University Nice Côte d'Azur
Master 1, Informatique et Interactions

Graphs
February-March 2020

Exercises : Spanning Trees and Matchings

To be returned for **March 30th 2020**.

The goal of this homework is to learn basic concepts on graphs, how to prove some properties and how to execute algorithms.

1 Minimum Spanning Tree *vs.* Shortest Path Tree

A tree is a connected acyclic graph.

Question 1 *Let $T = (V, E)$ be a tree and $u, v \in V$ be two distinct vertices. Show that there exists a unique path between u and v in T.* *hint : proof by contradiction.*

Let $G = (V, E)$ be a connected graph with a weight function $w : E \to \mathbb{R}^+$. Let $P = (v_1, v_2, \cdots, v_\ell)$ be any path in G. The weight (or *length*) of P equals $\sum_{1 \leq i < \ell} w(v_i v_{i+1})$. Given two vertices $u, v \in V$, the **distance** $dist_G(u, v)$ between u and v equals the minimum length of a path between u and v (if $u = v$, then $dist_G(u, v) = dist_G(u, u) = 0$).

Let $d > 9$. Let us consider the following weighted graph $H = (V, E)$ with $d + 1$ nodes $V = \{v_0, v_1, \cdots, v_d\}$ such that, for every $1 \leq i \leq d$, there is an edge with weight/length $w(v_0, v_i) = d$ from v_0 to v_i, and, for any $1 \leq i < d$, there is an edge with weight/length $w(v_i, v_{i+1}) = i$ between v_i and v_{i+1}. This graph H is depicted in Figure 1.

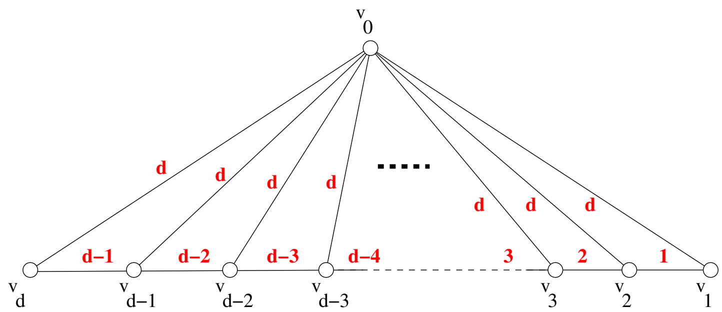

FIGURE 1 – The graph H with $d+1$ vertices ($d > 9$). A red number indicates the weight/length of the arc it is close to.

Question 2 *For every $0 \leq i \leq j \leq d$, give the distance $dist_H(v_i, v_j)$ between v_i and v_j in the graph H.* *Explain briefly by giving a shortest path for each pair of vertices.*

A tree $T = (V, E)$ *rooted in r* is simply a tree with a particular specified vertex, its *root*, $r \in V$. A spanning tree T of a graph $G = (V, E)$, such that T is rooted in $r \in V$, is a **shortest path tree** of G if, for every $v \in V$, the unique path between r and v in T has length $dist_G(r, v)$.

The goal of the following is to understand the difference between minimum spanning trees and shortest path trees.

Question 3 *Appy the Boruvska (or Kruskal, or Primm) Algorithm to the graph H. Explain how you proceed and what is the result that you obtain.*

Question 4 *Give d different minimum spanning trees of H.*

hint : consider the choices you had when applying the algorithm.

Question 5 *Let T be a spanning tree of H such that v_0 has degree at least two in T. Show that T is not a minimum spanning tree of H.*

Question 6 *Prove that there are exactly d distinct minimum spanning trees in H.*

hint : use Questions 4) and 5).

Question 7 *(*) Show that no minimum spanning tree of H is a shortest-path tree. That is, for any minimum spanning-tree T of G and for any $v \in V(H)$, T is not a shortest-path tree rooted in v.* *hint : the fact that $d > 9$ is important here.*

2 Maximum and Perfect Matchings

A **matching** M in a graph $G = (V, E)$ is any subset $M \subseteq E$ of pairwise disjoint edges (they do not share vertices), i.e., for all distinct $e, f \in M$, $e \cap f = \emptyset$. Given a set $F \subseteq E$ of edges, let $V(F) = \{u \in V \mid \exists e \in F, u \in e\}$ be the set of vertices that are the end of at least one edge in F, i.e., the set of vertices "touched" by some edge in F. A matching $M \subseteq E$ of G is **perfect** if $V(M) = V$, i.e., each vertex is "touched" by an edge of the the matching M. A matching $M \subseteq E$ is **maximum** in G if, for every matching M' in G, $|M'| \leq |M|$.

Question 8 *Prove that if G admits a perfect matching M, then M is a maximum matching.*

hint : what is the size of a perfect matching in function of $|V|$?

2.1 Case of bipartite graphs

Given a graph $G = (V, E)$, a set of vertices $S \subseteq V$ is said **stable** (or *independent set*) if, every two distinct vertices of S are not adjacent in G, i.e., for all $u, v \in S$, $\{u, v\} \notin E$. A graph $G = (V, E)$ is **bipartite** if V can be partitioned into two stable sets, i.e., there exist $A, B \subseteq V$ such that $V = A \cup B$, $A \cap B = \emptyset$ and A and B are stable. If G is bipartite, let us note $G = (A, B, E)$ to explicit two possible stable sets A and B.

Given a bipartite graph $G = (A, B, E)$ and a matching $M \subseteq E$ of G, M **saturates** A if $A \subseteq V(M)$, i.e., if each vertex of A is "touched" by an edge in M.

Question 9 *Let $G = (A, B, E)$ be a bipartite graph and assume that $|A| \leq |B|$. Prove that every maximum matching M in G is such that $|M| \leq |A|$.*

Deduce that, if there exists a matching saturating A, then it is a maximum matching in G.

Question 10 *Let $G = (A, B, E)$ be a bipartite graph. Let $S \subseteq A$ and assume that $N(S) = \{v \in V \mid \exists u \in S, \{u, v\} \in E\} \subseteq B$, i.e., $N(S)$ is the set of vertices (necessarily in B) that have a neighbor in S, is such that $|S| > |N(S)|$. Show that no matching in G saturates A.*

Next two questions aim at understanding how to compute a matching saturating A in a bipartite graph $G = (A, B, E)$, or showing that no such matching exists.

Algorithm 1 : Hungarian method [Kuhn 1955]

Require: A bipartite graph $G = (A \cup B, E)$.
Ensure: A matching M saturating A or a set $S \subseteq A$ such that $|S| > |N(S)|$.

$M \leftarrow \emptyset$.
while A is not saturated by M **do**
  Let $a_0 \in A$ be any vertex not covered by M. Set $X = \{a_0\}$.
  Let $Continue = True$.
  while $N(X)$ saturated by M and $Continue$ **do**
    $Y \leftarrow \{a_0\} \cup \{a \mid \exists b \in N(X), \{a, b\} \in M\}$.
    if $X \subset Y$ **then**
      $X \leftarrow Y$
    else
      $Continue = False$.
  if $\exists b_0 \in N(X)$ not covered by M **then**
    Let P be an M-augmenting path between a_0 and b_0 ;
    $M \leftarrow (M \setminus E(P)) \cup (E(P) \setminus M)$.
  else
    return X
return M.

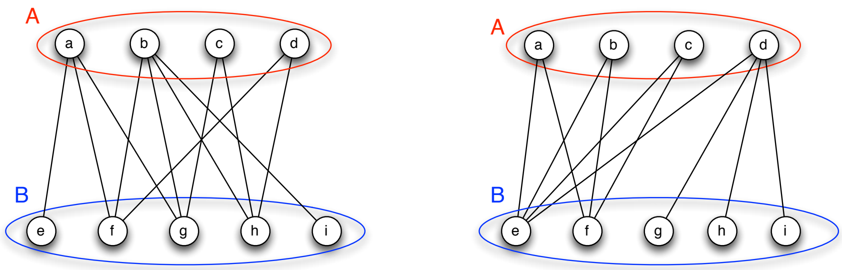

FIGURE 2 – Two examples of connected bipartite graphs.

Question 11 *Apply Algorithm 1 on the graph depicted in Figure 2 (left). Apply it in the graph depicted in Figure 2 (right). In both cases, describe the basic steps and what is the result that you obtain.*

Question 12 *(*) Prove the following theorem :*

Theorem 1 (Hall) *A bipartite graph $G = (A, B, E)$ admits a matching saturating A if and only if $\forall S \subseteq A$, $|S| \leq |N(S)|$.* *hint : have a look on the lecture notes.*

2.2 General graphs

To conclude this homework, we aim at generalizing previous notions to graphs that are not necessarily bipartite. Here, we focus on cubic graphs.

Let $G = (V, E)$ be an undirected **cubic** graph. That is, every vertex of G has degree 3. An example of a cubic graph is depicted in Figure 3 where bold edges represent a perfect matching.

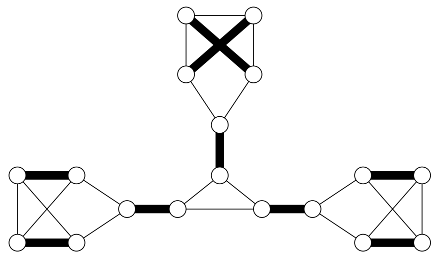

FIGURE 3 – A cubic graph with a perfect matching.

Given a graph $G = (V, E)$ and $S \subseteq V$, the subgraph $G[S]$ of G **induced by** S, is the subgraph of G with vertex-set S and all edges of G that have both ends in S, i.e., $V(G[S]) = S$ and $E(G[S]) = E \cap (S \times S)$. A **component** of $G - S$ is any connected component of $G[V \setminus S]$, i.e., any maximal connected component of the subgraph obtained from G by removing the vertices in S. A component C of $G - S$ is **odd** if it has an odd number of vertices. A (sub)graph H is **2-edge connected** if removing any edge from H lets the resulting subgraph connected, that is, for every $e \in E(H)$, the graph obtained from H by removing the edge e (but keeping its ends) is still connected.

We recall that $2|E| = \sum_{v \in V} deg(v)$, where $deg(v)$ denotes the degree of the vertex v.

Question 13 *Show that, if $C \subseteq V$ has odd size, then there is an odd number of edges between C and $V \setminus C$.*

Question 14 *Deduce from Question 13 that, if G is cubic and 2-edge connected, there are at least three edges between any set $S \subseteq V$ and an odd component C of $G - S$.*

Let $imp(G)$ denote the number of connected components of G with an odd number of vertices. In what follow, the following theorem can be used without proving it.

Theorem 2 (Tutte) *A graph G admits a perfect matching if and only if $\forall S \subseteq V$, $|S| \geq imp(G - S)$.*

Question 15 *(*) Using the theorem of Tutte above, deduce from Question 14, that if G is cubic and 2-edge connected, then G admits a perfect matching.*

Question 16 *Give an example of a cubic graph without perfect matching. Justify your answer.*