University Nice Côte d'Azur
Master 1, Informatique et Interactions

Graphs
February-March 2020

Exercises : Kernelization algorithms

To be returned for **May 25th 2020**.

The goal of this homework is to study parameterized algorithms for various graph parameters. All exercises are independent.

1 Introduction

Let us consider a decision problem Π that takes as input a graph G and a parameter $k \in \mathbb{N}$. As an example, Π may be the problem of deciding if G has a vertex cover of size at most k. We write $\Pi(G,k) = YES$ if the answer of the problem with input G and parameter k is positive.

A *kernelization algorithm* for the problem Π parameterized by k is an algorithm that computes, in time polynomial in the size of G, an instance (G',k') of Π such that

— $k' \leq k$;
— $\Pi(G,k) = YES$ if and only if $\Pi(G',k') = YES$, and
— the size of G' is upper bounded by $f(k)$ where f is any function depending only on k.

Question 1 *Show that, if Π admits a kernelization algorithm for the parameter k, then Π admits a Fixed Parameter Tractable algorithm when parameterized by k.*

2 Vertex Cover

Let $k \in \mathbb{N}$ be a fixed integer. Given a graph $G = (V,E)$, a *vertex cover* of G is any set $Q \subseteq V$ such that $e \cap Q \neq \emptyset$ for every $e \in E$. In this section, we consider the problem Π that, given a graph G, aims at deciding if G has a vertex cover Q such that $|Q| \leq k$.

A *matching* of G is any set $M \subseteq E$ of pairwise disjoint edges. Recall that a matching with maximum size can be computed in polynomial time in any graph.

Question 2 *Let G be a graph and M be a matching. Show that, if $|M| > k$, then $\Pi(G,k) = NO$.*

Question 3 *Let $G = (V,E)$ be a graph and $M \subseteq E$ be a matching with maximum size. Let V_M be the set of vertices of the edges of M. Show that $V \setminus V_M$ is a stable set (i.e., there are no edges between two vertices in $V \setminus V_M$).*

Given a bipartite graph $G = (V = (A,B),E)$, a matching M of G *saturates* A if every vertex of A is an endpoint of some edge in M. Let us recall the following result :

Theorem 1 (König, 1931) *In any bipartite graph, the size of a maximum matching is equal to the minimum size of a vertex cover. Moreover, a vertex cover with minimum size can be computed in polynomial time.*

Let $G = (V,E)$ be a graph, without isolated vertex (each vertex is incident to at least one edge), and with a maximum matching M of size at most k. Let us consider the bipartite graph S induced by V_M and $V \setminus V_M$ (i.e., keep only the edges of G between V_M and $V \setminus V_M$). Let X be a minimum vertex cover of S and M_S be a maximum matching of S (as computed by König's theorem).

Question 4 — *Show that* $|M_S| \leq |M|$.
— *Show that* $|X| \leq k$.
— *Show that, if* $X \subseteq V \setminus V_M$*, then* $X = V \setminus V_M$.
— *Show that, if* $X \subseteq V \setminus V_M$*, then* $|V| \leq f(k)$ *for some function* f *that must be explicited.*

Let $G = (V, E)$ be a graph. We say that G admits a *crown decomposition* if its vertex-set can be partitioned into 3 sets $V = C \cup H \cup R$ (C is for "crown", H for "head" and R for "reminder") such that :
— C is an stable set (i.e., there are no edges between two vertices in C) ;
— H separates C from R, i.e, there is no edges between vertices in C and vertices in R, and
— in the bipartite graph induced by C and H (i.e., keep the vertices in $C \cup H$ and the edges of G between C and H), there is a matching saturating H.

Question 5 *Let* $G = (V, E)$*,* M *and* X *be defined as in previous question. Let us assume that* $H = X \cap V_M \neq \emptyset$*. Let* $M^* \subseteq M_S$ *be the edges in* M_S *with an end in* H *and let* $C = V(M^*) \setminus X$*. Show that* $(C, H, V \setminus (C \cup H))$ *is a crown decomposition of* G*.*

Question 6 *Let* (C, H, R) *be a crown decomposition of* G *with* $|H| \leq k$*. Show that* $\Pi(G, k) = YES$ *if and only if* $\Pi(G - (C \cup H), k - |H|) = YES$*.*

Question 7 *Deduce from previous questions a kernelization algorithm for the vertex cover problem.*

3 Feedback Vertex Set

A *forest* is an acyclic graph and a *tree* is a connected forest, i.e., a connected acyclic graph.

Given a graph $G = (V, E)$, a set $F \subseteq V$ is a *feedback vertex set* if $G - F$ is a forest. That is, a set $F \subseteq V$ is a feedback vertex set of G if F intersects every cycle in G. The problem of computing a minimum feedback vertex set is a classical NP-complete problem.

A graph is r*-regular* if all its vertices have the same degree r.

Question 8 *What are the* r*-regular graphs, for* $r \in \{0, 1, 2\}$ *? Justify your answer.*

Let $r \in \mathbb{N}$. In this section, we consider the problem Π, parameterized by $k \in \mathbb{N}$, that, given an r-regular graph G, aims at deciding if G has a feedback vertex set F such that $|F| \leq k$.

Question 9 *Solve the problem* Π *in the case* $r \in \{0, 1, 2\}$*.*

From now on, let us assume $r \geq 3$. Let $G = (V, E)$ be a r-regular graph and $F \subseteq V$ be a feedback vertex set of G. Let $U = V \setminus F$ and let $H = G[U]$ be the subgraph of G induced by U. Recall that H is a forest. Let $V_{\leq 1}$ be the set of vertices of H with degree at most 1 in H. Similarly, Let V_2 be the set of vertices of H with degree equal to 2 in H. Finally, let $V_{\geq 3}$ be the set of vertices of H with degree at least 3 in H. Note that $U = V_{\leq 1} \cup V_2 \cup V_{\geq 3}$.

Question 10 *Prove that* $|V_{\geq 3}| \leq |V_{\leq 1}| - 2$*.*

Question 11 *Prove that* $(r-1)|V_{\leq 1}| \leq r|F|$ *and that* $(r-2)|V_2| \leq r|F|$*.*

Question 12 *Deduce from previous questions that* $|V| \leq 8|F|$*.* *(recall that* $r \geq 3$*)*

Question 13 *Give a kernelization algorithm for the feedback vertex set problem* Π *in* r*-regular graphs.*

4 Dominating set

A graph $G = (V, E)$ is C_4-free if G has no cycle of size exactly 4 as subgraph (not necessarily induced). That is, if G is C_4-free, there are no four vertices a, b, c, d that form a cycle, i.e., with the edges ab, bc, cd, da (Note that, even if there are more edges such as ac and/or bd, such a cycle cannot exist).

Let $v \in V$. Recall that $N(v) = \{u \in V \mid uv \in E\}$ is the set of neighbors of v, $N[v] = N(v) \cup \{v\}$ and, for any set $S \subseteq V$, $N[S] = \bigcup_{v \in S} N[v]$ and $N(S) = N[S] \setminus S$.

A set $D \subseteq V$ is a *dominating set* of a graph G if, for every vertex $v \in V$, $v \in D$ or there is $u \in D$ with $uv \in E$. That is, D is a dominating set if every vertex of G is in D or has some neighbor in D, i.e., $V = N[D]$. The problem of computing a minimum dominating set is a well known NP-complete problem.

In this section, we consider the problem Π, parameterized by $k \in \mathbb{N}$, that, given an C_4-free graph G, aims at deciding if G has a dominating set D such that $|D| \leq k$.

Question 14 *Let $G = (V, E)$ be a C_4-free graph and let $v \in V$ with degree at least $2k + 1$. Let D be a dominating set of G with $|D| \leq k$. Show that $v \in D$.*

hint : Show that, if $v \notin D$, at least one neighbor of v is not in $N[D]$. The fact that G is C_4-free is important.

Let $S \subseteq V$ be the set of vertices with degree at least $2k + 1$.

Question 15 *What can you conclude if $|S| > k$?*

From now on, let us assume that $|S| \leq k$. Let $R = V \setminus N[S]$, that is, R is the set of vertices that are not dominated by S.

Question 16 *Show that, if $\Pi(G, k) = YES$, then $|R| \leq 2k^2$.*

hint : note that there are no edges between S and R, and that all vertices of $V \setminus S$ have degree at most $2k$.

Question 17 *Deduce from previous question that, if $\Pi(G, k) = YES$, then $|N(R)| \leq 4k^3$.*

Let G' be the graph obtained as follows. Start from G and remove all vertices of $N(S) \setminus N(R)$. Finally, for every $v \in S$, add one vertex x_v adjacent only to v.

Question 18 *Bound the size of G' by a function of k if $\Pi(G, k) = YES$.*

Question 19 *Show that $\Pi(G, k) = YES$ if and only if $\Pi(G'; k) = YES$.*

Question 20 *Give a kernelization algorithm for the dominating set problem Π in C_4-free graphs.*

5 Longest Cycle

Computing a longest cycle in a graph is a well-known NP-complete problem since it generalizes the Hamiltonian cycle problem.

In this section, we consider the problem Π, parameterized by $k \in \mathbb{N}$, that, given an integer $\ell \in \mathbb{N}$ and a graph G with a given vertex cover Q of size k, aims at deciding if G has a cycle of length at least ℓ. Note that, this time, the parameter is not the size ℓ of the solution but the size of a given vertex cover Q of G.

Question 21 *Show that, if $\ell > 2k$ then $\Pi(G, k) = NO$.*

hint : show that at most k vertices of $V \setminus Q$ belong to any cycle.

To reduce the size of the graph, let us use Algorithm 1.

Algorithm 1 Reduction procedure

Require: A graph $G = (V, E)$ and a vertex cover $Q \subseteq V$.
Ensure: A set of vertices $M \subseteq V \setminus Q$.

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M ← ∅.
for every pair x, y ∈ Q, x ≠ y, of distinct vertices of Q do
    Let N_xy = (N(x) ∩ N(y)) \ (Q ∪ M) be the set of common vertices of y and x not in Q
    and not already in M.
    if |N_xy| ≤ k + 1 then
        M ← M ∪ N_xy, i.e. add N_xy to M.
    else
        Let N be any subset of size k + 1 of N_xy.
        M ← M ∪ N, i.e. add N to M.
return M.
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Let $G' = G[M \cup Q]$ be the subgraph of G induced by Q and M.

Question 22 *Bound the size of G' by a function of k.*

Question 23 *Show that, if $\Pi(G', k) = YES$ then $\Pi(G, k) = YES$.*

Question 24 *Show that, if $\Pi(G, k) = YES$ then $\Pi(G', k) = YES$.*

Question 25 *Give a kernelization algorithm for the longest cycle problem Π parameterized by the size of a vertex cover.*