The Komornik-Loreti constant is transcendental

Jean-Paul Allouche and Michel Cosnard

Let q be a non-integer real number > 1, and let $\lfloor q \rfloor$ be its integer part. The sum $\sum_{n=0}^{\infty} a_n/q^n$ is called a *q-expansion* of the positive real number x if

- for all $n \geq 0$, we have $0 \leq a_n \leq \lfloor q \rfloor$,
- $x = \sum_{n=0}^{\infty} \frac{a_n}{q^n}$.

Any real number x such that $0 \leq x \leq q\lfloor q \rfloor/(q-1)$ admits such a decomposition (use the *greedy algorithm*). But the decomposition is not necessarily unique: for example if $\tau = (1+\sqrt{5})/2$ is the golden ratio, then $1 = 1/\tau + 1/\tau^2$. In a recent paper appearing in this MONTHLY [4], Komornik and Loreti introduced an interesting constant by proving the following result.

Theorem 1 (Komornik-Loreti) *There exists a smallest $q \in (1,2)$ for which there exists a unique expansion of 1 as $1 = \sum_{n=1}^{\infty} \delta_n q^{-n}$, with $\delta_n \in \{0,1\}$. Furthermore, for this smallest q, the coefficient δ_n is equal to 0 (respectively 1) if the sum of the binary digits of n is even (respectively odd). This number q can be then obtained as the unique positive solution of $1 = \sum_{n=1}^{\infty} \delta_n q^{-n}$. It is equal to 1.787231650...*

In the electronic abstract of [4], the authors ask whether the number $q = 1.787231650...$ in Theorem 1 above is irrational. The purpose of this note is to prove, as a simple consequence of a result of Mahler, that q is transcendental.

Theorem 2 *The number $q = 1.787231650...$ defined as the smallest number in $(1,2)$ for which there exists a unique expansion of 1 as $1 = \sum_{n=1}^{\infty} \delta_n q^{-n}$, with $\delta_n \in \{0,1\}$, is transcendental.*

Proof. Mahler proved in [5, p. 363] that $F(a)$ is transcendental for every algebraic number a, with $0 < |a| < 1$, where F is defined for $|z| < 1$ by

$$F(z) = \prod_{n=0}^{\infty} \left(1 - z^{2^n}\right).$$

It is easy to see that

$$F(z) = 1 + \sum_{n=1}^{\infty} (-1)^{\delta_n} z^n$$

where $(\delta_n)_{n\geq 1}$ is the sequence in Theorem 1. Since δ_n takes values 0 and 1, we have $(-1)^{\delta_n} = 1 - 2\delta_n$. Hence

$$F(z) = 1 + \frac{z}{1-z} - 2\sum_{n=1}^{\infty} \delta_n z^n.$$

Taking $z = q^{-1}$, we obtain

$$F\left(\frac{1}{q}\right) = 1 + \frac{1}{q-1} - 2\sum_{n=1}^{\infty} \delta_n q^{-n} = 1 + \frac{1}{q-1} - 2 = \frac{2-q}{q-1}. \tag{1}$$

If q were algebraic, this would imply that the left-hand side of (1) is transcendental (from Mahler's result), although the right-hand side would be algebraic. □

Remarks

- To learn more about q-expansions (often called β-expansions), the reader should study the two seminal papers [7] and [9]. For more results and open problems, as well as for many links between q-expansions of real numbers, discrete dynamical systems, ergodic theory, combinatorics on words, language theory, number theory (in particular Pisot and Salem numbers), and even *quasi-crystals*, the reader can look at the references given in [2].
- The sequence (δ_n) in Theorem 1 is known as the *Prouhet-Thue-Morse sequence*. It was introduced by Thue ([10], [11]) at the beginning of the century (see also Prouhet's paper [8]) and rediscovered by Morse in the 20's [6]. This sequence occurs in many apparently unrelated fields [3].
- The authors got interested in this question while working on the paper [4], and showing that combinatorial properties of the q-expansion of 1 in case of uniqueness are reformulations of properties of kneading sequences for iterations of unimodal continuous functions; see [1] and [2] and references therein.

References

[1] J.-P. Allouche and M. Cosnard, Itérations de fonctions unimodales et suites engendrées par automates, *C. R. Acad. Sci. Paris, Série I* **296** (1983) 159–162.

[2] J.-P. Allouche and M. Cosnard, Non-integer bases, iteration of continuous real maps, and an arithmetic self-similar set, Preprint, 1999.

[3] J.-P. Allouche and J. Shallit, The ubiquitous Prouhet-Thue-Morse sequence, in: *Sequences and their applications, Proceedings of SETA'98*, C. Ding, T. Helleseth and H. Niederreiter (Eds.), Springer Verlag, 1999, pp. 1–16.

[4] V. Komornik and P. Loreti, Unique developments in non-integer bases, *Amer. Math. Monthly* **105** (1998) 636–639.

[5] K. Mahler, Arithmetische Eigenschaften der Lösungen einer Klasse von Funktionalgleichungen, *Math. Annalen* **101** (1929) 342–266. Corrigendum **103** (1930) 532.

[6] M. Morse, Recurrent geodesics on a surface of negative curvature, *Trans. Amer. Math. Soc.* **22** (1921) 84–100.

[7] W. Parry, On the β-expansions of real numbers, *Acta Math. Acad. Sci. Hung.* **11** (1960) 401–416.

[8] E. Prouhet, Mémoire sur quelques relations entre les puissances des nombres, *C. R. Acad. Sci. Paris Sér. I* **33** (1851) 225.

[9] A. Rényi, Representations for real numbers and their ergodic properties, *Acta Math. Acad. Sci. Hung.* **8** (1957) 477–493.

[10] A. Thue, Über unendliche Zeichenreihen, *Norske vid. Selsk. Skr. Mat. Nat. Kl.* **7** (1906) 1–22. Reprinted in *Selected Mathematical Papers of Axel Thue*, T. Nagell, editor, Universitetsforlaget, Oslo, 1977, pp. 139–158.

[11] A. Thue, Über die gegenseitige Lage gleicher Teile gewisser Zeichenreihen, *Norske vid. Selsk. Skr. Mat. Nat. Kl.* **1** (1912) 1–67. Reprinted in *Selected Mathematical Papers of Axel Thue*, T. Nagell, editor, Universitetsforlaget, Oslo, 1977, pp. 413–478.

CNRS, LRI, Bâtiment 490, F-91405 Orsay Cedex (France)
allouche@lri.fr

LORIA, 615 rue du Jardin Botanique, F-54602 Villers-lès-Nancy Cedex (France)
cosnard@loria.fr