Non-integer bases, iteration of continuous functions, and an arithmetic self-similar set

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1 Introduction

In a recent paper appeared in this Journal [14, Theorem 1, Lemma 1, and Theorem 3], Komornik and Loreti prove the following results.

Theorem 1 ([14]) *There exists a smallest $q \in (1, 2)$, for which there exists a unique expansion of 1 as $1 = \sum_{n=1}^{\infty} \delta_n q^n$, with $\delta_n \in \{0, 1\}$. Furthermore, for this smallest q, the coefficient δ_n is equal to 0 (resp. 1) if the sum of the binary digits of n is even (resp. odd). This number q can be then obtained as the unique positive solution of $1 = \sum_{n=1}^{\infty} \delta_n q^n$. It is equal to* 1.787231650...

Call a binary sequence $(\varepsilon_n)_{n \geq 1}$ *admissible* if it satisfies

$$\begin{array}{llll} \varepsilon_{n+1}\ \varepsilon_{n+2} \ldots & < \varepsilon_1\ \varepsilon_2 \ldots & \text{whenever} & \varepsilon_n = 0, \\ \overline{\varepsilon_{n+1}}\ \overline{\varepsilon_{n+2}} \ldots & < \varepsilon_1\ \varepsilon_2 \ldots & \text{whenever} & \varepsilon_n = 1, \end{array}$$

where $\overline{0} = 1$ and $\overline{1} = 0$, and "$<$" is the lexicographical order. Using results of [12] the authors of [14] prove that Theorem 1 above is a consequence of the following.

Theorem 2 ([14]) *The smallest admissible sequence is the sequence $(\delta_n)_{n \geq 1}$ where δ_n is equal to 0 (resp. 1) if the sum of the binary digits of n is even (resp. odd).*

The purpose of this note is threefold: first indicate that Theorem 2 was essentially given by the authors (see [3] and [1, Chapter 2]) in the framework of iteration of continued fractions, second give a proof based on a result of Mahler that the number q in Theorem 1 above is transcendental, and finally recall how a natural self-similar arithmetic set (already mentioned in [3] and [1, Chapter 3]) enters the picture.

We are very grateful to J. Shallit who was the first to notice the similarity of the result of Komornik and Loreti and the result of the authors, and who attracted our attention on Reference [14].

2 Iteration of continuous functions

Let f be a continuous function from the interval $[0,1]$ into itself. We suppose that f is unimodal, i.e., that there exists a $c \in (0,1)$ such that f is increasing on $(0,c)$ and decreasing on $(c,1)$. With a point $x \in [0,1]$ we associate its orbit $(f^{(n)}(x))_{n\geq 0}$, and its *kneading sequence* (a standard reference is [9]) $(\varphi(f^{(n)}(x)))_{n\geq 0}$, where φ is the characteristic function of the interval $[c,1]$. (A small problem arises for the value of φ at c, but we will not address it here.)

Cosnard proved (see [10]) that a binary sequence $(\alpha_n)_{n\geq 0}$ such that $\alpha_0 = 1$ and $\alpha_1 = 0$ is the kneading sequence of the point 1 for some continuous unimodal function f such that $f(1) = 0$ if and only if

$$\alpha_0 = 1, \ \alpha_1 = 0, \ \text{and} \ \forall k \geq 0, \ \overline{\tau(\alpha)} \leq \sigma^k(\tau(\alpha)) \leq \tau(\alpha),$$

where $\tau(\alpha) = (\tau(\alpha)_n)_{n\geq 1}$ is the binary sequence defined by $\tau(\alpha)_n = \sum_{j=0}^{n-1} \alpha_j \bmod 2$, and σ is the shift: for any sequence $(\beta_n)_n$, the sequence $(\sigma(\beta)_n)_n$ is defined by $\sigma(\beta)_n = \beta_{n+1}$

We note that τ is a bijection: τ^{-1} is defined by $\tau^{-1}((\beta_n)_{n\geq 1}) = (\alpha_n)_{n\geq 0}$, where $\alpha_0 = \beta_1$, and for all $n \geq 1$, $\alpha_n = \beta_n + \beta_{n+1} \bmod 2$. Hence, up to shifting indices, properties of kneading sequences of the point 1 for continuous unimodal functions f such that $f(1) = 0$ can be studied by studying the set of sequences $(\gamma_n)_{n\geq 0}$, called *extremal* in [3], such that

$$\gamma_0 = \gamma_1 = 1, \ \text{and} \ \forall k \geq 0, \ \overline{\gamma} \leq \sigma^k(\gamma) \leq \gamma.$$

The following definition and theorem are given in [3] with slightly different notations (details of the proof are given in [1, Chapter 2]).

Definition ([3])

Define Γ to be the set of extremal sequences:

$$\Gamma = \left\{\gamma = (\gamma_n)_{n\geq 0}, \ \gamma_0 = \gamma_1 = 1, \ \text{and} \ \forall k \geq 0, \ \overline{\gamma} \leq \sigma^k(\gamma) \leq \gamma\right\}.$$

Let $q \geq 2$ be an integer. A sequence $\theta = (\theta_n)_{n\geq 0}$ is called a *q-mirror sequence* if

$$\forall \ell \geq 0, \ \forall j \in [0, q2^\ell - 2], \ \theta_{q2^\ell + j} = \overline{\theta}_j \ \text{ and } \ \forall \ell \geq 0, \ \theta_{q2^\ell - 1} = 1.$$

Define the map φ on periodic sequences of the form $(u_0 \ldots u_d \ 0)^\infty$ by

$$\varphi((u_0 \ldots u_d \ 0)^\infty) = (u_0 \ldots u_d \ 1 \ \overline{u}_0 \ldots \overline{u}_d \ 0)^\infty.$$

Theorem 3 ([3]) *Let θ be a q-mirror sequence with $\theta_0 = \theta_1 = 1$ and $q \geq 2$. Let m be the binary word $m = \theta_0\theta_1 \ldots \theta_{q-2}$ if $q \geq 3$, and $m = 110$ if $q = 2$. We have the following properties.*

(i) *The sequence θ belongs to Γ if and only if the sequence $(m0)^\infty$ belongs to Γ and has minimal period q if $q \geq 3$, and 4 if $q = 2$.*

(ii) *If the sequence θ belongs to Γ, then*

$$\Gamma \cap \{\gamma,\ (m0)^\infty \leq \gamma < \theta\} = \{\varphi^t((m0)^\infty),\ t = 0, 1, \ldots\}.$$

(iii) *If the sequence θ belongs to Γ, then, for any binary sequence η such that $\eta > \theta$, the set $\Gamma \cap \{\gamma,\ \theta < \gamma < \eta\}$ is non-countable.*

(iv) *The set of sequences that are in Γ and periodic is dense in Γ.*

(v) *Every non-countable subset of Γ contains a subinterval isomorphic to Γ as ordered set of binary sequences.*

Remark 1 It was furthermore noted in [3] that the only 2-mirror sequence in Γ is the sequence $110100110010110\ldots$ which is the shifted sequence of the Thue-Morse sequence, and that all q-mirror sequences are 2-automatic (about the Thue-Morse sequence see [17, 18, 16], about automatic sequences, see for example [8, 6, 7, 11, 2]).

Remark 2 A transformation similar to τ was used in [13], but only periodic and anti-periodic sequences were studied in this paper.

Corollary 1 *The least nonperiodic element of the set Γ is the (unique) 2-mirror sequence $110100110010110\ldots$ which is the shifted sequence of the Thue-Morse sequence.*

Proof.

It suffices to note that the sequence $(1100)^\infty$ is the least element of Γ: namely this sequence trivially belongs to Γ; any sequence in Γ begins with 11, and hence is larger than $1100\ldots$; finally if a sequence in Γ begins with 1100, it must be equal to $(1100)^\infty$ (see [1, Lemma 2, pages 26–27], see also [14, Lemma 2]). $\square$

Remark 3 Another proof that the sequence $11010011001\ldots$ belongs to Γ was given in [4].

3 Theorem 2 is identical to Corollary 1

It suffices to show the following proposition.

Proposition 1 *A binary sequence is admissible if and only if it is in Γ, and either equal to 1^∞ or nonperiodic.*

Proof.

The sequence 1^∞ is clearly admissible. A sequence that is both nonperiodic and belongs to Γ is clearly admissible, since all inequalities must be strict: if $\sigma^k(\gamma) = \gamma$, then γ is periodic; if $\overline{\gamma} = \sigma^k(\gamma)$, then $\sigma^{2k}(\gamma) = \sigma^k(\sigma^k(\gamma)) = \sigma^k(\overline{\gamma}) = \overline{\sigma^k(\gamma)} = \overline{\overline{\gamma}} = \gamma$, and γ is periodic.

Conversely, suppose that the sequence γ is admissible. We first note that if γ is equal to 1^∞, then it is clearly extremal. We hence can suppose that γ is not equal to the sequence 1^∞. Furthermore γ_0 must be equal to 1, otherwise the admissibility condition would imply that all γ_k's are zero, which is clearly not possible. Finally note that if an admissible sequence is ultimately equal to 1^∞, it cannot contain any 0, hence is equal to the sequence 1^∞ itself.

We prove that for all $k > 0$ we have $\sigma^k(\gamma) < \gamma$ (the proof of the inequality $\overline{\gamma} < \sigma^k(\gamma)$ is similar). If $\gamma_{k-1} = 0$ this is clear from the definition of admissibility. If $\gamma_{k-1} = 1$, either the sequence γ begins with k symbols 1 (i.e., $\gamma_0 = \gamma_1 = \ldots = \gamma_{k-1} = 1$), or it is of the form $\gamma_0\gamma_1 \ldots \gamma_d 0 1^b \gamma_k \gamma_{k+1} \ldots$ for some $b \geq 1$, and some $d \geq 1$. In the first case, it is clear that $\sigma^k(\gamma) < \gamma$, since γ^k begins with strictly fewer symbols 1 than γ. In the second case the definition of admissibility implies that $1^b\gamma_k\gamma_{k+1} \ldots < \gamma_0\gamma_1 \ldots$. Hence $\gamma_k\gamma_{k+1} \ldots < 1^b\gamma_k\gamma_{k+1} \ldots < \gamma_0\gamma_1 \ldots$ $\square$

Remark 4 This easy proposition is actually given without proof in [12, Remark 1 page 379].

4 Where number theory enters the picture

We first answer a question posed in the electronic abstract of [14]: see
`http://www.maa.org/pubs/monthly_augsept98_toc.html`
where the authors ask whether the number $q = 1.787231650...$ in Theorem 1 is irrational.

Theorem 4 *The number $q = 1.787231650...$ defined as the smallest number in $(1, 2)$ for which there exists a unique expansion of 1 as $1 = \sum_{n=1}^{\infty} \delta_n q^n$, with $\delta_n \in \{0, 1\}$, is transcendental.*

Proof.

Mahler proved in [15, page 363] that $F(a)$ is transcendental for every algebraic number a, with $0 < |a| < 1$, where F is defined for $|z| < 1$ by

$$F(z) = \prod_{n=0}^{\infty} \left(1 - z^{2^n}\right).$$

It is easy to see that

$$F(z) = 1 + \sum_{n=1}^{\infty} (-1)^{\delta_n} z^n$$

where $(\delta_n)_{n \geq 1}$ is the sequence in Theorem 2, i.e., the shifted sequence of the Thue-Morse sequence. Since δ_n takes values 0 and 1, we have $(-1)^{\delta_n} = 1 - 2\delta_n$. Hence

$$F(z) = 1 + \frac{z}{1-z} - 2\sum_{n=1}^{\infty} \delta_n z^n.$$

Taking $z = q^{-1}$ where q is the number introduced in Theorem 1, we obtain

$$F(\frac{1}{q}) = 1 + \frac{1}{q-1} - 2\sum_{n=1}^{\infty} \delta_n q^{-n} = 1 + \frac{1}{q-1} - 2 = \frac{2-q}{q-1}.$$

If q were algebraic, this would imply that the lefthandside of the above equality is transcendental (from Mahler's result), although the righthand side would be algebraic. $\Box$

Suppose now we associate with any sequence $(\gamma_n)_{n\geq 0}$ the binary number

$$x(\gamma) = \sum_{n=0}^{\infty} \gamma_n 2^{-n-1} \in [0,1].$$

Let

$$\Gamma' := \{x \in [0,1],\ \forall k \geq 0,\ 1 - x \leq \{2^k x\} \leq x\}$$

where $\{x\}$ denotes the fractional part of the real number x. We clearly have that $x(\gamma)$ belongs to Γ' if and only if either γ belongs to Γ or $\gamma = (10)^\infty$. Namely any number $x(\gamma)$ in Γ' must satisfy $1 - x(\gamma) \leq x(\gamma)$, i.e., $\gamma_0 = 1$. Then either $\gamma_1 = 1$ and the sequence γ is extremal, or $\gamma_1 = 0$ and the same proof as in [1, Lemma 2, pages 26–27], or [14, Lemma 2] gives that $\gamma = (10)^\infty$.

Define a q-mirror number as a real number in $[0,1]$ having a q-mirror binary expansion. Define the map φ' on the real numbers of the form $\frac{2a}{2^q-1}$, with $a < 2^{q-1} - 1$, by

$$\varphi'\left(\frac{2a}{2^q-1}\right) = \frac{2^{q+1}a + 2^{q+1} - 2 - 2a}{2^{2q}-1}.$$

Translating Theorem 3, we obtain the following.

Theorem 5 *Let $g \geq 1/2$ be a q-mirror number, with $q \geq 2$. Define $r(g) = \frac{[2^q g]}{2^q-1}$. We have the following properties.*

(i) *The number g belongs to Γ' if and only if the rational number $r(g)$ belongs to Γ' and it cannot be written $\frac{a}{2^j-1}$ with $j < q$, and $0 < a < 2^j - 1$.*

(ii) *If the number g belongs to Γ', then*

$$\Gamma' \cap \{x,\ r(g) \leq x < g\} = \{(\varphi')^t(r(g)),\ t = 0, 1, \ldots\}.$$

Furthermore the sequence $(\varphi')^t(r(g))$ increases to g as $t \to \infty$.

(iii) *If the number g belongs to Γ', then for any number h in Γ' such that $h > g$, the set $\Gamma' \cap \{x,\ g < x < h\}$ is non-countable.*

(iv) *The set of rationals having a purely periodic binary expansion and that belong to Γ' is dense in Γ'.*

(v) *Every non-countable subset of Γ' contains a subinterval isomorphic to Γ' as ordered set of real numbers, i.e., Γ' is a self-similar set.*

Remark 5 This theorem and generalizations where base 2 is replaced by any integer base are given in [1, Chapter 3].

5 Conclusion

More links between q-expansions of real numbers, discrete dynamical systems, and number theory (in particular Salem numbers) can be found in [5].

References

[1] J.-P. Allouche, *Théorie des Nombres et Automates*, Thèse d'État, 1983, Université Bordeaux I.

[2] J.-P. Allouche, Automates finis en théorie des nombres, *Exposition. Math.* **5** (1987) 239–266.

[3] J.-P. Allouche, M. Cosnard, Itérations de fonctions unimodales et suites engendrées par automates, *C. R. Acad. Sci. Paris, Série I* **296** (1983), 159–162.

[4] J.-P. Allouche, J. Currie, J. Shallit, Extremal infinite overlap-free binary words, *Electronic J. Combinatorics* **5** (1) (1998).

`http://www2.combinatorics.org/Volume_5/v5i1toc.html`

`http://www.emis.de/journals/EJC/Volume_5/v5i1toc.html`

[5] D. W. Boyd, The beta expansion for Salem numbers, in: J. Borwein et al. eds., Organic mathematics. Proceedings of the workshop, Simon Fraser University, Burnaby, Canada, December 12-14, 1995. Providence, RI: American Mathematical Society. CMS Conf. Proc. **20** (1997) 117–131.

`http://www.cecm.sfu.ca/organics/papers/boyd/index.html`

[6] G. Christol, Ensembles presque périodiques k-reconnaissables, *Theoret. Comput. Sci.* **9** (1979) 141–145.

[7] G. Christol, T. Kamae, M. Mendès France, G. Rauzy, Suites algébriques, automates et substitutions, *Bull. Soc. Math. France* **108** (1980) 401–419.

[8] A. Cobham, Uniform tag sequences, *Math. Systems Theory* **6** (1972) 164–192.

[9] P. Collet, J.-P. Eckmann, *Iterated maps on the interval as dynamical systems*, Progress in Physics, Birkhäuser, 1980.

[10] M. Cosnard, Étude de la classification topologique des fonctions unimodales, *Ann. Inst. Fourier* **35** (1985) 59–77.

[11] F. M. Dekking, M. Mendès France, A. J. van der Poorten, Folds!, *Math. Intelligencer* **4** (1982) 130–138, 173–181, 190–195.

[12] P. Erdős, I. Joó, V. Komornik, Characterization of the unique expansions $1 = \sum q^{-n_i}$ and related problems, *Bull. Soc. Math. France* **118** (1990) 377–390.

[13] L. Jonker, Periodic orbits and kneading invariants, *Proc. London Math. Soc.* **3** (1979) 428–450.

[14] V. Komornik, P. Loreti, Unique developments in non-integer bases, *Amer. Math. Monthly* **105** (1998) 636–639.

[15] K. Mahler, Arithmetische Eigenschaften der Lösungen einer Klasse von Funktionalgleichungen, *Math. Annalen* **101** (1929), 342–266. Corrigendum **103** (1930), 532.

[16] M. Morse, Recurrent geodesics on a surface of negative curvature, *Trans. Amer. Math. Soc.* **22** (1921) 84–100.

[17] A. Thue, Über unendliche Zeichenreihen, *Norske vid. Selsk. Skr. Mat. Nat. Kl.* **7** (1906) 1–22. Reprinted in *Selected Mathematical Papers of Axel Thue* T. Nagell, editor, Universitetsforlaget, Oslo, 1977, pp. 413–478.

[18] A. Thue, Über die gegenseitige Lage gleicher Teile gewisser Zeichenreihen, *Norske vid. Selsk. Skr. Mat. Nat. Kl.* **1** (1912) 1–67. Reprinted in *Selected Mathematical Papers of Axel Thue*, T. Nagell, editor, Universitetsforlaget, Oslo, 1977, pp. 413–478