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Laurent Baratchart, Martine Olivi

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Abstract: This paper deals with best rational approximation of prescribed McMillan degree to matrix-valued functions in the real Hardy space of the complement of the unit disk endowed with the Frobenius L_2-norm. We describe the topological structure of the set of approximants in terms of inner–unstable factorizations. This allows us to establish a two–sided tangential interpolation equation for the critical points of the criterion, and to prove that the rank of the error $F - H$ is at most $k - n$ when F is rational of degree k and H is critical of degree n. In the particular case where $k = n$, it follows that $H = F$ is the unique critical point, and this entails a local uniqueness result when approximating near–rational functions.

Key-words: Matrix rational approximation in H_2, Topology of rational matrices, Critical point analysis, Poles and zeros of rational matrix functions, Identification of linear dynamical systems.

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Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles, B.P. 93, 06902 Sophia Antipolis Cedex (France)
Téléphone : 04 92 38 77 77 - International : +33 4 92 38 77 77 — Fax : 04 92 38 77 65 - International : +33 4 92 38 77 65

Approximation rationnelle matricielle dans H_2, à degré de Mac-Millan fixé: points critiques et rang de l'erreur.

Résumé : Ce rapport traite de l'approximation rationnelle matricielle à degré de Mac-Millan fixé dans l'espace de Hardy réel du complémentaire du disque unité muni de la norme de Frobenius L_2. La structure topologique de l'ensemble des approximants est décrite en termes de factorisation intérieure–instable. Cela permet d'établir une équation d'interpolation tangentielle bilatérale satisfaite par les points critiques du critère, et de prouver que si F est rationnelle de degré k et H un point critique de degré n alors le rang de l'erreur $F-H$ est au plus égal à $k-n$. Il s'ensuit que lorsque $k=n$, $H=F$ est l'unique point critique. Il en découle un résultat d'unicité locale pour l'approximation de fonctions proches d'être rationnelles.

Mots-clés : Approximation rationnelle matricielle dans H_2, Structure topologique des matrices rationnelles, Etude des points critiques, Poles et zeros d'une matrice rationnelle, Identification de systèmes dynamiques linéaires.

1 Introduction.

This paper deals with best rational approximation of prescribed McMillan degree to matrix-valued functions in the real Hardy space $\bar{H}_2^{p\times m}$ of the complement of the unit disk endowed with the Frobenius L_2-norm. In other words, given a $p \times m$–valued function $F(z)$ with real Fourier coefficients belonging to the Hardy class of exponent 2 of the complement of the disk, we are concerned with minimizing

$$\|F-H\|_2^2 = \frac{1}{2\pi}\mathbf{Tr}\left\{\int_0^{2\pi}[F-H](e^{it})\,[F-H]^*(e^{it})\,dt\right\} \qquad (1)$$

as H ranges over $p \times m$ rational matrices of McMillan degree at most n with real coefficients that are analytic for $\{|z| \geq 1\}$ including at infinity. Here, the symbol **Tr** stands for the trace and the superscript * denotes transpose–conjugate. We may in fact dispense with the analyticity requirement on H for it is an elementary remark (see below) that any minimizer of (1) actually belongs to $\bar{H}_2^{p\times m}$ when F does.
Changing z into $1/z$ yields an equivalent formulation in the Hardy space $H_2^{p\times m}$ of the disk, and our choice of working with functions analytic outside rather than inside the unit circle is merely a question of convention; the one we use sometimes simplifies the matter because it avoids considering poles at infinity.
Compared to L^∞ meromorphic approximation which has been extremely popular ever since its connection to operator theory was first established in [1], and subsequently carried over to the matrix case notably in [2], [35], [25], [52] and [41], rational H_2 approximation is an underdeveloped subject. In the so-called *scalar case*, that is when $p = m = 1$, the H_2 problem is classical in approximation theory and was considered in [49], [21], [37], [20], [17], [45], [5], [10], and [13], for instance. Several generalizations to the matrix case are of course possible; the one considered here is naturally akin to System Theory since the McMillan degree is the state–dimension of the linear control system whose tranfer–function is H. In this connection, members of $\bar{H}_2^{p\times m}$ may be considered as transfer-functions of discrete-time systems whose impulse response is square summable. In view of the degree preserving isometry

$$H \longrightarrow \frac{1}{\sqrt{\pi}(s-1)}\,H\left(\frac{s+1}{s-1}\right)$$

between $\bar{H}_2^{p\times m}$ and its right half–plane analog (see e.g.[28], [5]), this approximation problem may equivalently be stated on the imaginary line, assuming this time that F is the Laplace transform of some member of $L_2(0,\infty)^{p\times m}$. Therefore, though we shall be working entirely on the circle, everything translates to the half–plane setting and to transfer–functions of continuous time systems whose impulse response is square summable. The present introduction may thus be understood in either setting replacing Fourier series by Fourier integrals.

Incentives to study this problem lie mostly with approximate modelling and identification of linear dynamical systems. While rational approximations to transfer functions are generally seeked because finite-dimensional linear models dwell within a rich and effective theory, specific connections between the error in the time and frequency domain respectively are induced by L_2 *criteria*. To illustrate this, observe from Parseval's identity that any minimizer of (1) also minimizes the L_2 error between impulse reponses; this quantity is in turn equivalent (equal in the scalar case) both to the operator norm $L_1^m \to L_2^p$ and to the operator norm $L_2^m \to L_\infty^p$ of $F-H$ in the time domain [19], [5] [47]. Another application of Parseval's theorem shows that $\|F-H\|_2$ is also the variance of the Euclidean norm of the difference between the output of F and the output of H when both are driven by the same white noise input. To lend perspective to the discussion, let us briefly digress on the more general case where the input is an arbitrary stationary process. Applying the spectral theorem to the shift operator on the Hilbert space of the process allows one to compute the variance of the output error as a weighted L_2 integral:

$$\frac{1}{2\pi}\mathbf{Tr}\left\{\int_0^{2\pi}[F-H](e^{it})\,d\Lambda\,[F-H]^*(e^{it})\right\}, \qquad (2)$$

where the non-negative matrix-valued measure Λ is the so-called spectral measure of the input process (that reduces to Lebesgue measure times identity when the latter is white noise) and F now has to belong, say, to the weighted Hardy space associated to the measure $\mathbf{Tr}\Lambda$ [44] [26] [27]. Though we shall not prove anything concerning this extended type of approximation, we want to emphasize that the spectral theorem, as applied to shift operators, stresses deep links between time and frequency representations and the isometric character of this theorem (that may be viewed as a far-reaching generalization of

Parseval's relation) is a fundamental reason why L_2 approximation problems arise in System Theory. This underlying motivation is pervasive as we consider functions with real coefficients only. However, all the results hold as well in complex rational approximation to functions with complex Fourier coefficients, and the proofs are *mutatis mutandis* the same. In spite of this similarity, the complex version does not subsume the real one because the best complex approximation to a function with real Fourier coefficients need not have real coefficients.

In this paper, we shall characterize the critical points of the map $H \to \|F-H\|_2^2$ on the manifold of rational matrices of degree n in terms of inner-unstable factorizations introduced in [18] for strictly noncyclic functions and carried over to operator-valued functions in [22]. This will provide us with a generalization of the interpolation property of stationary points sometimes known as Walsh's theorem in the scalar case [1] (see *e.g.* [42]). Next, we use these equations to prove that *the rank of the error $F - H$ is at most $k - n$ when F is rational of degree k and H is critical of degree n.* In the particular case where $k = n$, it follows that $H = F$ not only is the best approximant but also the unique critical point, in other words the local approximation problem is equivalent to the global one in this case. This has been known for a while in the scalar case [46]; to our knowledge, however, the matrix version remained open and appears in the literature in connection with the numerics of recursive identification (see e.g. [38], pp. 293-294). In fact, the present work stems from the author's interest in the so–called consistency problem [9].

By perturbation, we finally derive from what precedes the following local uniqueness property: *for F_0 a rational matrix of degree n and K a compact subset of rational matrices of degree n containing F_0 the best approximation to F in degree n is unique and is also the unique critical point of $\|F-H\|_2^2$ on K, as soon as F is close enough to F_0 in L_2-norm.* In spite of its weakness, this seems to be the first result available on uniqueness in the matrix case (some more is known in the scalar case, see [13] and [12]). Note that we did not assert, this time, uniqueness of a critical point over the whole manifold of

[1] With all the consideration which is due to Walsh's major mathematical achievements, the origin of this terminology seems obscure as this author treats the problem with fixed poles only. The earliest references on free poles known to the authors are [21] in case they are simple and [37] in general.

matrices of degree n. Such a property would require L_∞ rather than L_2-small perturbations, and is beyond the scope of the present paper.

2 The H_2 approximation problem.

Let $\mathbb{T}$ be the unit circle and $L_2(\mathbb{T})$ the real Hilbert space of square-summable functions satisfying the conjugate–symmetry $f(e^{-i\theta}) = \overline{f(e^{i\theta})}$ or, equivalently, whose Fourier coefficients are real. Let also $L_\infty(\mathbb{T})$ be the Banach space of essentially bounded functions with this conjugate–symmetry. The Hardy space H_2 (resp.H_∞) of the unit disk is the closed subspace of $L_2(\mathbb{T})$ (resp.$L_\infty(\mathbb{T})$) consisting of functions whose Fourier coefficients (a_n) satisfy $a_n = 0$ when $n < 0$; symmetrically, the conjugate Hardy space $\bar{H}_2$ (resp. $\bar{H}_\infty$) consists of functions for which $a_n = 0$ when $n > 0$, and we further single out in $\bar{H}_2$ the subspace $\bar{H}_{2,0}$ of functions such that $a_0 = 0$. Note the orthogonal decomposition

$$L_2 = H_2 \oplus \bar{H}_{2,0}.$$

It is well-known (see e.g. [30]) that members of H_2 turn out to be the non-tangential limits on $\mathbb{T}$ of functions holomorphic in the unit disk including at infinity, and satisfying the growth condition

$$\sup_{r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta \, < \, \infty.$$

Members of H_∞ correspond to bounded holomorphic functions in this process. Symmetrically, members of $\bar{H}_2$ are nontangential limits of functions holomorphic outside the unit disk and satisfying an analogous growth condition for $r > 1$; the subspace $\bar{H}_{2,0}$ then consists of functions vanishing at ∞. Thus, f belongs to H_2 (resp. to $\bar{H}_2$) if, and only if, it can be written as

$$f(z) = \sum_{k \geq 0} a_k \, z^k \quad (\text{ resp. } \; f(z) = \sum_{k \geq 0} a_k \, z^{-k} \;), \tag{3}$$

with

$$a_k \in \mathbb{R} \;\text{ and }\; \sum_{k \geq 0} a_k^2 \, < \, \infty;$$

moreover, $f \in \bar{H}_2$ belongs to $\bar{H}_{2,0}$ if and only if $a_0 = 0$. Note that (3) is the Taylor expansion at 0 (resp. at ∞) and at the same time the Fourier expansion if we substitute $z = e^{i\theta}$.
The space $L_2^{p\times m}(\mathbb{T})$ becomes a real Hilbert space when endowed with the scalar product

$$< F, G >= \frac{1}{2\pi}\mathbf{Tr}\left\{\int_0^{2\pi} F(e^{it})G^*(e^{it})\,dt\right\}. \qquad (4)$$

The corresponding norm will be denoted by $\| \ \|_2$ and the orthogonal decomposition

$$L_2^{p\times m} = H_2^{p\times m} \oplus \bar{H}_{2,0}^{p\times m} \qquad (5)$$

is still valid. We shall designate by

$$\pi^+ : L_2^{p\times m}(\mathbb{T}) \to H_2^{p\times m} \quad \text{and} \quad \pi^- : L_2^{p\times m}(\mathbb{T}) \to \bar{H}_{2,0}^{p\times m}$$

the orthogonal projections, obtained by setting to zero the Fourier coefficients of negative and non-negative index respectively.
We also equip the Banach space $L_\infty^{p\times m}(\mathbb{T})$ with the norm

$$\|F\|_\infty = \sup_\theta ||F(e^{i\theta})||,$$

where $||\ .\ ||$ denotes the operator norm $\mathbb{C}^m \to \mathbb{C}^p$. This makes $L_\infty^{p\times p}$ into a Banach algebra.
Clearly, a rational matrix belongs to $\bar{H}_2^{p\times m}$ if, and only if, the poles lie in the open unit disk and the degree of the numerator of each entry is not greater than the degree of the corresponding denominator; the matrix then belongs automatically to $\bar{H}_\infty^{p\times m}$ as well.
The *McMillan degree* of a rational matrix H can be defined in many different ways using for instance Hankel ranks, or Smith-McMillan forms, or else coprime factorizations, or also local multiplicities (see *e.g.* [32], [14], [4], [33]) and even intersection theory [40] [39]; when H is analytic at infinity, one may as well characterize the degree as the minimal possible value of n allowing us to write

$$H(z) = C(zI - A)^{-1}B + D, \qquad (6)$$

where A, B, C, D are matrices in $\mathbb{C}^{n\times n}$, $\mathbb{C}^{n\times m}$, $\mathbb{C}^{p\times n}$, and $\mathbb{C}^{p\times m}$ respectively. A 4-tuple (A, B, C, D) satisfying (6) with A of minimal size is called a *minimal* *realization*, and it is well-known that any other minimal realization is of the form $(PAP^{-1}, PB, CP^{-1}, D)$ for some invertible $P \in \mathbb{C}^{n\times n}$. In particular, the characteristic polynomial $\det(zI - A)$ depends on H only, and is called the *polynomial of poles* of H. Its roots are, indeed, the poles of H (see *e.g.* [32] [33]). The McMillan degree will be denoted by deg; whereas it agrees with the usual notion of degree for polynomials, it is no longer so for polynomial matrices and we shall write pdeg to mean the *polynomial degree* of such a matrix which is, by definition, the *maximum* of the degrees of its entries. Two properties of the McMillan degree that we tacitly use are invariance under a Möbius transform of the variable and invariance under taking the inverse when the later is defined.

We only consider rational matrices that are conjugate–symmetric; this is equivalent to the requirement that the entries be rational functions with real coefficients, and the matrices A, B, C, D appearing in (6) can likewise be chosen real in this case.

The matrix $\bar{H}_2$ rational approximation problem in degree n may now be stated as follows :

Given $F \in \bar{H}_2^{p\times m}$ and an integer $n \geq 1$, minimize $\|F - H\|_2$ as H ranges over rational functions analytic in the complement of the open unit disk, of McMillan degree at most n.

Because our approach to this problem is based on differentiation, we need to set up a few pieces of notation in differential geometry. Throughout, the words smooth and C^∞ are used interchangeably. If M is a manifold, modelled on some Banach space, the tangent space to M at x will be denoted by $\mathbf{T}_x M$. If $f : M_1 \to M_2$ is a smooth map between two manifolds, the symbol $\mathbf{D}f(x)$ is used to mean the derivative of f at $x \in M_1$, which is a linear map $\mathbf{T}_x M_1 \to \mathbf{T}_{f(x)} M_2$, and we denote by $\mathbf{D}f(x).v$ the effect of $\mathbf{D}f(x)$ on the vector v. In case $f(x_1, x_2, ..., x_k)$ is a function of k arguments, $\mathbf{D}_j f(x)$ designates the partial derivative with respect to x_j. If $f : M \to \mathbb{R}$, we say that $x \in M$ is *critical* if $\mathbf{D}f(x) = 0$. For these and other basic notions in differential geometry (such as charts, submanifolds, embeddings, bundles and the like), we refer *e.g.* to [36].

We shall begin our study by accounting for the remark, made in the introduction, that the $\bar{H}_2$ rational approximation problem is equivalent to minimizing (1) over *all* rational functions of McMillan degree at most n. To see this, observe that such a minimizer, say H_{opt}, has to lie in $L_2^{p\times m}(\mathbb{T})$ for it cannot have poles on the circle since the objective function would then be infinite. Therefore, applying (5) to $z^{-1}H_{opt}$ and noting that multiplication by z is an isometry yields an orthogonal decomposition

$$H_{opt} = H^- + H^+ \quad \text{with} \quad H^- \in \bar{H}_2^{p\times m} \quad \text{and} \quad H^+ \in zH_2^{p\times m}.$$

Since $F \in \bar{H}_2^{p\times m}$, we obtain from orthogonality :

$$\|F - H_{opt}\|_2^2 = \|F - H^-\|_2^2 + \|H^+\|_2^2. \tag{7}$$

However, H^- and H^+ have no pole in common, neither at finite nor infinite distance, so that expressing the McMillan degree as the sum of the local degrees at the poles [32] yields

$$\deg H^- + \deg H^+ = \deg H_{opt}.$$

In particular $\deg H^- \leq n$, whence (7) contradicts the optimality of H_{opt} unless $H^+ = 0$, showing that $H_{opt} \in \bar{H}_2^{p\times m}$ as desired.

Let us continue with general comments on the problem. Existence of a best approximant is established in [7], and it is shown in [8] that the problem is *locally normal*, namely local *minima* have exact degree n unless F has degree less than n. Uniqueness of a best approximant may fail to hold [7] but is nevertheless true in generic situations : a general result by Stechkin [15] entails that a best approximant is unique for F in some subset of second category in $\bar{H}_2^{p\times m}$. This was improved in [8] where it is shown that this subset may be chosen open and dense. Thus, we see that the argument of the *global minimum* is usually uniquely defined. There may however be lots of *local minima*, and we shall study more generally the critical points of the squared L_2 error $H \to \|F-H\|_2^2$ as H ranges over the manifold $\Sigma_{p,m}(n)$ of $p\times m$ rational matrices of McMillan degree n that are analytic in the complement of the open unit disk [16] [29] [27]. This manifold is smoothly embedded in $\bar{H}_2^{p\times m}$ [3] and, *provided F is not rational of degree less than n*, local best approximants lie among these critical points by normality.

The study is easily reduced to the case where F vanishes at infinity. Indeed, writing the Taylor expansions of F and H as

$$F(z) = \sum_{k=0}^{\infty} F_k z^{-k}, \qquad H(z) = \sum_{k=0}^{\infty} H_k z^{-k},$$

we get from the Parseval formula:

$$\|F - H\|_2^2 = \sum_{k=0}^{\infty} \mathbf{Tr}\left\{[F_k - H_k][F_k - H_k]^*\right\}.$$

We may differentiate this relation with respect to $H_0 = H(\infty)$, since the value at infinity does not affect the degree by (6), and equating the derivative to zero leads to $H_0 = F_0$. Thus we see that H is critical for F if and only if $H' = H - F_0$ is critical for $F - F_0$, and H' now lies on the $n(m+p)$ dimensional submanifold $\Sigma^0_{p,m}(n)$ of $\Sigma_{p,m}(n)$ consisting of functions vanishing at infinity.
In what follows, we limit ourselves to the case where F belongs to $\bar{H}^{p\times m}_{2,0}$ for simplicity. By what we just said however, the results carry over immediately to the F's which are analytic, though not zero, at infinity.

3 Structure of the set of approximants.

We first describe the manifold $\Sigma^0_{p,m}(n)$ in way which suits our purpose better than classical parametrizations in terms of nice realizations [16] [29] [27].
Recall that $Q \in H^{p\times p}_\infty$ is said to be *inner* if the matrix $Q(e^{it})$ is unitary a.e. on the unit circle. Naturally associated to Q is the space $QH^p_2 \subset H^p_2$ which is invariant by the shift operator (*i.e.* the multiplication by z), and it is a celebrated theorem by Beurling and Lax that any closed and shift-invariant subspace of H^p_2 which has full range (*i.e.* which is of complex dimension p a. e. when evaluated pointwise on the disk) arises in this manner from some inner matrix Q which is unique up to a right orthogonal factor (see *e.g.* [22] or [30]). This, actually, is a real version of the Beurling-Lax theorem because we work in the real Hardy space so we require conjugate–symmetry throughout; however, the classical proofs given in the references apply *mutatis mutandis* to this case (see *e.g.*[11]). If P is another matrix-valued function in $H^{p\times m}_\infty$, the subspace generated by Q and P has full range, since it contains QH^p_2. Thus, *if it is closed*, there exists some inner $\Theta \in H^{p\times p}_\infty$ such that

$$QH^p_2 + PH^m_2 = \Theta H^p_2.$$

We call Θ a *left g.c.d.* of P and Q, and it is defined up to right multiplication by some orthogonal matrix. When $\Theta = I_p$, we say that Q and P are *left* *coprime*; this equivalently means that there exists $X \in H_2^{p\times p}$ and $Y \in H_2^{m\times p}$ such that

$$QX + PY = I_p. \tag{8}$$

Right *g.c.d.*'s and right coprimeness are defined in a symmetric way. The notion of *g.c.d.* is usually attached to the submodule generated by Q and P in H_∞^p rather than H_2^p [22], namely to the representation

$$QH_\infty^p \, + \, PH_\infty^m \, = \, \Theta H_\infty^p.$$

Such a representation holds if and only if the left handside in the above equation is a weak–* closed subspace of H_∞^p (see [24, Chapter II, Theorem 7.5] whose matrix version is easily derived), in which case $QH_2^p \, + \, PH_2^m$ is closed in H_2 and the two definitions of a *g.c.d.* coincide. When Q is rational, which is our only concern here, this weak–* closedness always holds because $QH_\infty^p \, + \, PH_\infty^m \subset QH_\infty^p \subset \det Q \, H_\infty^p$ has finite codimension, since $\det Q$ is a finite Blaschke product. More general *g.c.d.*'s must be considered if the range is not full [22], but we shall not need this.
To any $H \in \bar{H}_\infty^{p\times m}$, we associate the closed shift-invariant subspace of $[H_2^p]^T$

$$\mathcal{S}_L(H) = \{v \in [H_2^p]^T; \;\; vH \in H_2^m\}, \tag{9}$$

where the superscript "T'' means transpose. Note in passing that $\mathcal{S}_L(H)$ is just the kernel of the Hankel operator

$$\begin{array}{ccc} [H_2^p]^T & \longrightarrow & \bar{H}_{2,0}^m \\ v & \longrightarrow & \pi^-(vH). \end{array}$$

The analytic side of Fuhrmann's realization theory basically consists in applying the Beurling-Lax theorem to such subspaces and the following result, which is proved for instance in [11], is merely a specialization to rationals of the Douglas-Shapiro-Shields factorization [18] as generalized to matrix-valued strictly noncyclic functions in [22].

Theorem 1 *Any rational function H in $\bar{H}_{2,0}^{p\times m}$ can be factored as*

$$H = Q^{-1}P \tag{10}$$

where Q, P are rational, and Q is inner while P belongs to $H_\infty^{p\times m}$. The inequality $\deg Q \geq \deg H$ *prevails. We may further impose that Q and P be left coprime, and the decomposition is then unique up to a common left orthogonal factor; in this case, Q is characterized by the fact that $\mathcal{S}_L(H) = [H_2^p]^T\, Q$. This coprimeness condition holds if and only if* $\deg H = \deg Q$.

When $\deg H = \deg Q$, we call (10) the *left Douglas-Shapiro-Shields factorization* of H and we ensure uniqueness by requiring $Q(0) = I_p$. The importance of this factorization in our study stems from the two following facts :
(i) The factors Q and P in (10) will allow us to parametrize $\Sigma^0_{p,m}(n)$ and will provide a nice multiplicative description of the tangent space.
(ii) Left multiplication by the inner $Q \in H_\infty^{p\times p}$ is an isometry $H_2^p \to H_2^p$.
Our goal in this section is to account for fact *(i)* by giving a topological content to Theorem 1. More precisely, we shall endow the set of pairs (Q, P) with a vector bundle structure and represent $\Sigma^0_{p,m}(n)$ as an open subset. We denote by $\mathcal{I}_n^p$ the set of normalized inner functions of McMillan degree n; thus, $Q \in H_\infty^{p\times p}$ belongs to $\mathcal{I}_n^p$ if and onfy if :

$$\left\{\begin{array}{l} \deg Q = n, \\ Q(e^{it})Q(e^{it})^* = I_p, \quad \forall t \in \mathbb{R}, \\ Q(1) = I_p. \end{array}\right. \tag{11}$$

If $Q \in \mathcal{I}_n^p$, then $Q(1/z)$ belongs to $\bar{H}_\infty^{p\times p}$ and therefore posseses a minimal realization :

$$Q(1/z) = C(zI - A)^{-1}B + D. \tag{12}$$

Let $q = \det(zI - A)$ be the polynomial of poles whose degree is n and whose roots lie in the unit disk hereafter denoted by $\mathbf{U}$. Equation (12) expresses $Q(1/z)$ as the quotient by q of some $p \times m$ polynomial matrix, of polynomial degree at most n. Substituting back $1/z$ for z and multiplying by z^n throughout, we obtain a fractional representation

$$Q = D_Q/\tilde{q}, \tag{13}$$

where D_Q is a polynomial matrix with $\mathrm{pdeg}\, D_Q \leq n$ and where, by definition,

$$\tilde{q}(z) = z^n\, q(1/z)$$

is the reciprocal polynomial of q. Such a representation holds whenever $Q(1/z)$ is analytic at infinity; what is peculiar to the fact that Q is inner is the identity

$$\tilde{D}_Q D_Q = q\tilde{q} I_p, \tag{14}$$

where $\tilde{D}_Q = z^n \ D_Q^T(1/z)$ by definition. Granted $Q(1) = I_p$, this implies $\det Q = q/\tilde{q}$ [11, prop.1]; since no cancellation can occur between q and $\tilde{q}$, it follows in particular that

$$\deg(Q_1 Q_2) = \deg Q_1 + \deg Q_2$$

whenever Q_1 and Q_2 are inner of the same size.

Theorem 2 *The set $\mathcal{I}_n^p$, is a smooth manifold of dimension np embedded in $H_\infty^{p\times p}$, on which the coefficients of D_Q and q given by* (13) *are smooth functions.*

Proof: it follows from [3, thm 2.2 and prop. 2.2] that $\mathcal{I}_n^p$ is indeed a manifold of the right dimension smoothly embedded in $H_\infty^{p\times p}$, and that the image of $\mathcal{I}_n^p$ under the substitution $z \to 1/z$ is a submanifold of $\Sigma_{p,m}(n)$. Since local coordinates on the latter may be obtained from canonical realizations, the map associating to a function analytic at infinity its polynomial of poles is smooth. Substituting back z for $1/z$, we see that the coefficients of q are smooth functions of $Q \in \mathcal{I}_n^p$. Finally,

$$Q \to D_Q = Q\tilde{q}$$

is also smooth as multiplication is bilinear and continuous in H_∞. □

For $Q \in \mathcal{I}_n^p$, let

$$\mathcal{F}_L(Q) = \{P \in H_2^{p\times m}, \quad Q^{-1}P \in \bar{H}_{2,0}^{p\times m}\}.$$

We define our bundle in a set-theoretic manner as

$$\mathcal{B}_n = \{(Q,P), \quad Q \in \mathcal{I}_n^p, \quad P \in \mathcal{F}_L(Q)\}.$$

With the aid of Theorem 2, we shall endow $\mathcal{B}_n$ with a fibered structure :

Theorem 3 *When embedded in $H_\infty^{p\times p} \times H_2^{p\times m}$, the set $\mathcal{B}_n$ is a smooth vector bundle whose base space is $\mathcal{I}_n^p$ and whose fiber above Q is $\mathcal{F}_L(Q)$. The fiber $\mathcal{F}_L(Q)$ has dimension mn over $\mathbb{R}$ and consists of rational matrix-valued functions.*

Proof: First notice that $P \in \mathcal{F}_L(Q)$ if and only if each column belongs to

$$\mathcal{H}(Q) = H_2^p \ \cap \ Q\bar{H}_{2,0}^p,$$

which is the complement of QH_2^p in the orthogonal decomposition

$$H_2^p = QH_2^p \oplus \mathcal{H}(Q).$$

From realisation theory [22], we know that the dimension of $\mathcal{H}(Q)$ over $\mathbb{R}$ is equal to the McMillan degree of Q namely n. Consequently, $\mathcal{F}_L(Q)$ is a mn-dimensional real vector space. Moreover, if $P \in \mathcal{F}_L(Q)$ and if we set $Q^{-1}P = H \in \bar{H}_{2,0}^{p\times m}$, we get from (13)

$$\tilde{q}P = D_Q \ H. \tag{15}$$

The left hand-side of (15) lies in $H_2^{p\times m}$, while the right hand-side is the sum of a polynomial matrix of polynomial degree at most $n-1$ and of a member of $\bar{H}_{2,0}^{p\times m}$. Hence $\tilde{q}P$ is in fact a polynomial matrix, that we denote by N_P:

$$P = \frac{N_P}{\tilde{q}}, \quad \text{with pdeg}\, N_P \leq n-1,$$

showing in particular that $\mathcal{F}_L(Q)$ consists of rational elements.
By (13) and (14), we have $Q^{-1} = \tilde{D}_Q/q$ so that $P \in \mathcal{F}_L(Q)$ if and only if

$$\frac{\tilde{D}_Q N_P}{q\tilde{q}} \in \bar{H}_{2,0}^{p\times m}.$$

Upon changing z into $1/z$ and dividing by z, we see that this is equivalent to the requirement

$$\frac{D_Q \tilde{N}_P}{\tilde{q}q} \in H_2^{p\times m},$$

where $\tilde{N}_P = z^{n-1}N_P(1/z)$ by definition. Checking the poles shows the above condition to hold if and only if q divides $D_Q\tilde{N}_P$.
Thus, the pmn coefficients of the entries of N_P satisfy a homogeneous system $\mathcal{S}$ of pmn linear equations, obtained by equating the remainder of the Euclidean division of $D_Q\tilde{N}_P$ by q to zero. Since we know $\mathcal{F}_L(Q)$ has dimension mn, the solutions of $\mathcal{S}$ are obtained by choosing mn principal indeterminates and solving for the others using Cramer's *formulae*. Let $u \to Q(u)$ be a local parametrization of $\mathcal{I}_n^p$ where u ranges over an open set $\mathcal{U} \subset \mathbb{R}^{np}$. Since the coefficients of the system of equations $\mathcal{S}$ are rational in the coefficients of q and in the coefficients of the entries of D_Q, they depend smoothly on u by Theorem 2 and we can shrink $\mathcal{U}$ so that some principal minor doesn't vanish, or equivalently so that the same principal indeterminates can be chosen throughout $\mathcal{U}$. In this way, we construct a smooth map

$$\begin{array}{ccc} \mathcal{U} \times \mathbb{R}^{nm} & \stackrel{\tau}{\longrightarrow} & \mathcal{I}_n^p \times H_2^{p\times m} \\ (u,v) & \longrightarrow & (Q(u), P(u,v)) \end{array}$$

such that for any $u \in \mathcal{U}$, the partial map

$$\begin{array}{ccc} \mathbb{R}^{nm} & \stackrel{\tau_u}{\longrightarrow} & \mathcal{F}_L(Q(u)) \\ v & \longrightarrow & P(u,v) \end{array}$$

is the linear isomorphism which solves for P with the principal indeterminates v.
Since $\mathcal{I}_n^p$ is embedded in $H_\infty^{p\times p}$, there exists an open set $\mathcal{O}$ in the latter such that $Q(\mathcal{U}) = \mathcal{I}_n^p \cap \mathcal{O}$, where $Q(\mathcal{U})$ is a shorthand for the image of $\mathcal{U}$ under the map $u \to Q(u)$. Thus, the set $\mathcal{O} \times H_2^{p\times m} \cap \mathcal{B}_n$ is open in $\mathcal{B}_n$ and is the image of τ by construction. Since τ has continuous inverse (v is recovered as a subset of the coefficients of the entries of $N_{P(u,v)} = \tilde{q}(u)P(u,v)$), we see that each point of $\mathcal{B}_n$ has an open neighborhood which is homeomorphic to a product $\mathcal{U} \times \mathbb{R}^{nm}$ in a fiber preserving manner. Hence, $\mathcal{B}_n$ is a vector bundle.
To show smoothness, we have to establish that τ is an immersion, namely that its derivative is injective with complemented image in the Banach space $H_\infty^{p\times p} \times H_2^{p\times m}$. The last condition is automatic for $\operatorname{Im} \mathbf{D}\tau$ is finite-dimensional. As to injectivity, observe that $\mathbf{D}\tau(u,v)$ is the linear map

$$\begin{array}{ccc} \mathbb{R}^{np} \times \mathbb{R}^{nm} & \longrightarrow & H_\infty^{p\times p} \times H_2^{p\times m} \\ (h,k) & \longrightarrow & (\mathbf{D}Q(u).h\ ,\ \mathbf{D}_1P(u,v).h + P(u,k)) \end{array}$$

which is clearly one-to-one since by definition $\mathbf{D}Q(u)$ and $P(u,.)$ are. $\square$
Functions of the form $Q^{-1}P$ with $(Q,P) \in \mathcal{B}_n$ are rational by the above theorem, but not necessarily of degree n for Q and P may fail to be left coprime at certain points. However, we need only discard those in order to obtain the topological version of Theorem 1 we seek:

Corollary 1 *The map*

$$(Q,P) \longrightarrow Q^{-1}P$$

is a diffeomorphism from an open subset $\mathcal{C}_n$ of $\mathcal{B}_n$ onto $\Sigma^0_{p,m}(n)$. The set $\mathcal{C}_n$ consists of those pairs (Q,P) that are left coprime.

Proof: let $\mathcal{J}_p$ be the group of invertible elements in $L^{p\times p}_\infty(\mathbb{T})$. The map

$$\begin{array}{rccc} \rho: & \mathcal{J}_p \times H_2^{p\times m} & \longrightarrow & L_2^{p\times m}(\mathbb{T}) \\ & (S,R) & \longrightarrow & S^{-1}R \end{array} \tag{16}$$

is smooth since taking the inverse is a smooth operation in any Banach algebra and since multiplication $L_\infty(\mathbb{T}) \times H_2 \to L_2(\mathbb{T})$ is bilinear and continuous. As $\mathcal{B}_n$ is, by Theorem 3, smoothly embedded in $H^{p\times p}_\infty \times H_2^{p\times m}$ which is a closed subspace of $L^{p\times p}_\infty(\mathbb{T}) \times H_2^{p\times m}$, the restriction of ρ to $\mathcal{B}_n$ is likewise smooth. By Theorem 1, ρ maps $\mathcal{B}_n$ onto the set $S^0_{p,m}(n)$ of rational functions analytic in the complement of the open unit disk, vanishing at infinity and of degree *at most* n. Because $\Sigma^0_{p,m}(n)$ is open in $S^0_{p,m}(n)$ (see *e.g.* [7]), the set $\mathcal{C}_n = \rho^{-1}(\Sigma^0_{p,m}(n))$ is open in $\mathcal{B}_n$. Thanks to Theorem 1 again, and to the normalization $Q(1) = I_p$, we see that $\rho : \mathcal{C}_n \to \Sigma^0_{p,m}(n)$ is bijective and that Q and P are left coprime if and only if $(Q,P) \in \mathcal{C}_n$. Since the dimensions of $\mathcal{B}_n$ and $\Sigma^0_{p,m}(n)$ agree to $n(m+p)$, we need only prove that

$$\mathbf{D}\rho : \mathbf{T}_{(Q,P)}\,\mathcal{C}_n \longrightarrow \mathbf{T}_{Q^{-1}P}\,\Sigma^0_{p,m}(n)$$

is injective for we shall be done by the inverse function theorem then.
Now $\mathcal{B}_n$ is a vector bundle by Theorem 3; let again

$$(u,v) \xrightarrow{\tau} (Q(u), P(u,v)) \in \mathcal{C}_n$$

be a local parametrization where $v \to P(u,v)$ is linear. Applying the chain rule to the function $\rho \circ \tau$ gives for $(h,k) \in \mathbb{R}^{np} \times \mathbb{R}^{nm}$:

$$\mathbf{D}(\rho \circ \tau)(u,v).(h,k) = -Q^{-1} \underbrace{\mathbf{D}Q(u).h}_{\in H_\infty^{p\times p}} Q^{-1}P + Q^{-1} \underbrace{\mathbf{D}_1 P(u,v).h}_{\in H_2^{p\times m}} + Q^{-1}P(u,k), \tag{17}$$

where we have dropped at obvious places the dependency of Q on u and of P on u and v.
Assume for some (h,k) that

$$\mathbf{D}(\rho \circ \tau)(u,v).(h,k) = 0.$$

Multiplying by Q on the left and using (17) yields

$$\mathbf{D}Q(u).h\ Q^{-1}P = \mathbf{D}_1 P(u,v).h + P(u,k). \tag{18}$$

Observe that $\mathbf{D}Q(u).h$ lies in $H_\infty^{p\times p}$ whence *a fortiori* in $H_2^{p\times p}$, and that the right hand-side of (18) also lies in $H_2^{p\times m}$. This entails that each row of $\mathbf{D}Q(u).h$ belongs to $\mathcal{S}_L(Q^{-1}P)$. By the left coprimeness of Q and P, using Theorem 1, we deduce that $\mathbf{D}Q(u).h = JQ$ for some $J \in H_2^{p\times p}$. Differentiating the relation $QQ^* = I_p$ as a function $\mathcal{I}_n^p \times \mathcal{I}_n^p \to L_\infty^{p\times p}(\mathbb{T})$ gives

$$(\mathbf{D}Q(u).h)\ Q^* + Q\ (\mathbf{D}Q(u).h)^* = 0,$$

so that

$$JQQ^* + QQ^*J^* = J + J^* = 0;$$

as $J \in H_2^{p\times p}$ and $J^* \in \bar{H}_2^{p\times p}$, we conclude that J is a constant antisymmetric real matrix. Let us briefly appeal to the closed subalgebra $\mathcal{A} \subset H_\infty$ consisting of functions with continuous boundary values. Evaluating at 1 is a linear and continuous map $\mathcal{A} \to \mathbb{R}$ (whereas it is not $H_\infty \to \mathbb{R}$), and it is clear that $\mathcal{I}_n^p \subset \mathcal{A}^{p\times p}$. Differentiating $Q(1) = I_p$ yields $\mathbf{D}Q(u).h(1) = 0$ whence $J = 0$. Since $\mathbf{D}Q(u)$ is injective, we get $h = 0$, and thus $k = 0$ also for $P(u,.)$ is injective. $\square$

4 Critical points.

We have been working so far with left Douglas-Shapiro-Shields factorizations but it is clear by transposition that we might as well have considered right factorizations of the form $H = PQ^{-1}$, where this time Q is inner of size $m \times m$ and still $P \in H_\infty^{p\times m}$, and where left coprimeness is replaced by right coprimeness. We shall now take advantage of this symmetry. To avoid confusion, we shall designate by Q_0 and P_0 the left factors that we have been using up to now, while Q_1 and P_1 will stand for the right factors. Thus, we shall write

$$H = Q_0^{-1}P_0 = P_1Q_1^{-1}. \tag{19}$$

Note that the right factor Q_1 is obtained by applying the Beurling–Lax theorem to the closed shift–invariant subspace

$$\mathcal{S}_R(H) \;=\; \{v \in H_2^m; \quad H\,v \in H_2^p\}, \tag{20}$$

and is characterized by the fact that $\mathcal{S}_R(H) \;=\; Q_1\, H_2^m$. We shall also define $\mathcal{F}_R(Q_1)$ to be the set of matrix-valued functions $P_1 \in H_2^{p\times m}$ such that $P_1Q_1^{-1}$ belongs to $\bar{H}_{2,0}^{p\times m}$. As $\mathcal{F}_L(Q_0)$ was nm dimensional, $\mathcal{F}_R(Q_1)$ is np dimensional over $\mathbb{R}$.
The usefulness of the double factorization (19) appears in the following result.

Proposition 1 *Let H belong to $\Sigma_{p,m}^0(n)$ and* (19) *be its Douglas-Shapiro-Shields factorization. Then,*

$$\mathbf{T}_H\Sigma_{p,m}^0(n) \;=\; \{T \in \bar{H}_{2,0}^{p\times m}; \;\; Q_0TQ_1 \in H_2^{p\times m}\}. \tag{21}$$

Proof: by Corollary 1, the tangent space to $\Sigma_{p,m}^0(n)$ at H is the image under $\mathbf{D}\rho$ of $\mathbf{T}_{(Q_0,P_0)}\mathcal{C}_n$, where ρ was defined in (16). Using (19), and (17) where Q and P must now be replaced by Q_0 and P_0 respectively, we see that any member of $\mathbf{T}_H\Sigma_{p,m}^0(n)$ lies in $Q_0^{-1}H_2^{p\times m}Q_1^{-1}$. Hence, the left hand-side of (21) is included in the right. To prove equality, we will show that the dimensions agree. Pick any T in the right hand-side of (21) and decompose Q_0T according to (5) :

$$Q_0T = T^+ + T^-,$$

with $T^+ \in H_2^{p\times m}$ and $T^- \in \bar{H}_{2,0}^{p\times m}$. Multiplying on the left by $Q_0^{-1} \in \bar{H}_\infty^{p\times p}$, we conclude since T and $Q_0^{-1}T^-$ belong to $\bar{H}_{2,0}^{p\times m}$ that $Q_0^{-1}T^+ \in \bar{H}_{2,0}^{p\times m}$ also, that is to say $T^+ \in \mathcal{F}_L(Q_0)$. Moreover, we deduce from

$$\underbrace{Q_0TQ_1}_{\in H_2^{p\times m}} = \underbrace{T^+Q_1}_{\in H_2^{p\times m}} + T^-Q_1$$

that $T^{-}Q_1 \in \mathcal{F}_R(Q_1)$. Thus, T belongs to

$$Q_0^{-1}\left[\mathcal{F}_L(Q_0) + \mathcal{F}_R(Q_1)Q_1^{-1}\right]$$

and the dimension of this space is at most $nm+np = \dim \Sigma^0_{p,m}(n)$ by Theorem 3. □

We are now ready to derive a critical point equation for the map

$$\begin{array}{cccc} \Phi_F & : & \Sigma^0_{p,m}(n) & \longrightarrow & \mathbb{R} \\ & & H & \longrightarrow & \|F-H\|_2^2. \end{array}$$

Theorem 4 *Let $H \in \Sigma^0_{p,m}(n)$ and* (19) *be the Douglas-Shapiro-Shields factorizations of H. Then, H is a critical point of Φ_F if and only if there exists $G \in \bar{H}^{p\times m}_{2,0}$ such that*

$$F - H = Q_0^{-1}GQ_1^{-1}. \tag{22}$$

Proof: let $T \in \bar{H}^{p\times m}_{2,0}$ and decompose Q_0TQ_1 according to (5) :

$$Q_0TQ_1 = T^{+} + T^{-},$$

with $T^{+} \in H_2^{p\times m}$ and $T^{-} \in \bar{H}^{p\times m}_{2,0}$. Upon multipying by Q_0^{-1} on the left and Q_1^{-1} on the right, we deduce from (21) that $Q_0^{-1}T^{+}Q_1^{-1}$ belongs to $\mathbf{T}_H\Sigma^0_{p,m}(n)$. Because multiplying by an inner matrix on the left or on the right is an isometry, we thus obtain the orthogonal decomposition:

$$\bar{H}^{p\times m}_{2,0} \;=\; \mathbf{T}_H\Sigma^0_{p,m}(n) \;\oplus\; Q_0^{-1}\bar{H}^{p\times m}_{2,0}Q_1^{-1}. \tag{23}$$

Since $\Sigma^0_{p,m}(n)$ is embedded in $\bar{H}^{p\times m}_{2,0}$, the condition for H to be critical can be expressed as

$$< F - H \, , \, \mathbf{T}_H\Sigma^0_{p,m}(n) >= 0,$$

which is equivalent to (22) by (23). □

In the scalar case, a classical result states that a best approximant has to interpolate the function with order two at the reciprocal of its poles. It is perhaps interesting to point out that (22) is a natural generalization of this, for the zeroes of Q_0^{-1} and Q_1^{-1} are the reciprocal of their poles (by the inner property) whereas the latter are also the poles of H (by realization theory). The way in

which the directions of the zeroes are taken into account in the matrix case depends, as we now see, from the Douglas-Shapiro-Shields factorization.
We will find it useful to rewrite (22) in two steps as

$$\left\{ \begin{array}{l} Q_0F = G_0 + P_0, \quad G_0 \in \bar{H}^{p\times m}_{2,0} \\ G_0 = GQ_1^{-1}, \quad G \in \bar{H}^{p\times m}_{2,0}; \end{array} \right. \tag{24}$$

identifying $\Sigma^0_{p,m}(n)$ with an open subset of $\mathcal{B}_n$ as in Corollary 1, the first equation in (24) is to be interpreted as projecting F onto the fiber $\mathcal{F}_L(Q_0)$, which defines P_0 as a function of Q_0, whereas the second equation means that Q_0 itself is critical.

5 The case of nearly rational functions

The main technical result of the paper is this :

Proposition 2 *Let $F \in \Sigma^0_{p,m}(k)$ and H be a critical point of $\Phi_F: \Sigma^0_{p,m}(n) \longrightarrow \mathbb{R}$ for $n \leq k$. Then the rank of the rational matrix $F - H$ is at most $k - n$.*

As an immediate consequence, we obtain:

Theorem 5 *Let $F \in \Sigma^0_{p,m}(n)$ and H be a critical point of $\Phi_F: \Sigma^0_{p,m}(n) \longrightarrow \mathbb{R}$. Then $H = F$.*

Remark: on one hand, it follows from Theorem 5 and Corollary 1 that $\Phi_F \circ \rho : \mathcal{C}_n \to \mathbb{R}$ has a unique critical point at F if $F \in \Sigma^0_{p,m}(n)$. On the another hand, $\Phi_F \circ \rho$ extends naturally to all of $\mathcal{B}_n$ but it need *not* have a unique critical point there: if $n = 1$ and $p = m = 2$ for instance, it is not too difficult to show there is a *continuum* of degenerate *maxima*.
The proof of Proposition 2 requires several steps and will occupy the remaining of the section.

5.1 Algebraic preliminaries.

If $\mathcal{R}$ is a principal ideal domain and $M \in \mathcal{R}^{p\times m}$ a matrix of rank r, recall [31] [34] [43] there exist matrices U and V such that

$$UMV = \mathrm{diag}\,\{d_1, d_2, ..., d_r, 0, ..., 0\}, \quad d_1|d_2|...|d_r, \quad d_i \neq 0,$$

where U and V are invertible in $\mathcal{R}^{p\times p}$ and $\mathcal{R}^{m\times m}$ respectively and where the symbol $|$ means "divides". Matrices like U and V will be termed *unimodular*, to mean they are invertible and that the entries of the inverse belong to $\mathcal{R}$ or equivalently that their determinant is a unit in $\mathcal{R}$. The d_i's are called the *invariant factors* of M, and they are uniquely defined as $d_1...d_k$ is the *g.c.d.* of the $k \times k$ minors of M.
If N is a matrix over the quotient field of $\mathcal{R}$, we may write $N = M/d$, where d is a common denominator, and apply what precedes to M. This yields unimodular matrices U and V such that

$$UNV = \Lambda \tag{25}$$

with

$$\Lambda = \mathrm{diag}\,\{\phi_1/\psi_1, ..., \phi_r/\psi_r, 0, ..., 0\}, \tag{26}$$

ϕ_i and ψ_i being coprime elements in $\mathcal{R} - \{0\}$ satisfying the divisibility properties

$$\phi_1|\phi_2|...|\phi_r \quad \text{and} \quad \psi_r|\psi_{r-1}|...|\psi_1.$$

The matrix Λ is called the *Smith-McMillan* form of N and is uniquely defined, up to multiplication by units. Indeed, it is easy to check the following two properties (compare [48] [32]):

P1 $\psi_1...\psi_k$ is the least common denominator of all non zero $l \times l$ minors of N with $1 \leq l \leq k$;

P2 $\phi_1...\phi_k$ is the greatest common divisor of the numerators of all $k\times k$ minors of N once these minors are written over the common denominator $\psi_1...\psi_k$.

If L and M are two matrices in $\mathcal{R}^{p\times m}$ and $\mathcal{R}^{p'\times m}$ respectively, then $[\mathcal{R}^p]^T L + [\mathcal{R}^{p'}]^T M$ is a submodule of $[\mathcal{R}^m]^T$, say of rank r, whence has a basis whose elements can be stacked into a matrix E:

$$[\mathcal{R}^p]^T L \,+\, [\mathcal{R}^{p'}]^T M = [\mathcal{R}^r]^T E.$$

Clearly E is uniquely defined up to left multiplication by a unimodular matrix, and is a right *g.c.d* of L and M, because any right divisor of L and M is a right divisor of E as well. In the particular case where $r = p$ and E is unimodular, we say that L and M are right coprime. This means there exists $X \in \mathcal{R}^{m\times p}$ and $Y \in \mathcal{R}^{m\times p'}$ giving rise to the Bezout identity:

$$XL + YM = I_m. \tag{27}$$

Left coprimeness is defined similarly by transposing everything.
We need a basic lemma about coprime factorizations which is of constant use in Fuhrmann's realization theory [23] where it is phrased over the polynomial ring. To any $p \times m$ matrix N defined over the *quotient field* of $\mathcal{R}$, let us attach a submodule of $\mathcal{R}^m$ defined by

$$\mathcal{W}(N) = \{x \in \mathcal{R}^m;\quad Nx \in \mathcal{R}^p\}, \tag{28}$$

which is an algebraic analog to $\mathcal{S}_R$ defined in (20).

Lemma 1 *Let $L \in \mathcal{R}^{p\times m}$ and $M \in \mathcal{R}^{m\times m}$, with* $\det M \neq 0$*. Then*

$$\mathcal{W}(LM^{-1}) = M\mathcal{R}^m \tag{29}$$

if, and only if, L and M are right coprime.

Proof: it is clear that the right hand-side of (29) is included in the left hand-side. If L and M are right coprime, the Bezout identity (27) yields

$$XLM^{-1} + Y = M^{-1},$$

from which we deduce the reverse inclusion. Assume now that (29) holds and let E be a right *g.c.d* for L and M. We may write

$$L = TE \text{ and } M = ZE, \quad \text{with } T \in \mathcal{R}^{p\times m},\ Z \in \mathcal{R}^{m\times m}.$$

Note that E is invertible over the quotient field of $\mathcal{R}$, for E divides M which is full rank by hypothesis. If E is not unimodular, at least one of the columns of E^{-1}, say x, has an entry which is not in $\mathcal{R}$. Then, it is easily checked that Mx belongs to $\mathcal{W}(LM^{-1})$ but *not* to $M\mathcal{R}^m$. $\square$

Naturally asociated to a $p \times m$ matrix N, defined over the *quotient field* of $\mathcal{R}$, are the so-called *module of poles* and *module of zeros* given respectively (see [50],[51]) as:

$$\mathbf{X}(N) \;=\; \frac{\mathcal{R}^m}{\mathcal{W}(N)} \qquad \mathbf{Z}(N) \;=\; \mathbf{Tor}\left(\frac{\mathcal{R}^p}{N\,\mathcal{W}(N)}\right), \tag{30}$$

where **Tor** stands for the torsion submodule. These are finitely generated torsion modules, and as such can be decomposed as direct sums of cyclic submodules whose anihilators form a increasing sequence of ideals (see *e.g.* [31], [34]). If we denote by (26) the Smith-McMillan form of N over $\mathcal{R}$, it is plain that these decompositions are nothing but

$$\mathbf{X}(N) = \sum_{k=1}^{r} \frac{\mathcal{R}^m}{(\psi_r)} \qquad \mathbf{Z}(N) = \sum_{k=1}^{r} \frac{\mathcal{R}^p}{(\phi_r)};$$

note that taking **Tor** in (30) gets us rid of the structural drop in rank of N. If LM^{-1} is a right coprime factorization over $\mathcal{R}$, it follows from Lemma 1 that

$$\mathbf{X}(LM^{-1}) = \mathbf{X}(M^{-1}) = \frac{\mathcal{R}^m}{M\mathcal{R}^m} \qquad \mathbf{Z}(LM^{-1}) = \mathbf{Z}(L) = \mathbf{Tor}\left(\frac{\mathcal{R}^p}{L\mathcal{R}^m}\right). \tag{31}$$

Since a matrix and its transpose have the same McMillan form, they share the same module of poles and the same module of zeros. In particular, if $M_0^{-1}L_0$ is a left coprime factorization over $\mathcal{R}$, we also have a left-sided analog to (31):

$$\mathbf{X}(M_0^{-1}L_0) = \mathbf{X}(M_0^{-1}) = \frac{\mathcal{R}^p}{M_0\mathcal{R}^p} \qquad \mathbf{Z}(M_0^{-1}L_0) = \mathbf{Z}(L_0) = \mathbf{Tor}\left(\frac{\mathcal{R}^p}{L_0\mathcal{R}^m}\right). \tag{32}$$

We shall consider two instances of $\mathcal{R}$ namely the polynomial ring in one variable $\mathbb{R}[z]$, and the ring of germs of analytic functions at $a \in \mathbb{C}$ that we denote by $\mathcal{G}_a$. Specifically, members of $\mathcal{G}_a$ are power series

$$b(z) = \sum_{k=0}^{\infty} b_k (z-a)^k$$

in the variable $(z-a)$, with complex coefficients b_k, having positive radius of convergence. The quotient ring of $\mathbb{R}[z]$ is the field $\mathbb{R}(z)$ of rational functions,

whereas the quotient ring $\mathcal{M}_a$ of $\mathcal{G}_a$ is the collection of germs of meromorphic functions at a. The ring $\mathcal{G}_a$ is much more than a principal ideal domain as it is a discrete valuation ring: it has only one prime namely $(z-a)$, so that any element which is nonzero when evaluated at a is in fact invertible.
Consider a $p \times m$ rational matrix N and let us first set $\mathcal{R} = \mathbb{R}[z]$. If the Smith-McMillan form of N is denoted by (26), the polynomials $\phi = \phi_1 \cdots \phi_r$ and $\psi = \psi_1 \cdots \psi_r$ are called the *polynomial of zeros*[2] and the *polynomial of poles* respectively. When N is analytic at infinity, Fuhrmann's theory [23] [22] shows there is a realization (A, B, C, D) of N where A is multiplication by z on $\mathbf{X}(N)$, so this definition of the polynomial of poles agrees with the one we gave in section 2. Next, we set $\mathcal{R} = \mathcal{G}_a$ and we regard N as a meromorphic matrix-valued function in $\mathcal{M}_a^{p\times m}$ for each $a \in \mathbb{C}$. Though we shall not keep notational distinction between N and its germ at a, the context will always make our meaning clear. If we write

$$\phi_i/\psi_i = (z-a)^{\sigma_i(a)}\lambda_i, \quad \sigma_i(a) \in \mathbb{Z}$$

where $\lambda_i \in \mathcal{G}_a$ has neither poles nor zeros at a, we get from the divisibility properties

$$\sigma_1(a) \leq \sigma_2(a) \leq \ldots \leq \sigma_r(a),$$

and it is clear that

$$\operatorname{diag}\{(z-a)^{\sigma_1(a)}, \ldots, (z-a)^{\sigma_r(a)}, 0, \ldots, 0\}$$

is the Smith-McMillan form of N over $\mathcal{G}_a$ because the λ_i's are units in that ring and because a unimodular matrix over $\mathbb{R}[z]$ is *a fortiori* unimodular over $\mathcal{G}_a$. The $\sigma_i(a)$'s are called the *structural indices* of N at a, and a is a pole of N if and only if some index is negative. The sum of the negative indices is called the degree of the pole and agrees with its multiplicity in the polynomial of poles. Similarly, a is termed a *zero* of N if some index is positive, the degree of the zero being the sum of the positive indices which is also its multiplicity in the polynomial of zeros. Note that a pole may well be at the same time a zero.

[2] In control theory the zeros of the polynomial of zeros are usually called *transmission zeros*.

5.2 Lemmata

We gather in this subsection three technical results that are used in the proof of Proposition 2. We begin with a specialization of the Smith-Mc-Millan form over $\mathcal{G}_a$ to the effect that the matrix V in (25) may be designed as the product of some upper triangular matrix by a *constant* matrix.

Lemma 2 *Let N belong to $\mathcal{G}_a^{p\times m}$ and Λ be its Smith-McMillan form. There exists a nonsingular matrix $R \in \mathbb{C}^{m\times m}$ such that*

$$NR = U\Lambda W, \tag{33}$$

for some unimodular matrices $U \in \mathcal{G}_a^{p\times p}$ and $W \in \mathcal{G}_a^{m\times m}$ such that W is upper triangular. Moreover, any two members of $\mathcal{G}_a^{p\times m}$ may be simultaneously decomposed in this fashion using the same matrix R.

Remark: more is in fact true, namely the matrix R may be chosen in a dense open subset of $\mathbb{C}^{m\times m}$, but the proof of this stronger assertion would be too long a digression. This lemma is not valid over $\mathbb{R}[z]$ but nevertheless holds for nonsingular matrices there [6].

Proof: let us write the entries of N as

$$n_{i,j} = \sum_{k=0}^{\infty} n_{i,j}^{(k)}(z-a)^k \quad \text{with} \quad n_{i,j}^{(k)} \in \mathbb{C}.$$

When $m = 1$ we may take R and W to be identity since the matrix V is not needed in (25), so we suppose $m > 1$. If $N = 0$, any R will do. Otherwise, we may factor out the *g.c.d.* of the entries, and therefore assume that the first invariant is 1 :

$$\Lambda = \operatorname{diag}\{1, d_2, ..., d_r, 0, ..., 0\}.$$

One of the entries, say n_{i_0,j_0}, is then prime to $(z-a)$ which means that $n_{i_0,j_0}^{(0)} \neq 0$. Let us multiply N on the right by the matrix R_λ whose columns are those of the identity matrix except for the first one which is $(1,\lambda)^T$ where $\lambda = (\lambda_2, ..., \lambda_m)$ and $\lambda_i \in \mathbb{C}$. We obtain a new matrix $S \in \mathcal{G}_a^{p\times m}$ with entries

$$s_{i,j} = \sum_{k=0}^{\infty} s_{i,j}^{(k)}(z-a)^k$$

and the power series $s_{i_0,1}$ has constant coefficient

$$s_{i_0,1}^{(0)} = n_{i_0,1}^{(0)} + \sum_{1<j\leq m} n_{i_0,j}^{(0)}\lambda_j. \tag{34}$$

Because $n_{i_0,j_0} \neq 0$, it is clear that (34) is nonzero, and therefore that $s_{i_0,1}$ is invertible in $\mathcal{G}_a$, whenever λ^T takes values in some dense open subset of $\mathbb{C}^{m-1}$. In short, NR_λ has an invertible entry in the first column for generic λ. In particular, if N' is another matrix in $\mathcal{G}_a^{p\times m}$ whith Smith-McMillan form

$$\Lambda' = \mathrm{diag}\,\{1, d_2', ..., d_{r'}', 0, ...0\},$$

we may choose λ so that $S' = N'R_\lambda$ also has an invertible entry in the first column, say $s'_{i_0',1}$. Fix λ with this property. If $p=1$, then $i_0 = i_0' = 1$ and R_λ does the job for we can write

$$NR_\lambda = \mathrm{diag}\,\{1, 0, ..., 0\} \begin{pmatrix} s_{1,1} & s_{1,2}\dots s_{1,m} \\ 0 & I_{m-1} \end{pmatrix},$$

$$N'R_\lambda = \mathrm{diag}\,\{1, 0, ..., 0\} \begin{pmatrix} s'_{1,1} & s'_{1,2}\dots s_{1,m} \\ 0 & I_{m-1} \end{pmatrix}$$

where the first factor in each right hand-side is indeed the Smith-McMillan form of N and N'. If $p>1$, we perform elementary operations on the rows of NR_λ and $N'R_\lambda$ to obtain unimodular matrices Z and Z' such that

$$ZNR_\lambda = \begin{pmatrix} s_{i_0,1} & * \\ 0 & N_1 \end{pmatrix} \quad \text{and} \quad Z'N'R_\lambda = \begin{pmatrix} s'_{i_0',1} & *' \\ 0 & N_1' \end{pmatrix}, \tag{35}$$

where $*$ and $*'$ are row vectors in $\mathcal{G}_a^{1\times(m-1)}$ whose value is unimportant. Looking at the *g.c.d.*'s of the minors, we see that the Smith-McMillan forms of N_1 and N_1' are

$$\Lambda_1 = \mathrm{diag}\,\{d_2, .., d_r, 0, ..., 0\} \quad \text{and} \quad \Lambda_1' = \mathrm{diag}\,\{d_2', .., d_{r'}', 0, ..., 0\}$$

respectively. By induction on p, there exists a nonsingular $R_1 \in \mathbb{C}^{(p-1)\times(p-1)}$ and unimodular matrices U_1, U_1', W_1, W_1' where the last two are upper triangular, such that

$$N_1R_1 = U_1\Lambda_1W_1 \quad \text{and} \quad N_1'R_1 = U_1'\Lambda_1'W_1'.$$

The nonsingular complex matrix

$$R = R_\lambda \begin{pmatrix} 1 & 0 \\ 0 & R_1 \end{pmatrix}$$

and the unimodular matrices

$$U = Z^{-1} \begin{pmatrix} 1 & 0 \\ 0 & U_1 \end{pmatrix}, \quad U' = (Z')^{-1} \begin{pmatrix} 1 & 0 \\ 0 & U_1' \end{pmatrix}$$

$$W = \begin{pmatrix} s_{i_0,1} & *R_1 \\ 0 & W_1 \end{pmatrix}, \quad W' = \begin{pmatrix} s'_{i'_0,1} & *'R_1 \\ 0 & W_1' \end{pmatrix}$$

satisfy $NR = U\Lambda W$, $N'R = U'\Lambda' W'$, and W, W' are upper triangular as desired. □

Lemma 2 is intrumental for the next result which is the major technical ingredient in the proof of Proposition 2.

Lemma 3 *Let* L', $L \in \mathcal{G}_a^{p\times m}$ *and* $M \in \mathcal{G}_a^{m\times m}$ *satisfy* $\det\, M \neq 0$ *and*

$$L' = LM^{-1}. \tag{36}$$

Assume that

$$\mathbf{Ker}\{L'\} \cap \mathcal{G}_a^m \subset M\mathcal{G}_a^m, \tag{37}$$

where $\mathbf{Ker}\{L'\} \subset \mathcal{M}_a^m$ *is the kernel of the left multiplication by* L' *from* $\mathcal{M}_a^m$ *to* $\mathcal{M}_a^p$. *Then, the degree of* a *when viewed as a zero of* M *is not greater than its degree when viewed as a zero of* L.

Proof: we may assume $L \neq 0$. Let

$$\Lambda = \mathrm{diag}\,\{\phi_1, ..., \phi_r, 0, ..., 0\},$$

be the Smith-McMillan form of L over $\mathcal{G}_a$; we have, say

$$\phi_k = (z-a)^{\mu_k}, \quad \text{with} \quad 0 \le \mu_1 \le ... \le \mu_r. \tag{38}$$

Denote by

$$\Delta = \mathrm{diag}\,\{d_1, ..., d_m\}$$

the Smith-McMillan form of M over $\mathcal{G}_a$, noticing that it is full rank whence

$$d_k = (z-a)^{\alpha_k} \quad \text{with} \quad \alpha_{k+1} \geq \alpha_k \geq 0. \tag{39}$$

By Lemma 2, there exist a nonsingular $R \in \mathbb{C}^{m\times m}$, unimodular matrices $U, V \in \mathcal{G}_a^{p\times p}$ and $W, X \in \mathcal{G}_a^{m\times m}$, with W and X upper triangular, such that

$$LR = U\Lambda W \tag{40}$$

and

$$MR = V\Delta X. \tag{41}$$

From (36), (40) and (41), we obtain

$$U^{-1}L'V = \Lambda Y \Delta^{-1} \tag{42}$$

where $Y = WX^{-1}$ is upper triangular and unimodular with entries, say $y_{i,j}$. The entry (i,j) of $U^{-1}L'V$ is thus $\phi_i y_{i,j}/d_j$ if $i \leq r$ and 0 otherwise. The matrix $U^{-1}L'V$ belongs to $\mathcal{G}_a^{p\times m}$, so that

$$d_j \mid \phi_i y_{i,j} \quad \forall i,j, \;\; 1 \leq i \leq r, \; 1 \leq j \leq m. \tag{43}$$

Since $y_{i,i}$ is a unit, we deduce that $\Pi_{k=1}^r d_k \mid \Pi_{k=1}^r \phi_k$. If $r = m$ we are done. If $r < m$, the matrix $U^{-1}L'V$ may be block-decomposed as

$$U^{-1}L'V = \begin{pmatrix} Z_1 & Z_2 \\ 0 & 0 \end{pmatrix} \tag{44}$$

where $Z_1 \in \mathcal{G}_a^{r\times r}$ is upper triangular with non-zero entries on the diagonal while $Z_2 \in \mathcal{G}_a^{r\times(m-r)}$. We shall denote by Z the matrix $(Z_1 \; Z_2)$.
We also define $Y_1 \in \mathcal{G}_a^{r\times r}$ and $Y_2 \in \mathcal{G}_a^{r\times(m-r)}$ to be the two submatrices of Y given by

$$\begin{aligned} Y_1 &= (y_{i,j}) \quad \text{for} \quad 1 \leq i \leq r \;\; \text{and} \;\; 1 \leq j \leq r, \\ Y_2 &= (y_{i,j}) \quad \text{for} \quad 1 \leq i \leq r \;\; \text{and} \;\; r+1 \leq j \leq m, \end{aligned} \tag{45}$$

noting that Y_1 is unimodular. Now, set $S = Z_1^{-1}Z_2 \in \mathcal{M}_a^{r\times(m-r)}$ and observe from (42) and (44) that S may be expressed as

$$S = \text{diag}\{d_1 \dots d_r\}\, Y_1^{-1} Y_2 \,\text{diag}\{d_{r+1}^{-1} \dots d_m^{-1}\}. \tag{46}$$

Consider $v \in \mathbf{Ker}\{Z\} \cap \mathcal{G}_a^m$; then $Vv \in \mathbf{Ker}\{L'\} \cap \mathcal{G}_a^m$ by the definition of Z. Using the hypothesis (37), $Vv \in M\mathcal{G}_a^m$ and (41) implies $v \in \Delta\mathcal{G}_a^m$. In other words, *each vector in* $\mathbf{Ker}\{Z\} \cap \mathcal{G}_a^m$ *has an i-th component which is divisible by* d_i. Since we can embed each $s \in \mathcal{W}(S)$ (see definition (28) with $\mathcal{R} = \mathcal{G}_a$) into the vector

$$\begin{pmatrix} -Ss \\ s \end{pmatrix} \in \mathbf{Ker}\{Z\} \cap \mathcal{G}_a^m,$$

we get in particular from the last fact that

$$\mathcal{W}(S) \subset \Delta_{m-r}\, \mathcal{G}_a^{m-r}, \quad \text{with} \quad \Delta_{m-r} = \text{diag}\,\{d_{r+1}, ..., d_m\}.$$

In another connection, it is clear from (46) that S can be factored as

$$S = \Omega\Delta_{m-r}^{-1} \quad \text{with} \quad \Omega = \text{diag}\{d_1 \ldots d_r\}\, Y_1^{-1} Y_2 \in \mathcal{G}_a^{r\times(m-r)}. \tag{47}$$

Hence, we conclude from Lemma 1 that Ω and Δ_{m-r} are right coprime :

$$J\Omega + K\Delta_{m-r} = I_{m-r}, \quad \text{for some } J \in \mathcal{G}_a^{(m-r)\times r} \text{ and } K \in \mathcal{G}_a^{(m-r)\times(m-r)}. \tag{48}$$

Because we work in the local ring $\mathcal{G}_a$, (48) will hold if, and only if, the complex matrix

$$\begin{pmatrix} \Omega(a) \\ \Delta_{m-r}(a) \end{pmatrix} \tag{49}$$

has full column rank $m - r$. Let ℓ be the integer such that $d_1 = ... = d_\ell = 1$ and $d_k \neq 1$ for $\ell < k$. If $\ell = m$, the lemma is trivial for M is unimodular and consequently has no zeros. If $\ell < m$, assume first that $r \leq \ell$. Then

$$\Delta_{m-r}(a) = \text{diag}\{\underbrace{1, \ldots, 1}_{\ell-r}, \underbrace{0, \ldots, 0}_{m-\ell}\}$$

and (49) has full column rank if, and only if, the last $m - \ell$ columns of $\Omega(a)$ are linearly independent over $\mathbb{C}$. Equivalently by (47) the last $m - \ell$ columns of $Y_2(a)$ are linearly independent. Certainly then, in view of (45), there exists permutations $j_1, ..., j_{m-\ell}$ of $\{\ell + 1, \ldots, m\}$ and distinct row indices $i_1, ..., i_{m-\ell}$ in $\{1, ..., r\}$ such that

$$\Pi_{k=1}^{m-\ell} y_{i_k, j_k}(a) \neq 0.$$

But d_{j_k} divides $y_{i_k,j_k}\phi_{i_k}$ as we have shown, and we now see that y_{i_k,j_k} is prime to $(z-a)$. Hence d_{j_k} divides ϕ_{i_k} for $1 \leq k \leq m-\ell$. As j_k ranges over all the indices for which the d_j's are nontrivial, this implies that the degree of a when viewed as a pole of M^{-1} is majorized by the degree of a when viewed as a zero of L.

Assume now that $\ell < r$. Then $\Delta_{m-r}(a) = 0$ and $\Omega(a)$ must have full column rank. Since

$$\mathrm{diag}\{d_1, \ldots, d_r\}(a) = \mathrm{diag}\{\underbrace{1, \ldots, 1}_{\ell}, \underbrace{0, \ldots, 0}_{r-\ell}\},$$

this means by (47) that

$$X_1(a)\, Y_2(a) \tag{50}$$

has full column rank, where X_1 is the submatrix of Y_1^{-1} consisting of the first ℓ rows. Let us partition X_1 and Y_2 according to

$$X_1 = \begin{pmatrix} \underbrace{X_{11}}_{\ell} & \underbrace{X_{12}}_{r-\ell} \end{pmatrix} \quad \text{and} \quad Y_2 = \begin{pmatrix} Y_{21} \\ Y_{22} \end{pmatrix}.$$

Observe that X_{11} is unimodular and set $\Upsilon = Y_{21} + X_{11}^{-1}X_{12}Y_{22}$, noticing that $\Upsilon \in \mathcal{G}_a^{\ell \times (m-r)}$ has entries of the form:

$$\upsilon_{i,j} = y_{i,r+j} + \sum_{k=\ell+1}^{r} \lambda_k y_{k,r+j} \tag{51}$$

with $\lambda_k \in \mathcal{G}_a$. Now, (50) has full column rank if, and only if Υ does when evaluated at a. Certainly then, there exists a permutation $j_1, ..., j_{m-r}$ of $\{1, \ldots, m-r\}$ and distinct row indices $i_1, ..., i_{m-r}$ in $\{1, ..., \ell\}$ such that

$$\Pi_{k=1}^{m-r} \upsilon_{i_k,j_k}(a) \neq 0,$$

so that υ_{i_k,j_k} is prime to $(z-a)$ for each k. As d_{r+j_k} divides $\upsilon_{i_k,j_k}\phi_{i_k}$ by (51) and (43), we conclude that d_{r+j_k} divides ϕ_{i_k} for $1 \leq k \leq m-r$. Hence each d_j with $r+1 \leq j \leq m$, divides a different ϕ_i for some $i \in \{1, \ldots, \ell\}$. Moreover, we know by (43) that d_j divides ϕ_j for $\ell+1 \leq j \leq r$.

As $\{\ell+1, \ldots, m\}$ is the set of indices j for which the d_j's are nontrivial, this again entails that the degree of a when viewed as a pole of M^{-1} is majorized by the degree of a when viewed as a zero of L. □

The last auxiliary result that we need is an easy invariance property of the Douglas-Shapiro-Shields factors:

Lemma 4 *Let T belong to $\Sigma^0_{p,m}(n)$ and*

$$T = Q_0^{-1}P_0 = P_1 Q_1^{-1}$$

be the Douglas-Shapiro-Shields factorization. If S lies in $H_\infty^{m\times m'}$, then $\pi^-(TS)$ is rational of degree at most n. If in addition S is left coprime to Q_1, then T and $\pi^-(TS)$ share the same left inner denominator namely we have a left Douglas-Shapiro-Shields factorization of the form

$$\pi^-(TS) = Q_0^{-1}\Xi_0. \tag{52}$$

Proof: write $Q_0 = D_{Q_0}/\tilde{q}_0$ according to (13). Since $\mathrm{pdeg}\, D_{Q_0} \leq n$, we see on the one hand that $D_{Q_0}\pi^-(TS)$ belongs to $z^n \bar{H}_{2,0}^{p\times m}$. On the other hand, using $\pi^+ + \pi^- = \mathrm{id}$, we may write

$$\pi^-(TS) = TS - \pi^+(TS) \tag{53}$$

so that

$$D_{Q_0}\pi^-(TS) = \tilde{q}_0 P_0 S - D_{Q_0}\pi^+(TS)$$

also belongs to $H_2^{p\times m}$ and consequently is a polynomial. Because D_{Q_0} is an invertible rational matrix, $\pi^-(TS)$ is in turn rational.
Assume now that S and Q_1 are left coprime. There exist $X \in H_2^{m'\times m}$ and $Y \in H_2^{m\times m}$ such that $SX + Q_1 Y = I_m$, and multiplying by T on the left using $\pi^+ + \pi^- = \mathrm{id}$ yields

$$\pi^-(TS)X + \pi^+(TS)X + P_1 Y = T.$$

This equation shows that $\mathcal{S}_L(\pi^-(TS)) \subset \mathcal{S}_L(T)$, and the converse inclusion is trivial from (53), whence (52) follows from Theorem 1. □

5.3 Proof of Proposition 2

Let

$$H = Q_0^{-1}P_0 = P_1 Q_1^{-1}$$

be the Douglas-Shapiro-Shields factorizations of H and

$$F = S_0^{-1} R_0 = R_1 S_1^{-1},$$

that of F.
If Q_0 and S_0 are not right coprime, let Λ_0 be their g.c.d. :

$$\left\{ \begin{array}{lcl} Q_0 & = & Q_{0,1}\,\Lambda_0 \\ S_0 & = & S_{0,1}\,\Lambda_0 \end{array} \right. \tag{54}$$

Since

$$Q_{0,1}\,\pi^-(\Lambda_0 H) = Q_0 H - Q_{0,1}\,\pi^+(\Lambda_0 H),$$

lies in $H_2^{p\times m}$, we get

$$[H_2^p]^T Q_{0,1} \subset \mathcal{S}_L(\pi^-(\Lambda_0 H)). \tag{55}$$

Conversely, if $v \in \mathcal{S}_L(\pi^-(\Lambda_0 H))$, then $v\Lambda_0 \in \mathcal{S}_L(H) = [H_2^p]^T Q_0$ so that $v \in [H_2^p]^T Q_{0,1}$ by (54) because Λ_0 is nonsingular. Hence, we have equality in (55) and the left Douglas-Shapiro-Shields factorization of $\pi^-(\Lambda_0 H)$ is of the form $Q_{0,1}^{-1} P_{0,1}$. By the same argument, the left Douglas-Shapiro-Shields factorization of $\pi^-(\Lambda_0 F)$ is of the form $S_{0,1}^{-1} R_{0,1}$.
In a similar way, since

$$\pi^-(\Lambda_0 H) Q_1 = \Lambda_0 P_1 - \pi^+(\Lambda_0 H) Q_1$$

lies in $H_2^{p\times m}$, we have that

$$Q_1 H_2^p \subset \mathcal{S}_R\left(\pi^-(\Lambda_0 H)\right)$$

and thus the right Douglas-Shapiro-Shields factorization of $\pi^-(\Lambda_0 H)$ is of the form $P_{1,1} Q_{1,1}^{-1}$ where $Q_{1,1}$ is a left divisor of Q_1 :

$$Q_1 = Q_{1,1}\,\Lambda_1$$

for some inner Λ_1. Now, remember that H is critical so that (22) holds. Multiplying on the left by Λ_0 and applying π^- gives

$$\pi^-(\Lambda_0 F) - \pi^-(\Lambda_0 H) = Q_{0,1}^{-1}\,[G\Lambda_1^{-1}]\,Q_{1,1}^{-1}. \tag{56}$$

The rank of $\pi^-(\Lambda_0 F) - \pi^-(\Lambda_0 H)$ is equal to that of $F - H$ for it is the rank of G and, by what precedes, $\pi^-(\Lambda_0 F)$ and $\pi^-(\Lambda_0 H)$ have degree $k_0 = k - \deg \Lambda_0$ and $n_0 = n - \deg \Lambda_0$ respectively. Moreover, (56) is identical to (22) upon replacing $Q_{0,1}$ by Q_0, $Q_{1,1}$ by Q_1, $\pi^-(\Lambda_0 H)$ by H, $\pi^-(\Lambda_0 F)$ by F, and $G\Lambda_1^{-1}$ by G. Hence, we may have assumed from the start that Q_0 and S_0 are right coprime. A similar observation holds concerning the left coprimeness of Q_1 and S_1.
From these coprimeness assumptions, two important facts follow namely:

Fact 1 G_0, and G defined in (24) have the same left inner denominator in their Douglas-Shapiro-Shields factorization; this we deduce from Lemma 4 and its transpose by observing that $G_0 = \pi^-(Q_0 F)$, whence S_1 is the right inner factor of G_0, and that $G = G_0 Q_1 = \pi^-(G_0 Q_1)$.

Fact 2 $\mathbf{Ker}\{G_0\} \cap \mathcal{G}_a^m \subset Q_1 \mathcal{G}_a^m, \quad \forall a \in \mathbf{U}$; suppose indeed that $h \in \mathcal{G}_a^m$ satisfies $G_0 h = 0$, so that $Q_0 F h \in \mathcal{G}_a^p$ by (24). By coprimeness we can write

$$XQ_0 + YS_0 = I_p, \qquad X, Y \in H_2^{p\times p},$$

and this equation can be localized at a *i.e.* we may regard each entry of X, Y, Q_0, and S_0 as a member of $\mathcal{G}_a$. Multiplying on the right by F considered as an element of $\mathcal{M}_a^{p\times m}$, we obtain

$$XQ_0 F + YR_0 = F,$$

implying that $Fh \in \mathcal{G}_a^p$. Since $G_0 h = 0$, we have $Hh = Fh$ in view of (24); considering that $H = P_1 Q_1^{-1}$ is a coprime factorization of H over H_∞ and thus *a fortiori* over $\mathcal{G}_a$ (localize the Bezout identity), we deduce from Lemma 1 that $h \in Q_1 \mathcal{G}_a^m$ and this establishes **Fact 2**.

We now complete the proof of Proposition 2 using **Fact 1**, **Fact 2**, and Lemma 3.
By **Fact1**, the left Douglas–Shapiro–Shields factorizations of G and G_0 can be written

$$G = \Theta_0^{-1} L_0, \qquad G_0 = \Theta_0^{-1} L_0'.$$

Hence, the relation $G_0 = G\,Q_1^{-1}$ yields $L_0' = L_0\,Q_1^{-1}$. As L_0, L_0' and Q_1 are analytic in $\mathbf{U}$, we may localize this equation at each $a \in \mathbf{U}$. From **Fact 2**, we have that

$$\mathbf{Ker}\{L_0'\} \cap \mathcal{G}_a^m = \mathbf{Ker}\{G_0\} \cap \mathcal{G}_a^m \subset Q_1 \mathcal{G}_a^m.$$

We are thus in position to apply Lemma 3: the multiplicity of a viewed as a zero of Q_1 does not exceed its multiplicity when considered as a zero of L_0, and the latter is also its multiplicity as a zero of G by (32) since $G = \Theta_0^{-1} L_0$ is a left coprime factorization over $\mathcal{G}_a$ (localize the Bezout equation again). Summing over the zeros of Q_1, noticing they all lie in $\mathbf{U}$, we conclude that the polynomial of zeros of Q_1, say d, divides that of G. Let r be the rank of G. By property **P2** the polynomial of zeros of G, say ϕ, is the greatest common divisor of the numerators of all $r \times r$ minors of G once these minors are written over the polynomial of poles, say ψ. Since G vanishes at infinity, the valuation at infinity of a $r \times r$ minor is at least r so that $\deg \psi - \deg \phi \geq r$. Hence, $\deg \psi - \deg d \geq r$. Since the McMillan degree is equal to the degree of the polynomial of poles, this achieves the proof. □

Corollary 2 *Let $\hat{F} \in \Sigma_{p,m}^0(n)$, and $K \subset \Sigma_{p,m}^0(n)$ a compact set containing $\hat{F}$ in its interior. There exists $\alpha > 0$ such that*

$$\Phi_F : H \mapsto \|F - H\|_2^2$$

has a unique critical point in K whenever $F \in \bar{H}_{2,0}^{p\times m}$ satisfies $\|F - \hat{F}\|_2 < \alpha$. This critical point is non-degenerate and realizes the minimum of Φ_F on $\Sigma_{p,m}^0(n)$.

Proof: let $\psi : \mathcal{U} \to \Sigma_{p,m}^0(n)$ be a local parametrization of $\Sigma_{p,m}^0(n)$ around $\hat{F}$, where $\mathcal{U}$ is open in $\mathbb{R}^{n(m+p)}$. Define

$$\begin{array}{cccc} \Theta : & \bar{H}_{2,0}^{p\times m} \times \mathcal{U} & \longrightarrow & \mathcal{L}(\mathbb{R}^{n(m+p)}, \mathbb{R}) \\ & (F, u) & \mapsto & < F - \psi(u), \mathbf{D}\psi(u) > \end{array},$$

where $\mathcal{L}(\mathbb{R}^{n(m+p)}, \mathbb{R})$ denotes the set of linear forms on $\mathbb{R}^{n(m+p)}$. In the above notation, it is understood that the effect of $\Theta(F, u)$ on $x \in \mathbb{R}^{n(m+p)}$ is

$$< F - \psi(u), \mathbf{D}\psi(u).x >,$$

and clearly $H = \psi(u)$ is a critical point of Φ_F if, and only if, $\Theta(F, u)$ is the zero linear form.

Let $\hat{F} = \psi(\hat{u})$ so that
$$\Theta(\hat{F}, \hat{u}) = 0.$$
The partial derivative $\mathbf{D}_2\Theta(F, u)$ is a linear map $\mathbb{R}^{n(m+p)} \to \mathcal{L}(\mathbb{R}^{n(m+p)}, \mathbb{R})$ associating to each $h \in \mathbb{R}^{n(m+p)}$ the form whose effect on x is
$$\mathbf{D}_2\Theta(F, u).h\,(x) \;=\; < F - \psi(u), \mathbf{D}^2\psi(u).(h, x) > - < \mathbf{D}\psi(u).h, \mathbf{D}\psi(u).x > .$$
Evaluating at the point $(\hat{F}, \hat{u})$ yields
$$\mathbf{D}_2\Theta(\hat{F}, \hat{u}).h \;=\; - < \mathbf{D}\psi(\hat{u}).h, \mathbf{D}\psi(\hat{u}) >,$$
so that $\mathbf{D}_2\Theta(\hat{F}, \hat{u})$ is clearly an isomorphism. The implicit mapping theorem asserts that there exist open neighbourhoods $\mathcal{V}$ of $\hat{F}$ in $\bar{H}_{2,0}^{p\times m}$ and $\mathcal{U}_1 \subset \mathcal{U}$ of $\hat{u}$ such that
$$\Theta(F, u) = 0$$
has exactly one solution $u(F) \in \mathcal{U}_1$ for $F \in \mathcal{V}$ which is a smooth function of F such that $\mathbf{D}_2\Theta(F, u(F))$ remains nonsingular througout $\mathcal{V}$. Put differently, this means Φ_F has exactly one critical point in $\psi(\mathcal{U}_1)$ for $F \in \mathcal{V}$ and it is non-degenerate.
Now, let us shrink $\mathcal{V}$ to make and $\psi(\mathcal{U}_1) \subset K$. *We claim that Φ_F has no critical point in the corona $\mathcal{C} = K - \psi(\mathcal{U}_1)$ if F is sufficiently close to $\hat{F}$.* Indeed, let us denote by
$$P_H \;: \bar{H}_{2,0}^{p\times m} \longrightarrow \mathbf{T}_H\Sigma_{p,m}^0(n)$$
the orthogonal projection, so that H is critical for Φ_F if, and only if, $P_H(F - H) = 0$. Observe that $P_H(\hat{F} - H)$ does not vanish except at $H = \hat{F}$ by Theorem 5. Hence, by compactness, there exists $\alpha > 0$ such that
$$\|P_H(\hat{F} - H)\|_2 > \alpha \quad \text{for all } H \in \mathcal{C}.$$
If $\|F - \hat{F}\|_2 < \alpha$, we deduce that
$$\|P_H(F - H)\|_2 > \|P_H(\hat{F} - H)\|_2 - \|P_H(F - \hat{F})\|_2 > 0$$
whenever $H \in \mathcal{C}$, *thereby establishing the claim.*
Diminishing α if necessary, $\mathcal{V}$ will contain the ball of center $\hat{F}$ and of radius α and consequently $\|F - \hat{F}\|_2 < \alpha$ will imply that Φ_F has a unique critical point in K. Finally, for α sufficiently small, this point realizes the *minimum* of Φ_F by the continuity of best approximants established in [7, Prop.5]. □

6 Conclusion.

We took in this paper a few basic steps in H_2 matrix rational approximation, establishing a two–sided tangential interpolation equation for the critical points and deriving from this a bound on the rank of the error when the function to be approximated is itself rational. This enabled us to obtain a local uniqueness result when approximating near–rational functions. Concerning the uniqueness issue, which is of great importance from the computational viewpoint, it would be interesting to know whether the interpolation property we just mentioned entails, as in the scalar case, uniqueness or asymptotic uniqueness for certain classes of functions like matrix–valued Markov functions or exponentials [13] [12]. Note that a major technical piece is missing there, as there is no matrix analog so far to the index theorem [10]. Also, a natural generalization is to adjoin a weight in the *criterium* and this is quite important for System–theoretic applications. This generalization is still wide–open, even in the scalar case. Note, however, that the rank of the error may no longer be bounded as in Proposition 2 if a weight is added, even in the scalar case [46].

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Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles - B.P. 93 - 06902 Sophia Antipolis Cedex (France)

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