NUDELMAN INTERPOLATION, PARAMETRIZATIONS OF LOSSLESS FUNCTIONS AND BALANCED REALIZATIONS.

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Abstract: We investigate the parametrization issue for discrete-time stable all-pass multivariable systems by means of a Schur algorithm involving a Nudelman interpolation condition. A recursive construction of balanced realizations is associated with it, that possesses a very good numerical behavior. Several atlases of charts or families of local parametrizations are presented and for each atlas a chart selection strategy is proposed. These parametrizations allow for solving optimization problems within the fields of system identification and optimal control.

Keywords: Parametrization, Inner matrices, Interpolation algorithms, Scattering parameters, Differential geometric methods, Discrete-time systems, Multivariables systems.

1. INTRODUCTION

Lossless or stable allpass transfer functions play an important role in system theory mainly due to the Douglas-Shapiro-Shields factorization: any proper transfer function can be written as the product of a lossless function, which includes the dynamics of the system, and an unstable factor. In many problems, within the fields of identification, model reduction and optimal control, the unstable factor can be computed from the lossless one. These problems can thus be handled by optimization methods over the class of lossless functions of prescribed degree, or possibly a specified subclass. It is with such applications in mind, and in particular rational L^2 approximation, that we will approach this parametrization issue.

In view of these applications, the manifold structure of the class of lossless functions of fixed degree (Alpay *et al.*, 1994) will be used and parametrizations coming from an atlas of chart will be considered. An atlas of chart attached with a manifold is a collection of local coordinate maps (the charts), whose domains cover the manifold and such that the changes of coordinates are smooth. A search algorithm can be run through the manifold as a whole, using a local coordinate map to describe the manifold locally and changing from one coordinate map for another when necessary. Such a representation presents a lot of advantages. It ensures identifiability, takes into account stability and preserves the order.

In the literature, atlases of charts have been derived both from the state-space approach using nice selections and from the functional approach using interpolation theory and Schur type algorithms. A connection between these descriptions was found in the scalar case (Hanzon and Peeters, 2000) and generalized to the matrix case (Hanzon *et al.*, 2004). This work is in the same vein : atlases are constructed in which lossless functions are represented by balanced realizations built recursively from interpolation data. Instead of the Nevanlinna-Pick interpolation problems used in previous works we consider in this work the more general Nudelman interpolation problems. This very general framework allows to construct several atlases, including that of (Hanzon *et al.*, 2004), and to describe the subclass of real functions. For each particular atlas presented in this work, we propose a chart selection strategy which provides an adapted chart for a given lossless function. This last point, together with their nice numerical behavior, make these parametrizations an interesting tool for solving the optimization problems mentioned before.

2. PRELIMINARIES.

Let

$$J = \begin{bmatrix} I_p & 0 \\ 0 & -I_p \end{bmatrix}.$$

For any matrix function $F(z)$, we define

$$F^\sharp(z) = F(\bar{z})^*. \tag{1}$$

A $2p \times 2p$ rational matrix function $\Theta(z)$ is called J-inner if, at every point of analyticity z of Θ it satisfies

$$\Theta(z)^* J \Theta(z) \leq J, \quad |z| < 1, \tag{2}$$
$$\Theta(z)^* J \Theta(z) = J, \quad |z| = 1. \tag{3}$$

A $(p \times p)$ rational matrix function $F(z)$ is called lossless or stable all-pass (resp. inner), if and only if

$$F(z)F(z)^* \leq I_p, \quad |z| > 1 \text{ (resp. } |z| < 1), \tag{4}$$

with equality on the circle.

By analytic continuation, the identity on the circle extends almost everywhere, so that any rational lossless function $G(z)$ is invertible, its inverse being inner and given by $G(z)^{-1} = G^\sharp(z)$.

We denote by $\mathcal{L}_n^p$ the set of $(p \times p)$-lossless functions of McMillan degree n and by $\mathcal{U}_p$ the set of constant unitary matrices. The natural framework for these studies is that of complex functions. However, in most applications, systems are real-valued and their transfer functions T are real, that is satisfy the relation $\overline{T(z)} = T(\bar{z})$. We shall denote by $\mathcal{RL}_n^p$ the set of real $(p \times p)$-lossless functions. Even if the complex case includes the real case by restriction, a specific treatment is actually relevant and was the initial motivation for this work which notably improves (Marmorat *et al.*, 2003).

An important property of a lossless function is that if

$$G(z) = C(zI_n - A)^{-1}B + D,$$

is a balanced realization (always exists), then the associated *realization matrix*

$$R = \begin{bmatrix} D & C \\ B & A \end{bmatrix} \tag{5}$$

is *unitary* (see (Hanzon *et al.*, 2004) and the bibliography therein). Lossless function can be represented by unitary realization matrices.

Remark: The transfer functions that we consider are in discrete-time. However the parametrizations can be used for continuous-time systems using the usual Möbius transform.

Along with a $2p \times 2p$ rational function $\Theta(z)$ block-partitioned as follows

$$\Theta(z) = \begin{bmatrix} \Theta_1(z) & \Theta_2(z) \\ \Theta_3(z) & \Theta_4(z) \end{bmatrix}, \tag{6}$$

with each block of size $p \times p$, we associate the linear fractional transformations T_Θ which acts on $p \times p$ rational functions $F(z)$:

$$T_\Theta(F) = [\Theta_1 F + \Theta_2][\Theta_3 F + \Theta_4]^{-1}. \tag{7}$$

Linear fractional transformations occur extensively in representation formulas for the solution of various interpolation problems (Ball *et al.*, 1990). If $\Theta(z)$ is a J-inner matrix function, then the map T_Θ sends every lossless function onto a lossless function.

3. NUDELMAN INTERPOLATION FOR LOSSLESS FUNCTIONS

The problem is to find a $(p \times p)$ rational lossless function $G(z)$ which satisfies an interpolation condition of the form

$$\frac{1}{2i\pi} \int_{\mathbf{T}} G^\sharp(z) X \, (z\, I_d - W)^{-1} \, dz = Y, \tag{8}$$

where (X, W) is an observable pair and W is asymptotically stable (X is $p \times d$ and W is $d \times d$). Such a triple (W, X, Y) will be called Nudelman interpolation data. Note that if W is a diagonal matrix, this problem reduces to a Nevanlinna-Pick problem.

Let $\Theta_{W,X,Y}$ be the $(2p \times 2p)$ J-inner function builts from the interpolation data (W, X, Y) as follows:

$$\Theta_{W,X,Y}(z) = \left[I_{2p} - (z-1)C(z\,I_d - W)^{-1}P^{-1}(I_d - W)^{-*}C^*J\right] \quad (9)$$

where $C = \begin{bmatrix} X \\ Y \end{bmatrix}$ and P is the unique solution to the symmetric Stein equation

$$P - W^*PW = X^*X - Y^*Y. \quad (10)$$

Theorem 1. There exist a rational lossless function G satisfying the interpolation condition (8) if and only if the solution P of (10) is positive definite. In this case, any solution G can be represented by

$$G = T_\Theta(F),$$

for some lossless function F and where $\Theta = \Theta_{W,X,Y}H$, H being an arbitrary constant J-unitary matrix and $\Theta_{W,X,Y}$ being given by (9). Moreover,

$$\deg G = \deg F + d.$$

Proof. This result is a particular case of (Ball *et al.*, 1990, th.18.5.2), which describes all the Schur functions solution to a Nudelman interpolation problem. □

Let Λ and Π be $p \times p$ unitary matrices. Then the following important relations are satisfied

$$\begin{bmatrix} \Lambda & 0 \\ 0 & \Pi \end{bmatrix} \Theta_{W,X,Y} \begin{bmatrix} \Lambda^* & 0 \\ 0 & \Pi^* \end{bmatrix} = \Theta_{W,\Lambda X,\Pi Y} \quad (11)$$

$$T_{\Theta_{W,\Lambda X,\Pi Y}}(\Lambda F(z)\Pi) = \Lambda T_\Theta(F(z))\Pi^* \quad (12)$$

4. BALANCED REALIZATIONS

The aim of this section is to choose the arbitrary J-unitary factor H, so that the linear fractional transformation $\widetilde{G} = T_{\Theta_{W,X,Y}H}(G)$ gives rise to a simple construction of balanced realizations.

Let U and V be $(p+d) \times (p+d)$ unitary matrices partitioned as follows:

$$U = \begin{bmatrix} \alpha_U & M_U \\ k_U & \beta_U^* \end{bmatrix}, \quad V = \begin{bmatrix} \alpha_V & M_V \\ k_V & \beta_V^* \end{bmatrix}, \quad (13)$$

where k_U and k_V are $d \times d$, α_U, α_V, β_U and β_V are $p \times d$ and M_U and M_V are $p \times p$, and put

$$M = \begin{bmatrix} M_U & 0 \\ 0 & M_V \end{bmatrix}, \quad \alpha = \begin{bmatrix} \alpha_U \\ \alpha_V \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_U \\ \beta_V \end{bmatrix}. \quad (14)$$

Proposition 1. Let U and V be unitary matrices block-partitioned as in (13). Assume that $k_V\,z - k_U$ is invertible. The $(2p \times 2p)$ rational matrix function

$$\Phi_{U,V}(z) = M + \alpha(k_V\,z - k_U)^{-1}\beta^*J \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix} \quad (15)$$

is J-inner and has McMillan degree d.
A minimal matrix realization $\widetilde{R}$ of $\widetilde{G} = T_{\Phi_{U,V}}(G)$ can be computed from a minimal matrix realization R of G by

$$\widetilde{R} = \begin{bmatrix} U & 0 \\ 0 & I_k \end{bmatrix} \begin{bmatrix} I_d & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} V^* & 0 \\ 0 & I_k \end{bmatrix}, \quad (16)$$

where k is the McMillan degree of $G(z)$.

Proof. The case $d = 1$ has been studied in (Hanzon *et al.*, 2004). A similar approach can be developed when k_U and k_V are matrices. □

Proposition 2. Let (W, X, Y) be some Nudelman interpolation data. There exist unitary $(p+d) \times (p+d)$ matrices U and V and a $(2p \times 2p)$ J-unitary matrix $H_{W,X,Y}$ such that

$$\hat{\Theta}_{W,X,Y} \equiv \Theta_{W,X,Y}\,H_{W,X,Y} = \Phi_{U,V} \quad (17)$$

Proof. If (17) is satisfied, the function $\Phi_{U,V}$ cannot have a pole on the circle and can be rewritten

$$\Phi_{U,V}(z) = \left[I_{2p} - (z-1)\,\alpha(k_V\,z - k_U)^{-1}(k_V - k_U)^{-*}\alpha^*J\right]K$$

with $K = \Phi_{U,V}(1)$.

Comparing with (9), there must exist a transformation T such that $P = T^*T$, $k_U k_V^{-1} = TWT^{-1}$, $\alpha k_V^{-1} = CT^{-1}$. The matrix V being unitary, $\alpha_V^*\alpha_V + k_V^*k_V = I_d$, and thus

$$(T^{-*}Y^*YT^{-1} + I_d)^{-1} = k_V k_V^*. \quad (18)$$

Since $T^{-*}Y^*YT^{-1} + I_d$ is positive definite, a solution to (18) does exist. Let the matrices T and k_v be chosen such that for $Y = 0$, $T = I_d$ and $k_V = I_d$. We may choose for example the positive square roots: $T = P^{1/2}$ and $k_V = \left(T^{-*}Y^*YT^{-1} + I_d\right)^{-1/2}$. Then, we must have

$$\begin{cases} k_U &= TWT^{-1}k_V \\ \alpha_V &= \quad YT^{-1}k_V \\ \alpha_U &= \quad XT^{-1}k_V \end{cases} \quad (19)$$

and the first d columns of U and V are determined. Now, we must find M_U, β_U, M_V, β_V and H such that

$$\begin{bmatrix} M_U \\ \beta_U^* \end{bmatrix}^* \begin{bmatrix} M_U \\ \beta_U^* \end{bmatrix} = I_p \quad (20)$$

$$\begin{bmatrix} M_U \\ \beta_U^* \end{bmatrix} \begin{bmatrix} M_U \\ \beta_U^* \end{bmatrix}^* = I_{p+d} - \begin{bmatrix} \alpha_U \\ k_U \end{bmatrix} \begin{bmatrix} \alpha_U \\ k_U \end{bmatrix}^* \quad (21)$$

the same for M_V and β_V, and

$$H = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} = M + \alpha(k_V - k_U)^{-1}\beta^* J, \quad (22)$$

where M, α, and β are given by (14). Solving for these equations yields

$$H_1 H_1^* = \\ I_p + \alpha_U (k_V - k_U)^{-1}(I_d - k_V k_V^*)(k_V^* - k_U^*)^{-1}\alpha_U^*,$$

$$H_2 H_2^* = \\ I_p + \alpha_V (k_V - k_U)^{-1}(I_d - k_U k_U^*)(k_V^* - k_U^*)^{-1}\alpha_V^*,$$

and choosing for H_1 and H_2 the positive square roots, we have

$$\begin{cases} M_U H_1^* = I_p - \alpha_U k_V^*(k_V^* - k_U^*)^{-1}\alpha_U^*, \\ \beta_U^* H_1^* = (I_d - k_U k_V^*)(k_V^* - k_U^*)^{-1}\alpha_U^*, \end{cases} \quad (23)$$

$$\begin{cases} M_V H_2^* = I_p + \alpha_V k_U^*(k_V^* - k_U^*)^{-1}\alpha_V C^*, \\ \beta_v^* H_2^* = -(I_d - k_V k_U^*)(k_V^* - k_U^*)^{-1}\alpha_V^*. \end{cases} \quad (24)$$

Now, the matrices U and V are completely determined and we can put

$$H_{W,X,Y} = M + \alpha(k_V - k_U)^{-1}\beta^* J. \quad (25)$$

□

Note that the J-inner function $\hat{\Theta}_{W,X,Y}$ also satisfies (11) and (12).

Corollary 1. A unitary matrix realization $\widetilde{R}$ of $\widetilde{G} = T_{\hat{\Theta}_{W,X,Y}}(G)$ can be computed from a unitary matrix realization R of G by (16).

Remark. Note that if $Y = 0$, since the pair (X, W) satisfies $X^*X + W^*W = I_d$, we have that $P = I_d$ and $T = I_d$ too. Thus,

$$U = \begin{bmatrix} X & I_p - X(I_d - W^*)^{-1}X^* \\ W & -(I_d - W)(I_d - W^*)^{-1}X^* \end{bmatrix}$$

$$V = \begin{bmatrix} 0 & I_p \\ I_d & 0 \end{bmatrix}$$

so that the recursion (16) becomes

$$\left[\begin{array}{c|c} \widetilde{D} & \widetilde{C} \\ \hline \widetilde{B} & \widetilde{A} \end{array}\right] = \left[\begin{array}{c|cc} M_U D & X & M_U C \\ \hline \beta_U^* D & W & \beta_U^* C \\ B & 0 & A \end{array}\right]. \quad (26)$$

5. CHARTS FROM A SCHUR ALGORITHM.

Let $\sigma = ((X_1, W_1),\ (X_2, W_2), \dots, (X_l, W_l))$ be a sequence of observable pairs such that the W_j's are asymptotically stable, W_j is $n_j \times n_j$, X_j is $p \times n_j$, and

$$n = \sum_{j=1}^{l} n_j.$$

We further imposed the normalization condition (output normal pairs)

$$W_j^* W_j + X_j^* X_j = I_{n_j}. \quad (27)$$

A Schur algorithm associated with a sequence σ of observable pairs consists from a given lossless function G of degree n, in a recursive construction of lossless functions of decreasing degree $G_l = G, G_{l-1}, \dots$. Assume G_j has been constructed and put

$$Y_j = \frac{1}{2i\pi}\int_{\mathbf{T}} G_j^\sharp(z) X_j \left(z I_{n_j} - W_j\right)^{-1} dz.$$

If the solution P_j to the symmetric Stein equation

$$P_j - W_j^* P_j W_j = X_j^* X_j - Y_j^* Y_j$$

is positive definite, then from theorem 1, a lossless function G_{j-1} is defined by

$$G_j = T_{\hat{\Theta}_{W_j,X_j,Y_j}}(G_{j-1}).$$

If P_j is not positive definite, the construction stops.

A chart $(\mathcal{V}, \phi)$ of $\mathcal{L}_n^p$ attached with a sequence σ of observable pairs and with a chart $(\mathcal{W}, \psi)$ of $\mathcal{U}_p$, is defined as follows.

A function $G \in \mathcal{L}_n^p$ belongs to the domain $\mathcal{V}$ of the chart if the Schur algorithm allows to construct a complete sequence of lossless functions, $G = G_l, G_{l-1} \dots, G_0$, where G_0 is a constant lossless matrix in $\mathcal{W} \subset \mathcal{U}_p$.

The local coordinate map ϕ is defined by

$$\phi : G \in \mathcal{V} \to (Y_1, Y_2, \dots, Y_l, \psi(G_0)),$$

and the interpolation matrices Y_j are the *Schur parameters* of the function in the chart.

Theorem 2. A family of charts $(\mathcal{V}, \Phi)$ defines an atlas of $\mathcal{L}_n^p$ provided the union of their domains cover $\mathcal{L}_n^p$.

Atlases for the quotient $\mathcal{L}_n^p/\mathcal{U}_p$ are obtained using the properties (11) and (12). If G has Schur parameters $(Y_1, Y_2, \dots, Y_l)$ and constant unitary matrix G_0 in a given chart, and if $\Pi \in \mathcal{U}_p$, then $G\Pi^*$ has Schur parameters $(\Pi Y_1, \Pi Y_2, \dots, \Pi Y_l)$ and constant unitary matrix $G_0\Pi^*$. The quotient can be performed within a chart by imposing the last constant lossless matrix G_0 in the Schur algorithm to be the identity matrix.

6. SOME PARTICULAR ATLASES.

We present three atlases for which a chart selection strategy can be given. They all present some interest from the optimization viewpoint. The first one is for complex functions and it involves only schur steps in which the degree is increased by one. It allows for a search strategy of local minima by induction on the degree, which can be very helpful in some difficult optimization problems. The second one is the analog for real-valued functions. The third one involves only a schur step and is much more simple that the previous ones.

6.1 Adapted charts from realizations in Schur form.

6.1.1. The case of complex functions. Consider the charts associated with sequences of observable pairs (x_1, w_1), $(x_2, w_2), \dots, (x_n, w_n)$ in which the w_j's are complex numbers. In this case, the Nudelman interpolation condition (8) can be rewritten as a Nevanlinna-Pick interpolation condition

$$G(1/\bar{w}_j)^* x_j = y_j.$$

This is the atlas described in (Hanzon *et al.*, 2004). However, the normalization conditions differ. In (Hanzon *et al.*, 2004) the p-vectors x_j's have norm one, while in this work, the pairs (x_j, w_j)'s are input normal (27). Note that, in the case $d > 1$ the matrix X_j may fail to satisfy $X_j^* X_j$ positive definite and the normalization condition X_j unitary cannot be chosen. Moreover, condition (27) simplifies the procedure to obtain adapted charts.

An adapted chart for $G(z) \in \mathcal{L}_n^p$ will be a chart in which all the Schur parameters are null p-vectors. It is obtained as follows: let $(\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D})$ be a balanced realization of $G(z)$ in Schur form ($\widetilde{A}$ is upper triangular), and write it in the form

$$\widetilde{A} = \begin{bmatrix} w_n & \star \\ 0 & A_{n-1} \end{bmatrix}, \quad \widetilde{B} = \begin{bmatrix} \beta_n \\ B_{n-1} \end{bmatrix},$$
$$\widetilde{C} = \begin{bmatrix} x_n & \cdots \end{bmatrix},$$

where w_n is a complex number, x_n a column vector and β_n a row vector. Comparing with (26), choose the observable pair (x_n, w_n) and $y_n = 0$ and run the Schur algorithm with interpolation data (w_n, x_n, y_n). The lossless function G_{n-1} defined by $G = T_{\Theta_{w_n, x_n, y_n}}(G_{n-1})$ has realization $(A_{n-1}, B_{n-1}, C_{n-1}, D_{n-1})$, obtained by inverting (16), still in Schur form. This process can thus be repeated and we get a sequence of output normal observable pairs (x_j, w_j), the w_j's being the eigenvalues of $\widetilde{A}$. In the associated chart, the Schur parameters of $G(z)$ are $y_n = \ldots = y_1 = 0$.

6.1.2. The case of real functions. To deal with real functions we consider the charts associated with sequences of observable pairs (X_1, W_1), $(X_2, W_2), \dots, (X_n, W_n)$ in which the W_j's are either real numbers or real 2×2 matrices with complex conjugate eigenvalues.

To find an adapted chart for a given lossless function $G \in \mathcal{RL}_n^p$, we shall proceed as follows: we start from a balanced realization $(\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D})$ of $G(z)$ in which $\widetilde{A}$ is in real Schur form

$$\widetilde{A} = \begin{bmatrix} W_l & \star & \cdots & \star \\ 0 & W_{l-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \star \\ 0 & \cdots & 0 & W_1 \end{bmatrix},$$

where for $j = 1, \dots, l$, W_j is either a real number or a (2×2) block with complex conjugate eigenvalues. Let

$$\widetilde{C} = \begin{bmatrix} X_l \ \star \cdots \star \end{bmatrix},$$

where X_l is $p \times n_j$, the size of W_j. As previously, choose the observable pair (X_l, W_l) and $Y_l = 0$ and run the Schur algorithm from $G_l = G$ with interpolation data (W_n, X_n, Y_n). A new lossless function is obtained with a realization still in Schur form. Repeating this process, we get a sequence of output normal observable pairs (X_j, W_j), the W_j's being the diagonal blocks of $\widetilde{A}$, that index a chart in which the Schur parameters of $G(z)$ are all null matrices: $Y_l = \ldots = Y_1 = 0$.

The Schur algorithm attached with this sequence of interpolation data yields a Potapov factorization for real lossless functions, namely

$$G(z) = B_l(z) B_{l-1}(z) \cdots B_1(z),$$

where B_j is the real-valued lossless function

$$B_j(z) = \\ I_p - (z-1) X_j (z I_{n_j} - W_j)^{-1} (I_{n_j} - W_j^*)^{-1} X_j^*.$$

6.2 A one step Schur algorithm.

We only consider sequences of observable pairs composed with a single pair (X, W) in which W is $n \times n$ and X is $p \times n$. The recursion formula (16) has a particular interpretation in this context.

Let $G(z) = \mathcal{D} + \mathcal{C}(zI_n - \mathcal{A})^{-1}\mathcal{B}$ be a balanced realization of $G(z)$. Let Q be the unique solution to the Stein equation

$$Q - \mathcal{A}^* Q W = \mathcal{C}^* X, \tag{28}$$

and P the solution to (10) where Y is given by (8).

The contour integral (8) can be computed using

$$\left(\frac{1}{z}I_n - \mathcal{A}^*\right)^{-1} = z\sum_{j=0}^{\infty}(z\mathcal{A}^*)^j$$
$$(zI_n - W)^{-1} = z^{-1}\sum_{j=0}^{\infty}(z^{-1}W)^j,$$

and thus

$$Y = \frac{1}{2\pi}\int_{\mathbf{T}} \mathcal{D}^* X \left(\sum_{j=0}^{\infty}(z^{-1}W)^j\right) + \mathcal{B}^* z \left(\sum_{j=0}^{\infty}(z\mathcal{A}^*)^j\right) \mathcal{C}^* X \left(\sum_{j=0}^{\infty} z^{-j}W^j\right)\frac{dz}{z},$$

so that

$$Y = \mathcal{D}^* X + \mathcal{B}^* Q W. \tag{29}$$

Moreover, it is easily verified that

$$P = Q^* Q.$$

Equations (28) and (29) can be rewritten in a matrix form

$$\begin{bmatrix} Y \\ Q \end{bmatrix} = \begin{bmatrix} \mathcal{D}^* & \mathcal{B}^* \\ \mathcal{C}^* & \mathcal{A}^* \end{bmatrix} \begin{bmatrix} X \\ QW \end{bmatrix}, \tag{30}$$

Since $Y^*Y + Q^*Q = Y^*Y + P$ is positive definite we may define $\xi = (Y^*Y + Q^*Q)^{-1/2}$ and (30) yields

$$I_n = \begin{bmatrix} X\xi \\ QW\xi \end{bmatrix}^* \begin{bmatrix} \mathcal{D} & \mathcal{C} \\ \mathcal{B} & \mathcal{A} \end{bmatrix} \begin{bmatrix} Y\xi \\ Q\xi \end{bmatrix},$$

which can be completed into an analog of (16).

The observable pair (X, W) represents a lossless function $F(z) = D_F + X(zI_n - W)^{-1}B_F \in \mathcal{L}_n^p$ (see (Alpay *et al.*, 1994)). In this atlas, a chart of $\mathcal{L}_n^p$ can be defined by a lossless function of $\mathcal{L}_n^p$. The solution P to the Stein equation (10), which satisfies $0 \leq P \leq I_n$, measures the possibility to encode the given lossless function $G(z)$ in the chart associated with $F(z)$. An adapted chart for $G(z)$ will be the chart associated with the observable pair $(\mathcal{C}, \mathcal{A})$ of a balanced realization $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ of $G(z)$. Then $P = I_n$ and $Y = 0$. At the opposite, if P is only positive semi-definite, then $G(z)$ does not belong to the chart associated with $F(z)$.

7. CONCLUSION.

In this paper atlases of charts for the class of lossless function of fixed degree have been proposed. This differential geometric approach presents a lot of advantages since it ensures identifiability, takes into account the stability constraint and allows to run a search algorithm through the manifold as a whole. Moreover, lossless functions are represented by balanced realizations computed by a recursion formula from interpolation data which presents a nice numerical behavior. This formula is independent from the dimension of the interpolation data so that a same implementation can be used for different atlases.

This approach takes place into a very general framework, that of Nudelman interpolation and we hope it could allow to describe other classes of transfer functions, taking into account for example some particular structure of the realization coming from the physics. We also think that symmetric lossless function could be handled using two-sided Nudelman interpolation data.

These parametrizations have been used to handle the L^2 rational approximation problem. The atlas of section 6.1.1 has been first implemented, numerical examples from the literature and real-data simulations have been presented in (Marmorat *et al.*, 2002). This parametrization describes complex lossless function and even if it allows to approximate real-valued functions, a specific atlas is preferable. The atlas of section 6.2 has been recently implemented. It works for both complex and real-valued functions and its effectiveness has been demonstrated through the examples mentioned in (Marmorat *et al.*, 2002).

REFERENCES

Alpay, D., L. Baratchart and A. Gombani (1994). On the differential structure of matrix-valued rational inner functions. *Operator Theory : Advances and Applications*.

Ball, J.A., I. Gohberg and L. Rodman (1990). *Interpolation rational matrix functions*. Vol. 45 of *Operator Theory: Advances and Applications*. Birkhuser.

Hanzon, B. and R.L.M. Peeters (2000). Balanced parametrizations of stable siso all-pass systems in discrete-time. *MCSS* **13**, 240–276.

Hanzon, B., M. Olivi and R.L.M. Peeters (2004). Balanced realizations of discrete-time stable all-pass systems and the tangential schur algorithm. Technical Report 5111. INRIA.

Marmorat, J.-P., M. Olivi, B. Hanzon and R.L.M. Peeters (2002). Matrix rational h^2 approximation: a state-space approach using schur parameters. In: *Proceedings of the CDC02, Las-Vegas, USA*.

Marmorat, J.-P., M. Olivi, B. Hanzon and R.L.M. Peeters (2003). Schur parametrizations and balanced realizations of real discrete-time stable allpass systems. In: *Proceedings of the CDC03, Maui, HAWAI*.