Spy Game on Graphs

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Seminario del Doctorado en Ingeniería de Sistemas Complejos

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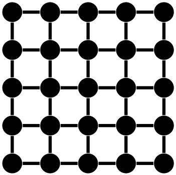

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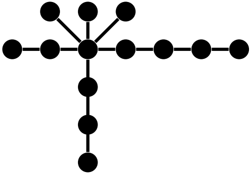

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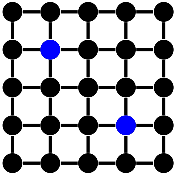

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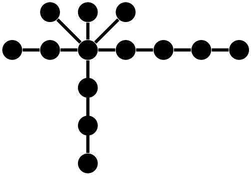

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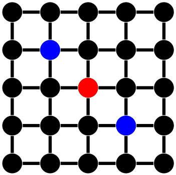

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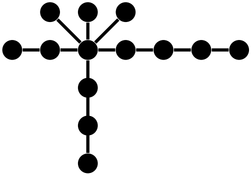

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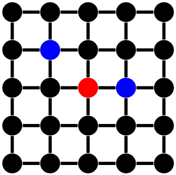

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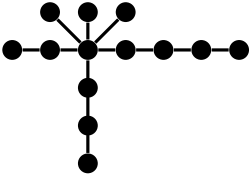

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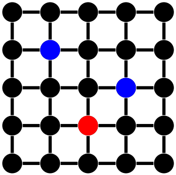

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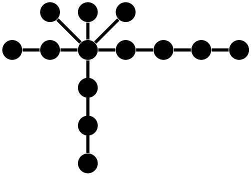

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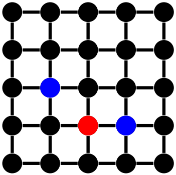

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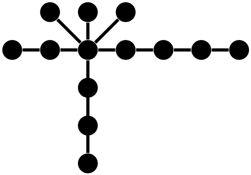

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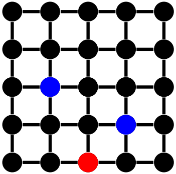

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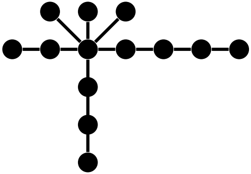

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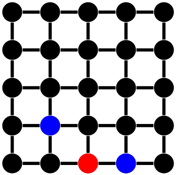

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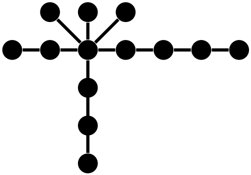

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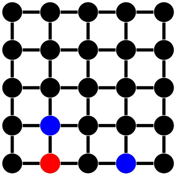

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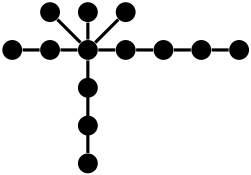

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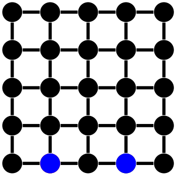

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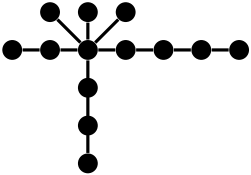

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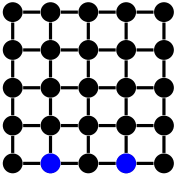

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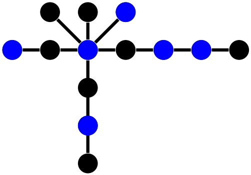

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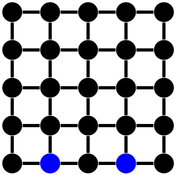

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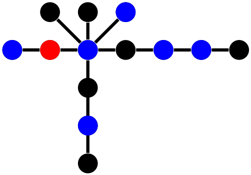

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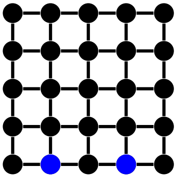

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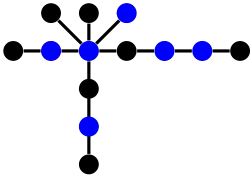

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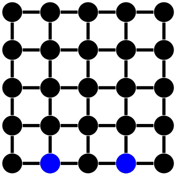

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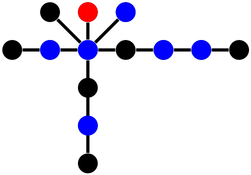

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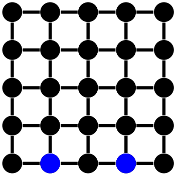

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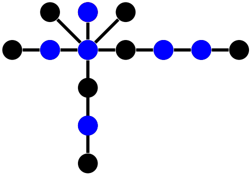

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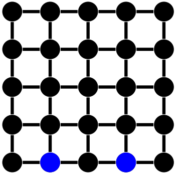

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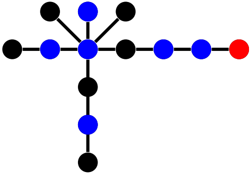

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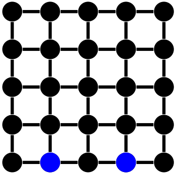

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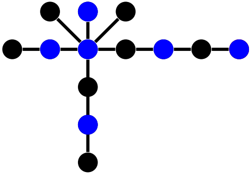

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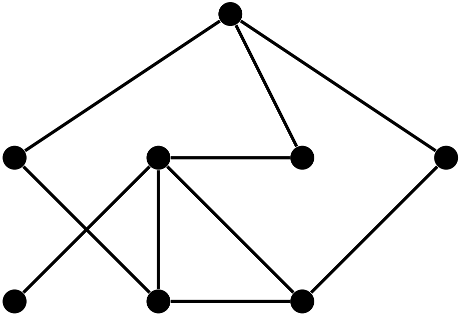
Spy (1^{st}) vs guards (2^{nd}) in a graph G.

Start : Spy placed at a vertex. Then, guards placed.

Turn-by-turn : Spy traverses up to $\mathbf{s} \geq 2$ edges. Guards traverse up to 1 edge.

Goal : Spy wants to be at least distance $\mathbf{d} + 1$ from all guards.

Ex : $\mathbf{s} = 2$ and $\mathbf{d} = 1$.



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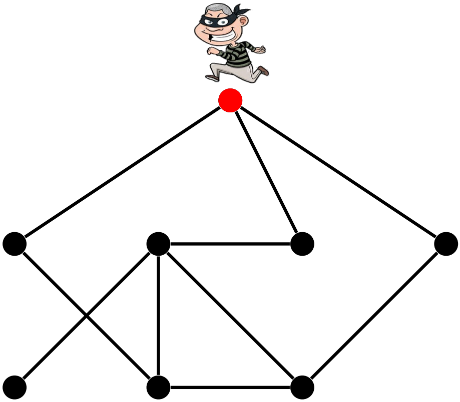

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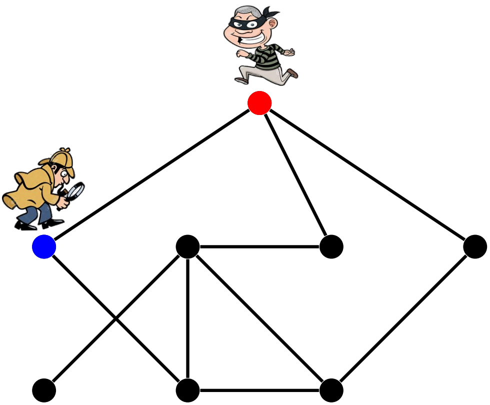

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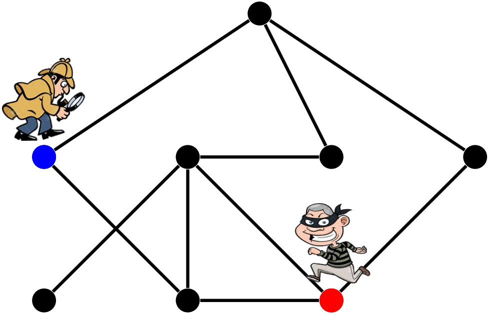

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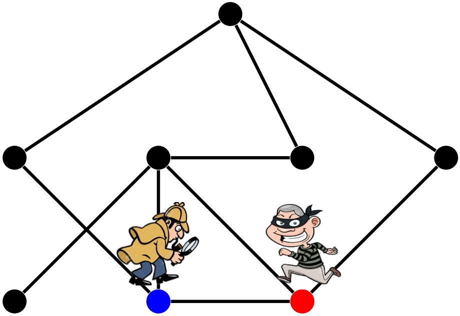

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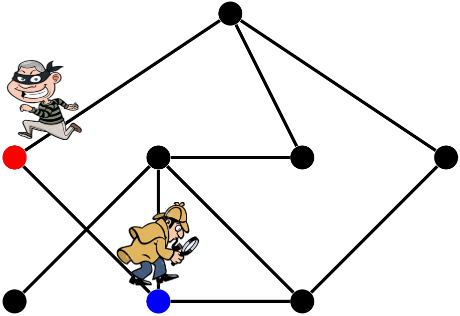

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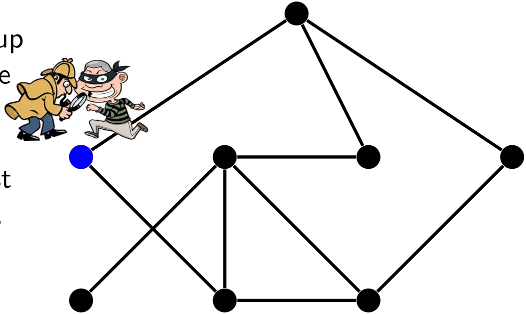

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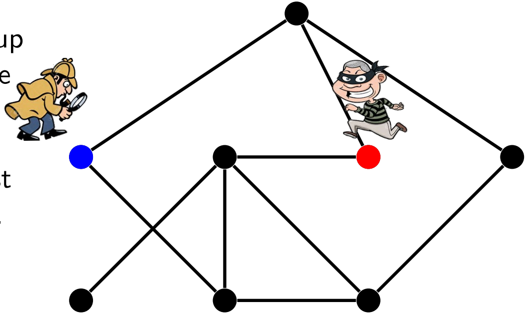

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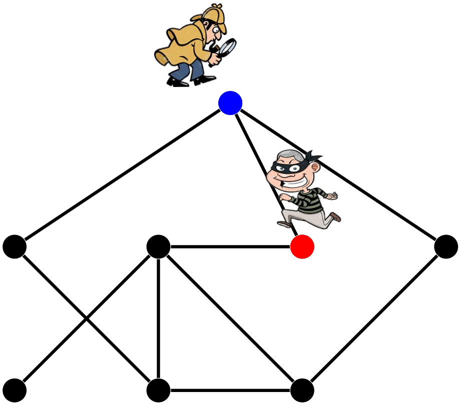

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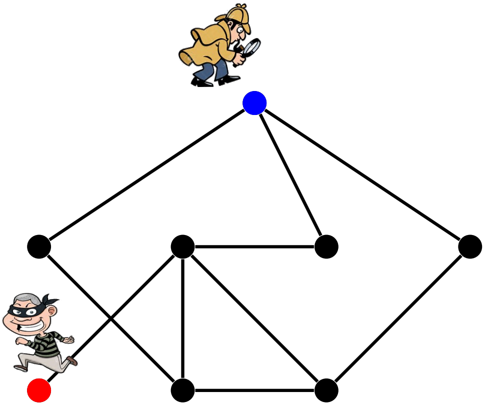

Guard Number : $gn_{s,d}(G)$

Definition

For all $s \geq 2$, $d \geq 0$ and a graph G, $gn_{s,d}(G)$ is the minimum number of guards guaranteed to win vs the spy.

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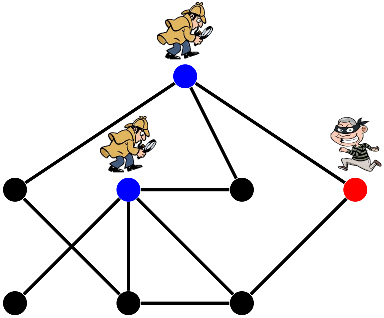

$$gn_{2,1}(G) = 2$$

$$gn_{s,1}(G) \leq \gamma(G)$$

Our Results : Computing *gn*

Complexity

Calculating $gn_{s,d}$ is NP-hard in general.

Tight bounds for paths

$gn_{s,d}(P_n) = \left\lceil \frac{n}{2d+2+q} \right\rceil$ where $q = \lfloor \frac{2d}{s-1} \rfloor$.

Almost tight bounds for cycles

$\left\lceil \frac{n+2q}{2(d+q)+3} \right\rceil \leq gn_{s,d}(C_n) \leq \left\lceil \frac{n+2q}{2(d+q)+1} \right\rceil$ where $q = \lfloor \frac{2d}{s-1} \rfloor$.

Polynomial time Linear Program for trees

Can calculate $gn_{s,d}(T)$ and a corresp. strategy in polynomial time.

Grids

$\exists \beta > 0$, s.t. $\Omega(n^{1+\beta}) \leq gn_{s,d}(G_{n \times n})$.

Related Work

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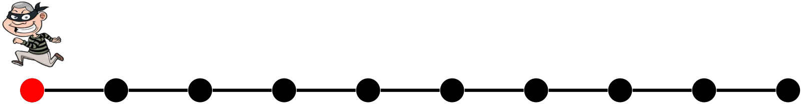

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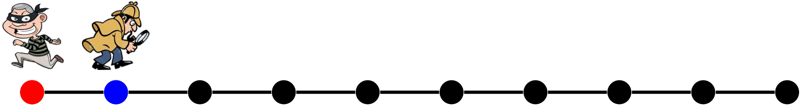

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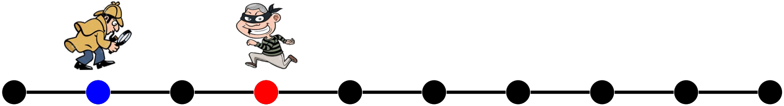

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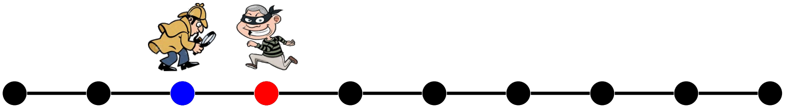

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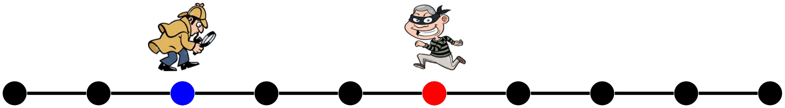

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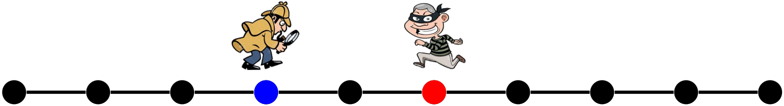

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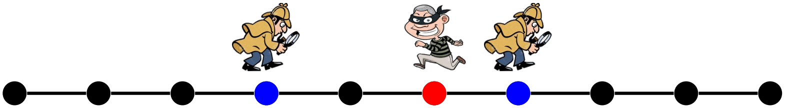

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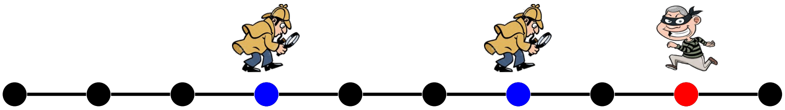

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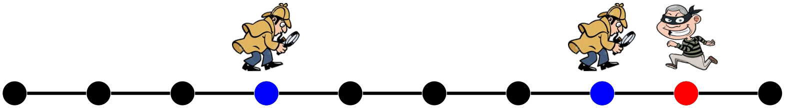

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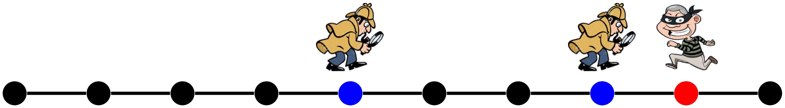

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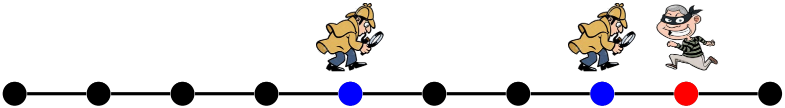

$$gn_{3,1}(P_{10}) = 2$$

Paths : Upper bound

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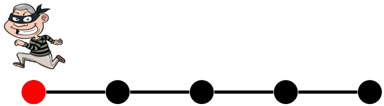

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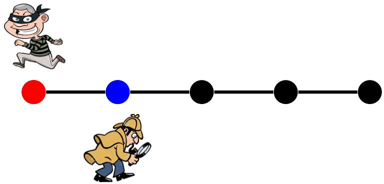

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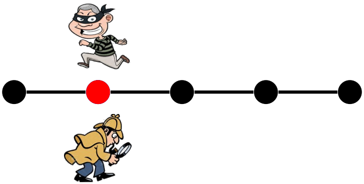

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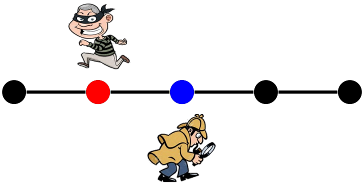

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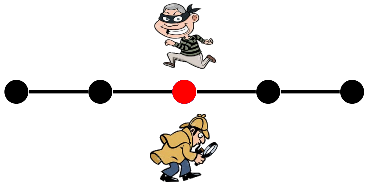

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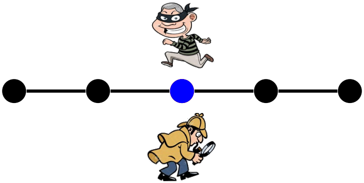

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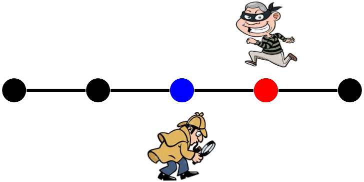

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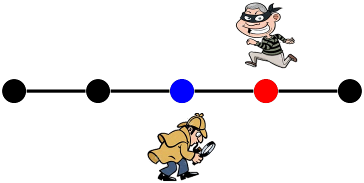

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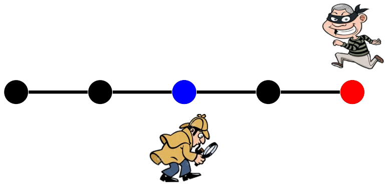

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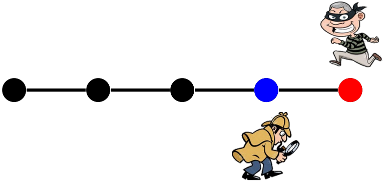

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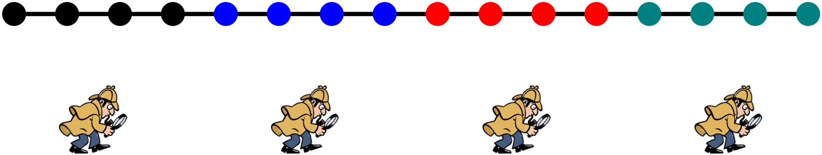


Guard Strategies

Spy-Positional Strategy :

$f : V^k \times V \Rightarrow V^k$ (Unrestricted strategy)

$\omega : V \Rightarrow V^k$ (Restricted or **Spy-Positional** strategy)

- Guards' positions depend only on position of spy.
- Unique configuration for guards for each position of spy.

Zonal Strategy :

Divide graph into subgraphs & assign certain number of guards to each.

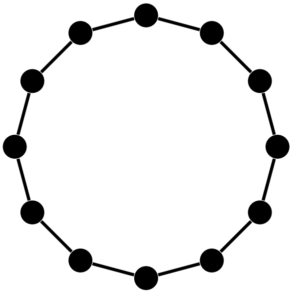
Optimal strategy in paths uses both strategies above.

Theorem

For all $s \geq 2$, $d \geq 0$ s.t. $2d < s - 1$, and a cycle C_n on n vertices, $gn_{s,d}(C_n) = \left\lceil \frac{n}{2d+3} \right\rceil$.

Ex : $s = 6$ and $d = 0$.

$$gn_{6,0}(C_{12}) = 4$$



Cycles : Upper Bound Case $2d < s - 1$

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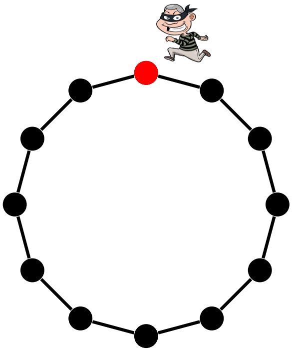

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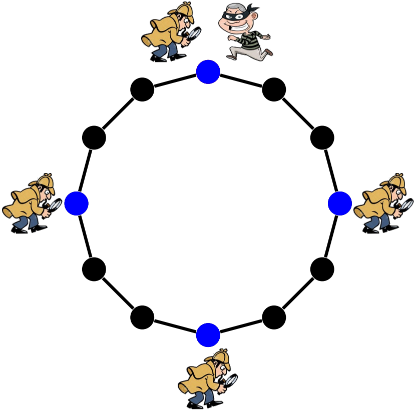

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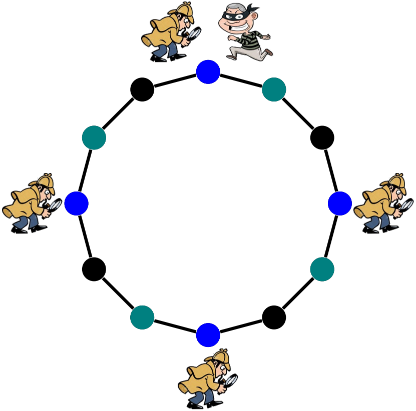

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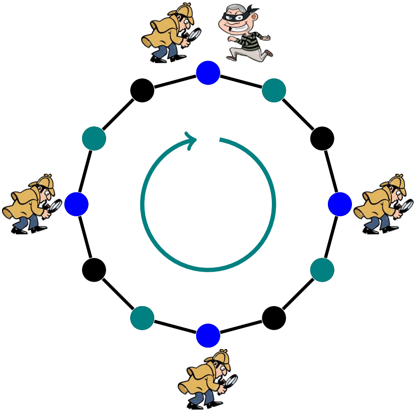

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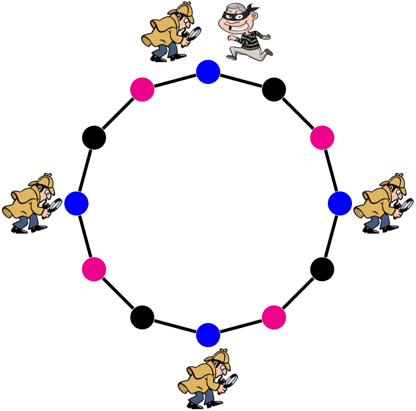

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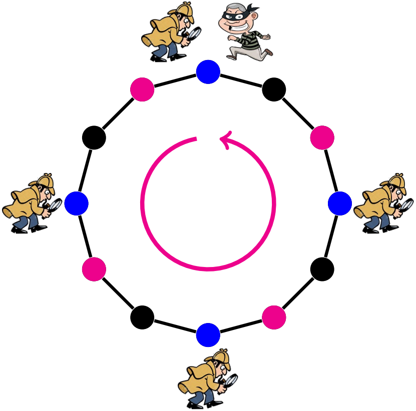


Cycles

Theorem

For all $s \geq 2$, $d \geq 0$ s.t. $q = 0$, and a cycle C_n on n vertices, $gn_{s,d}(C_n) = \left\lceil \frac{n}{2d+3} \right\rceil$.

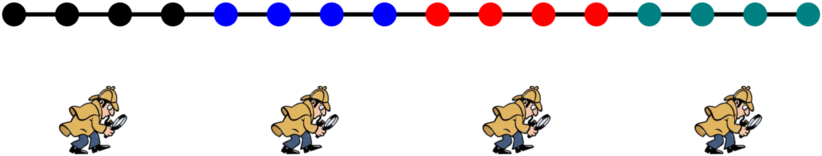
Theorem

For all $s \geq 2$, $d \geq 0$ s.t. $q \neq 0$, and a cycle C_n on n vertices, $\left\lceil \frac{n+2q}{2(d+q)+3} \right\rceil \leq gn_{s,d}(C_n) \leq \left\lceil \frac{n+2q}{2(d+q)+1} \right\rceil$.

Reminder : $q = \left\lfloor \frac{2d}{s-1} \right\rfloor$.

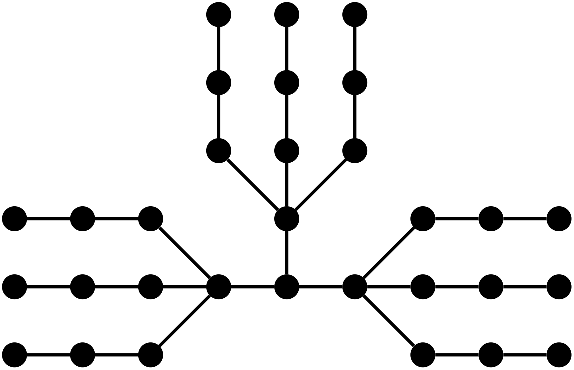
Trees are Harder

Zonal strategy in Paths : 1 guard per subpath of $2d + 2 + \lfloor \frac{2d}{s-1} \rfloor$ vertices.



Trees are Harder

Can't always divide tree into subtrees protected by a certain number of guards. No Zonal strategy in trees.



Example of a tree T where $s = 2$, $d = 1$ and $gn_{2,1}(T) = 4$.

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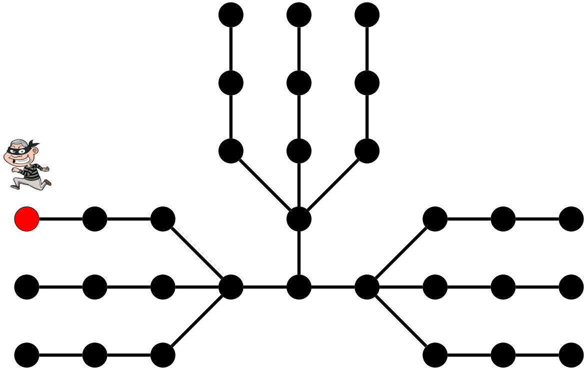

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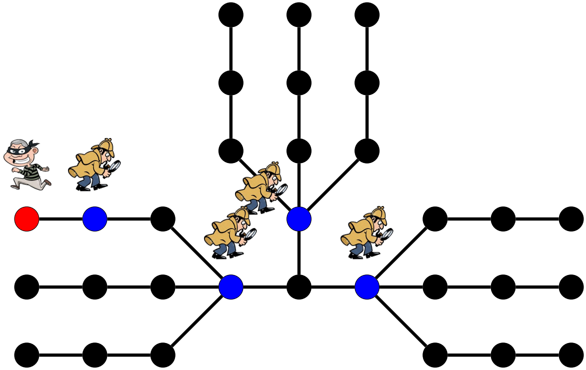

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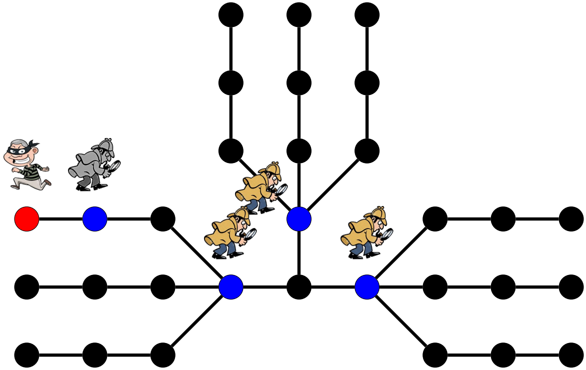

Example of a tree T where $s = 2$, $d = 1$ and $gn_{2,1}(T) = 4$.

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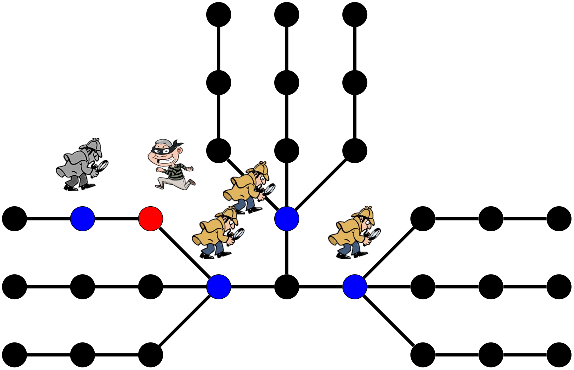

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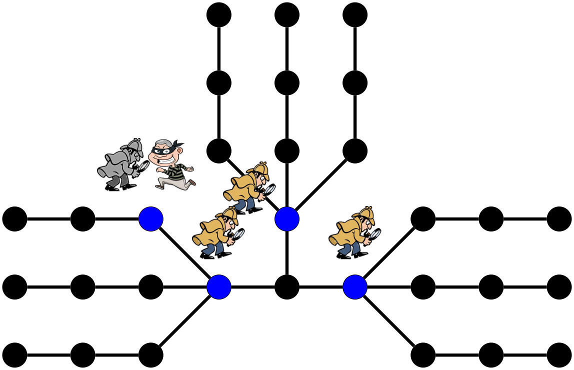

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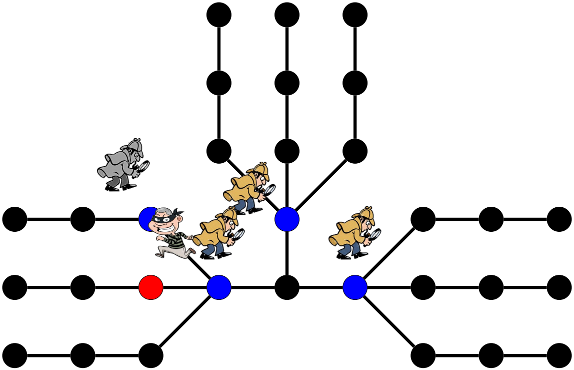

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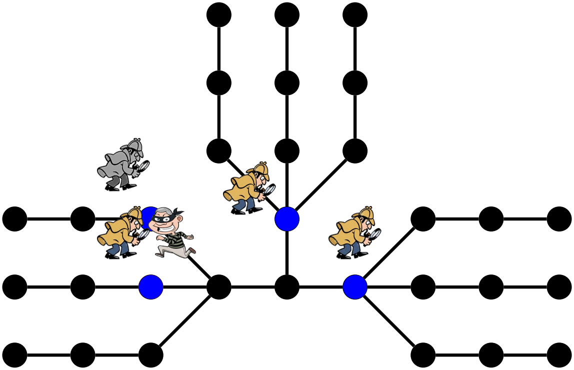

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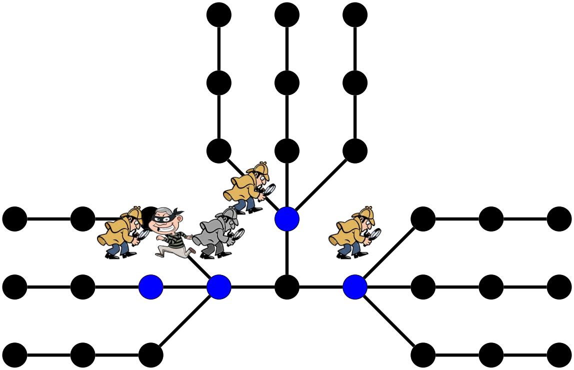

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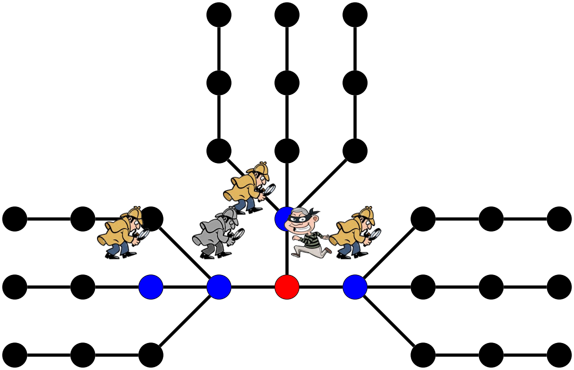

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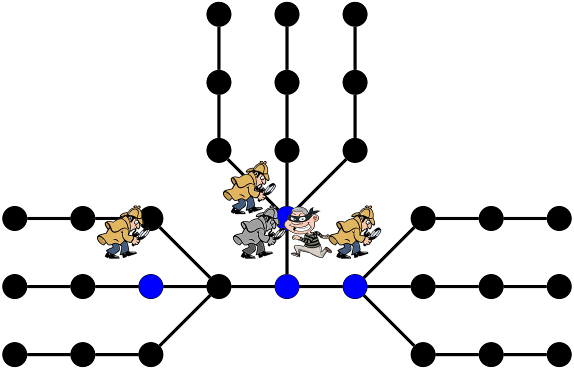

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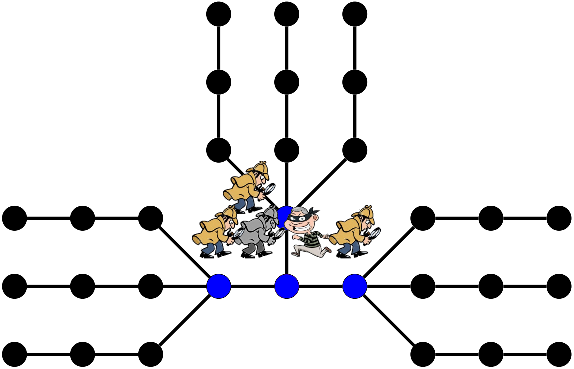

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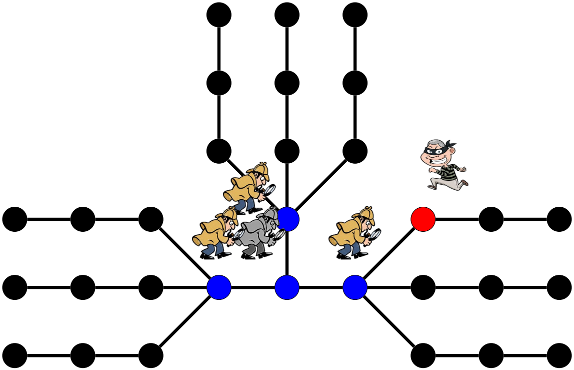

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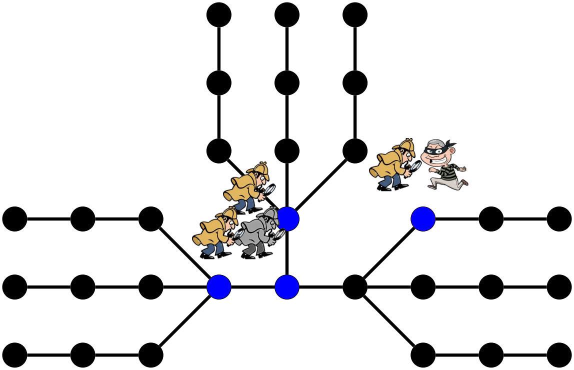

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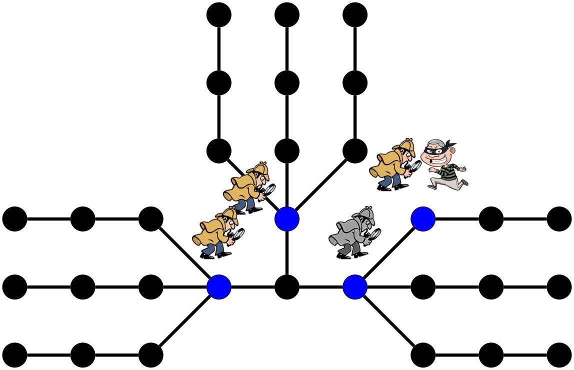

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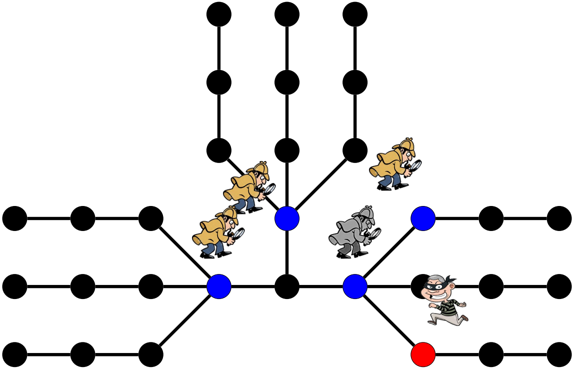

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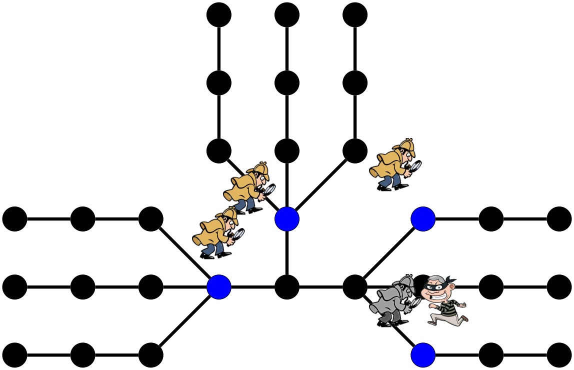

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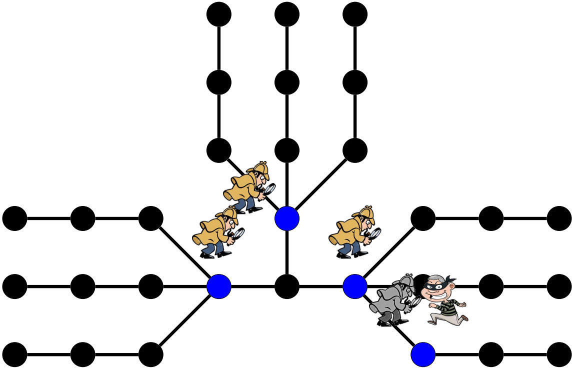

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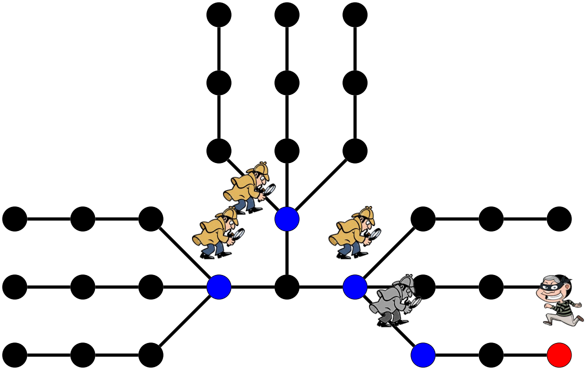

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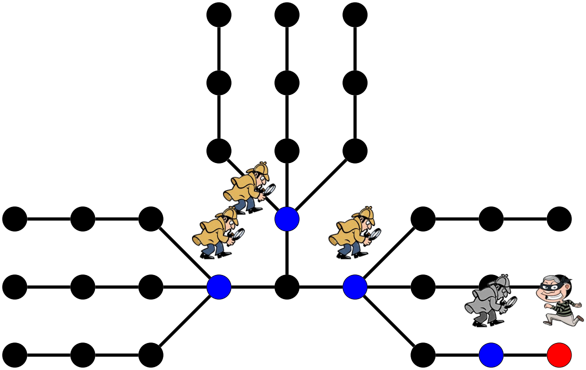

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- Guards may be fractional entities ; movements rep. by flows.
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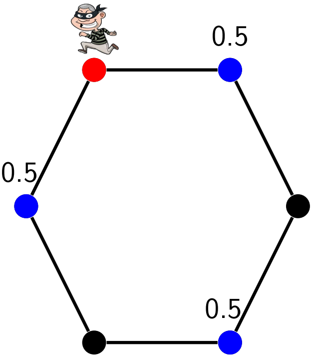

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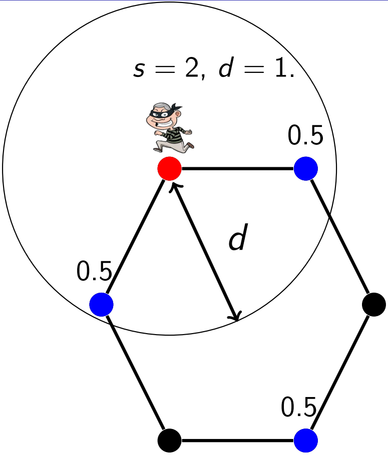

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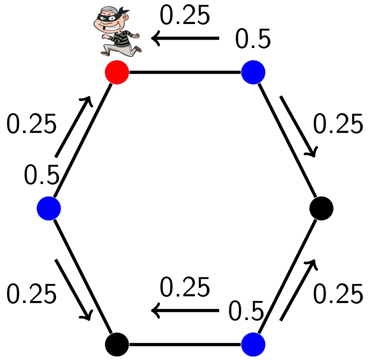

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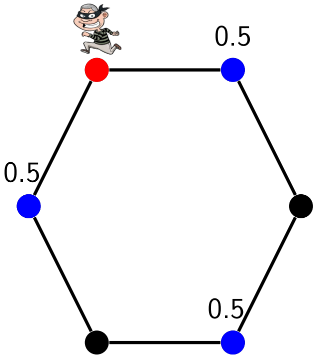

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- Optimal fractional strategy $\Rightarrow$ optimal integral strategy in trees.

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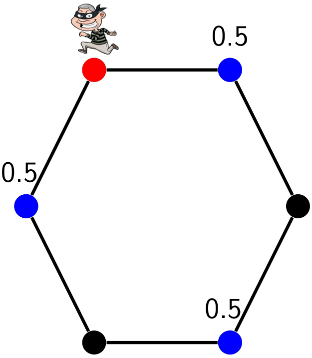

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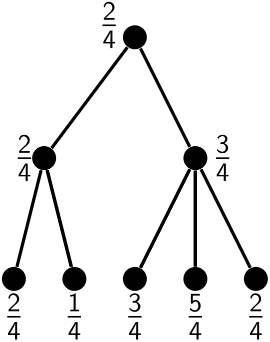
Opt. Fractional Strategy $\Rightarrow$ Opt. Integral Strategy in Trees

Theorem : Can transform optimal fractional strategy into optimal integral strategy in polynomial time.



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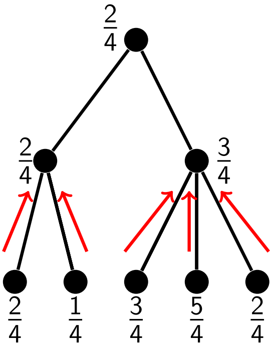

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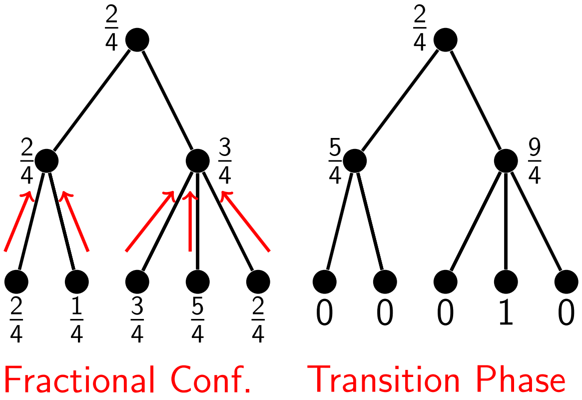

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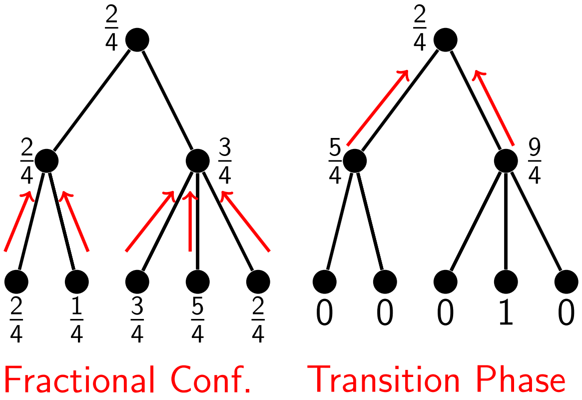

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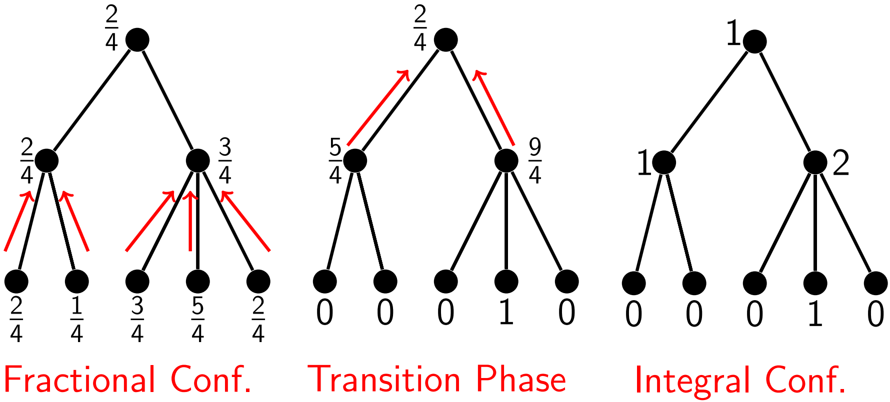

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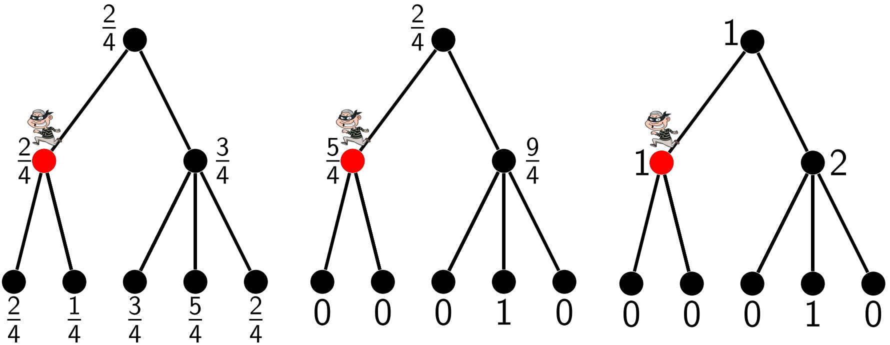

Fractional Conf. Transition Phase Integral Conf.

Tree's protection and guards' movements preserved.

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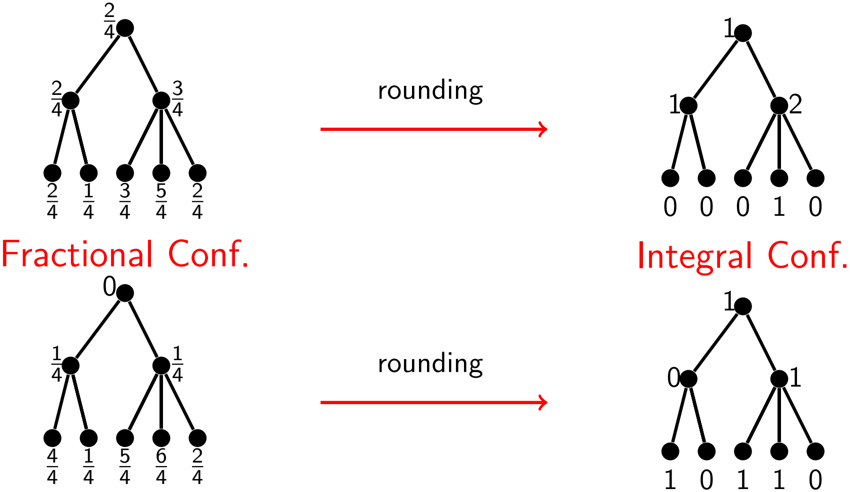

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Restricted Strategies

$f : V^k \times V \Rightarrow V^k$ (Unrestricted strategy)

$\omega : V \Rightarrow V^k$ (Restricted or **Spy-Positional** strategy)

- Guards' positions depend only on position of spy.
- Unique configuration for guards for each position of spy.

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Optimal fractional strategy $\Rightarrow$ optimal fractional Spy-Positional strategy in trees.

Can calculate optimal Spy-positional fractional strategies with Linear Program in polynomial time.

Linear Program to Compute Spy-Positional Strategy

Spy-positional strategy : $\omega : V \Rightarrow V^k$

$\omega_{x,u}$: quantity of guards on u when spy is on x.

$f_{x,x',u,u'}$: quantity of guards that go from u to u' when spy goes from x to x'.

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$$(1)\ \text{Minimize} \sum_{v \in V} \omega_{x_0,v}$$

Minimize number of guards.

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$$(2) \sum_{v \in N_d[x]} \omega_{x,v} \geq 1 \qquad \forall x \in V$$

Guarantees always at least 1 guard within distance d of spy.

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$$(3) \sum_{u' \in N[u]} f_{x,x',u,u'} = \omega_{x,u} \qquad \forall u \in V,\ x' \in N_s[x]$$
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Guarantees validity of moves of guards when spy moves.

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$O(n^4)$ real variables and constraints.

Main Result : *gn* in Trees

Theorem

$\forall s > 1$, $d \geq 0$ and all trees T, $gn_{s,d}(T)$ and a corresponding strategy can be calculated in polynomial time.

Idea of proof : Linear Program can compute opt. frac. Spy-positional strategy in polynomial time.

Run LP. From previous theorem, strategy is opt. frac.

Can transform opt. frac. into opt. int. in polynomial time.

Grids

Theorem

$\exists \beta > 0$, s.t. $\forall s > 1$, $d \geq 0$, $\Omega(n^{1+\beta}) \leq gn_{s,d}(G_{n\times n})$.

Idea of proof : Lower bound holds for fractional version.

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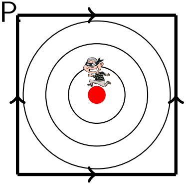
Idea of proof : Lower bound holds for fractional version.

Torus and grid have same order of number of guards.

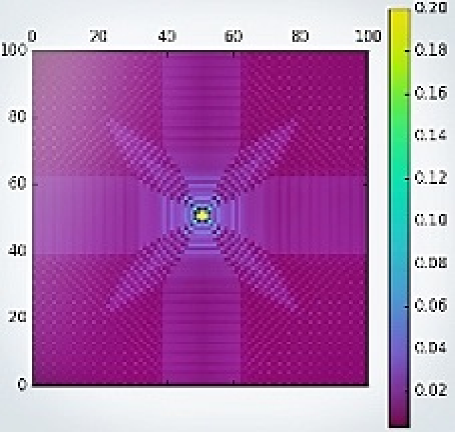
Theorem

$\exists \alpha \geq log(3/2) \approx 0.58$, s.t. $\forall s > 1$, $d \geq 0$,
$fgn_{s,d}(G_{n \times n}) \leq O(n^{2-\alpha})$.

Idea of proof : Density function $\omega^*(v) = \frac{c}{(dist(v,v_0)+1)^{log 3/2}}$ for a constant $c > 0$ satisfies LP.



Distribution of Guards in the Torus for an optimal symmetrical spy-positional strategy when $n = 100$, $m = 100$, $s = 2$ and $d = 1$



Further Work

- Determine $gn_{s,d}(G_{n \times n})$.
- Approximate $gn_{s,d}(G)$ in polynomial time in certain classes of graphs ?
- Fractional approach applied to other combinatorial games.

Gracias !