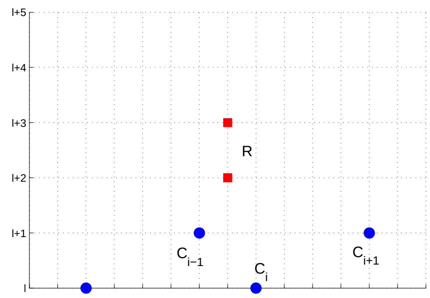

FIGURE 27. The typical configuration $B2'$

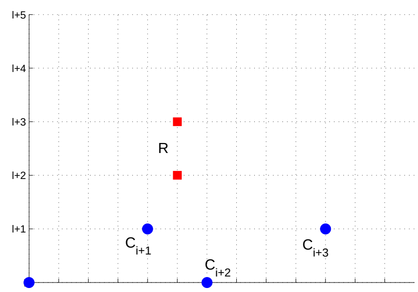

FIGURE 28. The configuration $B2''$

in comparison with their positions in the initial configuration O. It means that the configuration $B2''$ is equivalent to the configuration $B2$ shifting 1-vertex to the left.

Configuration B2'.
Let consider the configuration $B2'$. It is the robber's turn and all the possible positions for robber are the triangle points as illustrated in Figure 29

B2'.i. If the robber moves to the $\lhd$ points, all the cops shift 1-vertex to the left. The configuration of the cops and the robber changes to the configuration B2.

B2'.ii. If the robber moves to the $\rhd$ points, all the cops shift 1-vertex to the left except C_i remaining at his position. The configuration of the cops and the robber changes to the configuration A.

B2'.iii. If the robber moves to the $\triangle$ points, all the cops shift 1-vertex to the left except C_i shifting 1-vertex to the right. The configuration of the cops and the robber changes to the configuration B1.

Configuration B2". Let consider the configuration $B2''$. All possible positions for the robber are the triangle points as illustrated in Figure 30

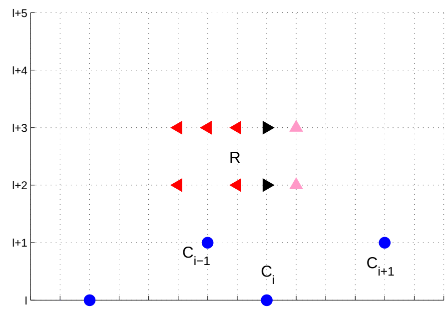

FIGURE 29. All possible positions for robber's move from the configuration B2'

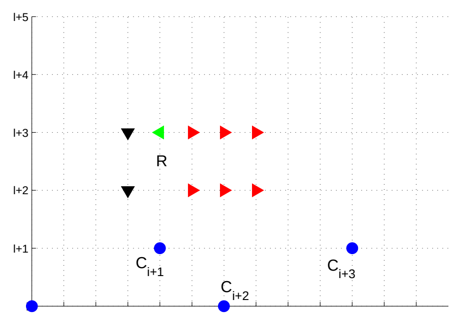

FIGURE 30. All possible positions for robber's move from the configuration $B2''$

B2".i. If the robber moves to the $\triangleright$ points, all the cops shift 1-vertex to the right. The configuration of the cops and the robber changes to the configuration B2.

B2".ii. If the robber moves to the $\triangleleft$ points, all the cops shift 1-vertex to the right except C_{i+1} remaining at his position. The configuration of the cops and the robber changes to the configuration C.

B2".iii. If the robber moves to the $\triangledown$ points, all the cops shift 1-vertex to the right except C_{i+1} shifting 1-veretx to the left. The configuration of the cops and the robber changes to the configuration B1.
We remark that the configuration of the cops and the robber always changes to one of the configurations A, $B1$, $B2$ or C from the configurations $B2'$ and $B2''$. By symmetry, analysis of the configuration $B2$ is analog to the configuration $B1$.

Case 2: Now we consider case 2 where $i = k - 1$. There are 2 sub-cases:
Case 2.1: k is even.
We only need to consider the position of the robber such that $ord(C_{k-1}) < ord(R) \leq ord(C_k)$ because if $ord(R) = ord(C_{k-1})$, it is the case 1 with $i = k - 2$.
If $ord(C_{k-1}) < ord(R) < ord(C_k)$, all the cops shift 1-vertex to the right except the cop C_k shifts 1-vertex to the left. Then the configuration of the cops and the robber changes to the configuration $B2 - border$ as illustrated in Figure 31 where the square points represent all the possible positions of the robber.

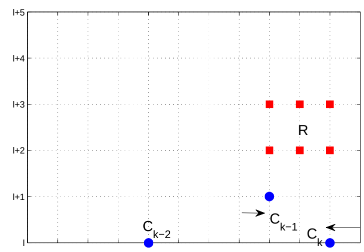

FIGURE 31. The configuration $B2 - border$

If $ord(R) = ord(C_k)$, all the cops shift 1-vertex to the right except the cop C_k remaining at his position. Then the configuration of the cops and the robber changes to the configuration $A - border$ as illustrated in Figure 32 where the square points represent all the possible positions of the robber.

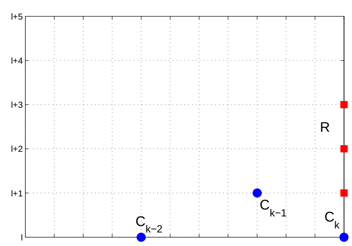

FIGURE 32. The configuration $A - border$

By analog to Case 1, we have the configuration $D1 - border$ as illustrated in Figure 33 where the square point represent all the possible positions of the robber.
By analog proof as Case 1, from the configuration $B2 - border$, the cops have the strategy to move after the robber's move so that the configuration of cops and the robber can change to one of the configurations $B2 - border$, $A - border$, C and the new $B1'$ as illustrated in Figure 34. In this particular case, $i = k - 2$.
By the analog analysis, from the configuration $B1'$, the configuration of the cops and the robber can change to one of the configurations $C, B1, B2$ or $B2 - border$.
Also by the analog analysis, from the configuration $A - border$, the configuration of the cops and the robber can change to one of the configurations $A - border$, $B2 - border$ and $D1 - border$.

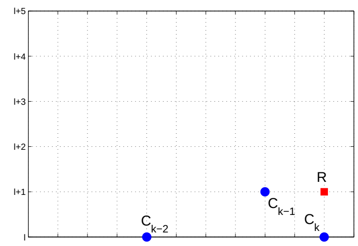

FIGURE 33. The configuration $D1 - border$

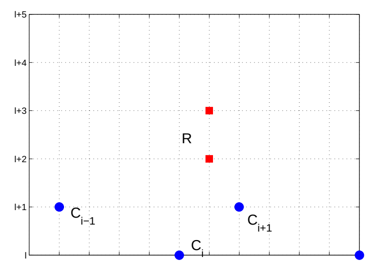

FIGURE 34. The configuration $B1'$

Also by the analog analysis, from the configuration $D1 - border$, the configuration of the cops and the robber can change to one of the configurations $B2 - border$ or $A - border$.

We also consider the configuration $B2'' - border$ as illustrated in Figure 35 where the square points represent all the possible positions of the robber.

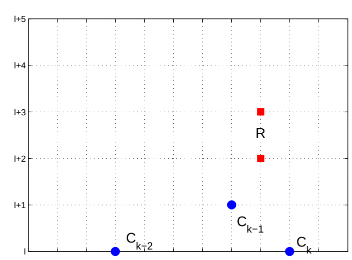

FIGURE 35. The configuration $B2'' - border$

By analog proof, we have that the configuration of cops and robber can changes to $C, B1, B2 - border$.

Case 2.2: k is odd. We only need to consider position of the robber such that $ord(C_{k-1}) < ord(R) \leq ord(C_k)$ because if $ord(R) = ord(C_{k-1})$, it is the case 1 with $i = k - 2$.

If $ord(C_{k-1}) < ord(R) < ord(C_k)$, all the cops shift 1-vertex to the right except

C_k shift 1-vertex to the left. Then the configuration of cops and robber changes to the configuration $B1 - border$ as illustrated in Figure 36 where the square points represent all the possible positions of the robber.

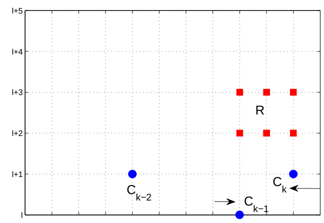

FIGURE 36. The configuration $B1 - border$

If $ord(R) = ord(C_k)$, all the cops shift 1-vertex to the right except the cop C_k remaining at his position. Then the configuration of the cops and the robber changes to the configuration $C - border$ as illustrated in Figure 37 where the square points represent the possible positions of the robber.

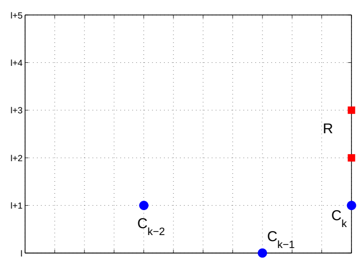

FIGURE 37. The configuration $C - border$

The analysis of the typical configuration $B1 - border$ and $C - border$ is analog as $B1$ and C.

Case 3: If $i = 0$, by symmetry, the analysis of $i = 0$ is analog as Case 2.1 where k even.

Let S be the set of the configurations O, A,$B1$,$B2$,C,$D1$,$D2$,$A-border$,$B1-border$, $B2-border$,$C-border$,$B1'$,$B2'$,$B2''$,$B2''-border$. We remark that after the robber's move, the cops always have the corresponding move such that the configuration of cops and robber belongs to S. Because there is no configuration in S such that the robber is on row l; hence, the cops have a strategy to avoid the robber move to row l by applying the corresponding move for each configuration and the robber's move. $\square$

Remark 2. *If the configuration of the cops and the robber belongs to* $X = \{A, B1, B2, C, A-border, B1-border, B2-border, C-border\}$*, the cops can change to their positions to the same as those in the configuration O after 1 move.*

Remark 3. *If the configuration of the cops and the robber belongs to* $Y = \{B1', B2', B2'', B2''-border\}$*, wherever the robber moves, the cops can move* *correspondingly and the configuration of the cops and the robber changes to a configuration in X.*

Remark 4. *If the robber stays on row $l+2$ or $l+3$, at the robber's turn, the configuration of the cops and the robber always belongs to $X \cup Y$.*

Proof. We will prove that with $k+4$ cops in square grid $n \times n$ where $n = 4k+1$, the cops have a strategy to capture the robber after finite steps. The $k+1$ cops called guard cops are put on rows 0 and 1 as follows: The cop C_i is put at $(4i, 0)$ if i is even and at $(4i, 1)$ if i is odd. In addition, 3 cops called search cops are put at the vertices $(0, 2)$, $(0, 3)$ and $(0, 4)$. There search cops will move from left (colum 0) to right (column n) and vice verse in case 3, 4 as below. Now, we consider 4 cases as follows:

Case 1: The robber is on row $l \leq 5$. All the cops will move up vertically until $ord(R) - ord(C_0) \leq 4$. In this situation, this configuration of the cops and the robber is analog to the configuration in cases 3 or 4.

Case 2: If at the initial configuration, the robber is on row 0 or 1. We will prove that there exists a step that the robber moves to row 3 while all the guard cops remain at their positions. By contradiction, suppose that the robber's movement is restricted in the rows 0, 1, 2. First, the three search cops (on the vertices (0,2), (0,3), (0,4)) move to (0,0), (0,1), (0,2). Then they move from the left (column 0) to the right (column n) to search three rows 0, 1, 2. Because the robber cannot pass through three search cops, so after one move of the cops, the search cops get closer 1-vertex to the right border. So the robber is captured after finite steps if he does not move to row 3. Hence, there must exist some step that the robber moves to row 3 and the next turn is the cops' turn. Hence, the configuration of the $k+1$ guard cops and the robber changes to Case 4. By maintaining the configuration of the guard cops and the robber belonging to case 1 or 3 or 4, there search cops can move back to (0,2), (0,3), (0,4).

Case 3: The robber is on row 4.

(a) If the robber's movement is restricted in row 4, $k+1$ guard cops remain at their initial positions.

(b) If at one step, the robber moves to line 2 or 3, the configuration of the guard cops and the robber changes to case 4.

(c) If at one step, the robber moves up to row 5 or 6, the configuration of the guard cops and the robber changes to Case 1. Then $k+1$ guard cops can move up at least 1 vertex until $ord(R) - ord(C_0) \leq 4$.

We remark that three search cops move on rows 2, 3, 4 from left to right; therefore, the robber cannot stays on row 4 infinitively. So the configuration of the guard cops and robber must change to Case 1 or Case 4 at some step.

Case 4: If the robber is on row 2 or 3.
We will prove that in Case 4, $k+1$ guard cops use the modified strategy from

Lemma 9 to avoid the robber move to row 0. Furthermore, after finite steps, cooperating with three searching cops, $k+1$ guard cops can move from rows 0, 1 to rows 1, 2 and avoid the robber move to row 1.
We define the modified strategy as:
(4-a) If the robber moves within the rows 2 and 3, the guarding cops use the strategy as Lemma 9.
(4-b) If the robber moves up 1-vertex from row 3 to row 4, or move up 2-vertex from row 2 to row 4, all $k+1$ cops move up 1-vertex from rows 0, 1 to rows 1, 2. So the type of configuration in $X \cup Y$ remains unchanged but the row l changes from 0 to 1. Applying the strategy as Lemma 9, the $k+1$ guard cops can avoid the robber move to row 1.
(4-c) If the robber moves up 2-vertex from row 3 to row 5, all the $k+1$ guard cops move up 1-vertex from rows 0, 1 to rows 1, 2. If in the next move of the robber:

(4-c-i) The robber moves to row 3 or 4, by considering the robber at $(abs(R), 5)$ as $(abs(R), 4)$, all the $k+1$ guard cops move as Lemma 9 and avoid the robber move to row 1.

(4-c-ii) If the robber moves to row $l \geq 5$ and the positions of the guard cops as the same as those in one of the configurations in X, the guard cops will change their positions to those in the configuration O. The configuration of the guard cops and the robber changes to case 1 (if $l > 5$)or case 2(if $l = 5$).

(4-c-iii) If the robber moves to row $l \geq 5$ and the positions of the guard cops is the same as those in one of the configurations in Y, the guard cops will change to the positions as those in the configuration X by considering the robber at $(abs(R), l)$ as $(abs(R), 4)$ and applying the strategy as Lemma 9. So the configuration of the guard cops and the robber changes to 4-c-ii.
We remark that if the robber moves up vertically 1-vertex or 2-vertex, the guard cops can also move up 1-vertex and avoid the robber move to row 1.

(4-d) Now let consider the case 4-d where the robber does not move up vertically 1-vertex or 2-vertex.
Because 3 search cops move from left (column 0) to right (column n), then there exists one step that the robber has to move to row $l \geq 5$. Because the robber cannot move up vertically 2-vertex, in order to move to row $l \geq 5$, he has to move to from row 3 to row 4 then from row 4 to row 5.
After the robber moves from row 3 to row 4, if the positions of the guard cops is the same as those in one of the configurations in X, the guard cops will change their positions to those in the initial configuration O. If the positions of guard cops are the same as those in one of the configurations in Y, the guard cops will change their positions to those in one of the configurations in X by applying the strategy in Lemma 9 and considering the position of the robber at $(abs(R), 4)$ as $(abs(R), 3)$. So when the robber is on row 4, the positions of the guard cops are the same as those in one of configurations of $X \cup \{O\}$.
After the robber moves from row 4 to row 5, if the positions of the guard cops are the same as those in the configuration O, all the guard cops will move up at least 1-vertex as Case 1. If the positions of guard cops are the same as those in one of

configurations in X, all the guard cops will change the positions the same as those in the configuration O.
Now the robber stays on row 5 and the $k+1$ guard cops stay at the positions as those in the initial configuration O.
If at the next move, the robber moves to some row $l \geq 5$, all the guard cops can move up 1-vertex as Case 1.
If at the next move, the robber moves to row 3 or 4, so the robber is still able to be captured by there search cops when they keep moving from the left to the right. Because after finite steps, there search cops finish searching from the left to the right, the robber has to escape by moving to row 5 and the next move he does not go back to row 3 and 4. Hence, obviously, the guard cops can move up 1-vertex and avoid the robber move to row 1.
We remark that after the guard cops move to row 1, if the robber on row 3,4, the configuration of the guard cops and the robber belongs S. If the robber on row $l \geq 5$, the configuration of the guard cops and the robber changes to Case 1.
So the guard cops can repeat the strategy to move up and after finite steps, the guard cops can move to rows $n-3$, $n-2$ and avoid the robber move to row $n-3$. Finally, the search cops search the rows $n-2, n-1, n$ and capture the robber after finite steps. $\square$

5. Conclusion

In this report, we present the novel results in the radius capture and faster robber variants with particular graphs such as square grid, k-chordal, outerplanar, k-outerplanar, triangulated.
In the radius capture variant, we proved that if G is k-chordal graph, $G \in CWRC(k-4)$ and $cn(G) \leq k-4$ with radius capture 1. We also proved that outerplanar graph $G \in CWRC(1)$ if and only if G admits no internal face whose length > 5. In addition, we found the sufficient conditions for k-graph and triangulated graphs to be in $CWRC(1)$. For future work, we will try to characterize the planar graphs in $CWRC(1)$. We have gained the preliminary result as if an outerplanr graph $G = (V, E)$ satisfying V can be ordered as $(v_1, v_2, ..., v_n)$ such that $(v_{n-1}, v_n) \in E$, $\forall i < n-1$, $\exists j > i, d(v_i, v_j) = 2, N[v_i] \cap X_i \subseteq N_2(v_j)$ where $X_i = \{v_i, v_{i+1}, .., v_n\}$, then $G \in CWRC(1)$.
In the faster robber variant, we found the relation between CW and $CWFR(k, k)$ and proved formally the upper bound of cops number in $n \times n$ square grid with speed of robber 2 and speed of cop 1 as $\lceil (n-1)/4 \rceil + 1$. Because there is big gap between the lower bound as $\Omega(\sqrt{logn})$ and the upper bound as $\lceil (n-1)/4 \rceil + 1$, we will try to find the better lower bound and upper bound for the cops' number in $n \times n$ square grid.

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